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INTRODUCTION

TO

Algebra;

IN WHICH

THE FUNDAMENTAL RULES

ARE CLEARLY DEMONSTRATED, AND THE WHOLE RENDERED EASY AND FAMILIAR TO EVERY CAPACITY.

WITH

AN APPENDIX,

CONTAINING

THE SOLUTION OF ONE HUNDRED ALGEBRAICAL QUESTIONS.

By ROBERT SHARP,

TEACHER OF MATHEMATICS, EDINBURGH.

EDINBURGH:

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1820.

TO

JOHN CHRISTISON, Esq. A. M.

HOUSE GOVERNOR OF GEORGE HERIOT'S HOSPITAL,

EDINBURGH,

AS AN EXPRESSION OF GRATEFUL REGARD,

THE FOLLOWING

Introduction to Algebra,

IS MOST RESPECTFULLY INSCRIBED,

BY HIS MOST OBEDIENT

HUMBLE SERVANT,

ROBERT SHARP.

EXAMPLES.

1. Reduce $\frac{b^2-y^2}{b^3-y^3}$ to its lowest terms.

$$b^2-y^2)b^3-y^3(b$$
$$\underline{b^3-by^2}$$
$$by^2-y^3)b^2-y^2$$

or $\frac{by^2-y^3}{y^2}=b-y)b^2-y^2(b+y$
$$\underline{b^2-by}$$
$$by-y^2$$
$$by-y^2$$

Therefore $b-y$ is the greatest common measure: now divide both terms of the fraction by it.

Thus, $b-y\Big)\frac{b^2-y^2}{b^3-y^3}=\frac{b+y}{b^2+by+y^2}=$ the fraction in its lowest terms.

2. Reduce $\frac{x^3-b^2x}{x^2+2bx+b^2}$ to its lowest terms.

$$x^2+2bx+b^2)x^3-b^2x(x$$
$$\underline{x^3+2bx^2+b^2x}$$
$$-2bx^2-2b^2x$$

now $-\frac{2bx^2-2b^2x}{-2bx}=x+b$

Then $x+b)x^2+2bx+b^2(x+b$
$$\underline{x^2+bx}$$
$$bx+b^2$$
$$bx+b^2$$

Therefore $x+b$ is the greatest common measure; and dividing both terms of the fraction by it, we have,

$x+b\Big)\frac{x^3-b^2x}{x^2+2bx+b^2}=\frac{x^2-bx}{x+b}=$ the fraction in its lowest terms.

EXAMPLES FOR PRACTICE.

Reduce $\dfrac{a^4 - x^4}{a^5 - x^2a^3}$ to its lowest terms.

Ans. $\dfrac{a^2 + x^2}{a^3}$.

2. Reduee $\dfrac{a^2 - 1}{ab + b}$ to its lowest terms.

Ans. $\dfrac{a - 1}{b}$.

3. Reduce $\dfrac{a^2 - b^2}{a^4 - b^4}$ to its lowest terms.

Ans. $\dfrac{1}{a^2 - b^2}$.

CASE VII. To add fractional quantities together.

RULE.—*Reduce the fractions to a common denominator, add the numerators together, and under their sum write the common denominator.*

EXAMPLES.

1. Add $\dfrac{x}{2}$ and $\dfrac{x}{3}$ together.

Thus, $\left.\begin{array}{l} x \times 3 = 3x \\ x \times 2 = 2x \end{array}\right\}$ new numerators,
$2 \times 3 = 6$ common denominator.

Now $\dfrac{3x}{6} + \dfrac{2x}{6} = \dfrac{5x}{6}$ their sum.

2. Add $\dfrac{2x}{3}$ and $\dfrac{3x}{5}$ together.

Thus $\left.\begin{array}{l} 2x \times 5 = 10x \\ 3x \times 3 = 9x \end{array}\right\}$ new numerators,
$3 \times 5 = 15$ common denominator.

Now $\dfrac{10x}{15} + \dfrac{9x}{15} = \dfrac{19x}{15}$ their sum.

ERRATA.

Page 2. *line* 7. for $a:b$, read $a.b$.

13. *Ex.* 5th, for $-4z^2$, read $-4z^4$.

14. *line* 16. for $a^3-2a^2b+2ab-b^3$, read $a^3-2a^2b+2ab^2-b^3$.

17. *Ex.* 3d, in the quotient, for $-b$, read $-3b$.

18. *line* 9. for $5y$, read $-5y$.

26. *line* 2. for $\frac{zc-zd-c^2+d^2}{c-d}$, read $\frac{zc-zd-c^2-d^2}{c-d}$.

28. *line* 4. for $\frac{15x^2+x}{3x}$, read $\frac{5x^2+x}{3x}$.

34. *Ex.* 3d, line 3. for $\overline{3a-4c}\times 4c=12ac-16bc$, read $\overline{3a-4b}\times 4c=12ac-16bc$.

98. *line* 2. from bottom, for $(xy=6)^4$, read $(xy=6)4$.

133. *line* 10. for $x=30205$ read 3.0205.

133. *line* 13. for $x^2+20x=20$, read $x^2+20x=100$.

PREFACE.

ALGEBRA, the subject of the present performance, is one of the most important and useful branches of the sciences, and may be justly considered as the key to all the rest. Geometry delights us by the simplicity of its principles, and the elegance of its demonstrations; Arithmetic is confined in its object, and partial in its application, but Algebra, or the analytic art, is general and comprehensive, and may be applied with success in all cases, where truth is to be obtained, and proper data can be established.

The Author's design in publishing the following pages, was to furnish Schools with a cheap and comprehensive *Text-book*, adapted to the present state of the improvements that have been made in the science. In the execution of this plan, he has studied perspicuity, conciseness, and utility.

Availing himself of the productions of his predecessors, wherever he found them to answer his purpose, he believes that very few things of a useful tendency, in former treatises on the subject of Algebra, are omitted in this compilation, and perhaps some things new will be found in different parts of the volume.

The Appendix contains a valuable selection of questions,† arranged according to the degrees of equations they produce, with their solutions at full length, which will be found of singular advantage to the young Algebraist, as the equations are fully reduced, and the different steps registered.

It will be of great use to Teachers as an exercise book, owing to the variety of questions it embraces, which may be given in whatever order the teacher thinks most proper.

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† Given in Alexander's Algebra, and not answered in Hill's Arithmetic.

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AN

INTRODUCTION

TO

ALGEBRA.

ALGEBRA is a general method of resolving mathematical problems, by means of equations; or, it is a method of computation, relative to all sorts of quantities, by means of certain indeterminate characters or symbols which have been invented for this purpose.

In algebraical inquiries some quantities are assumed as known or given, and others are unknown and to be found; the former are commonly represented by the first letters of the alphabet, a, b, c, d, &c. the latter by the last, u, x, y, z, &c.

The characters used to denote the operations are chiefly the following:—

$+$ signifies addition, and is named *plus*.

$-$ signifies subtraction, and is named *minus*.

$\times$ signifies multiplication, and is named *into*.

$\div$ signifies division, and is named *by*.

$\surd$ the mark of radicality denotes the square root; with a 3 before it, thus $\sqrt[3]{\;}$, the cube root; with a 4, thus $\sqrt[4]{\;}$, the fourth or biquadrate root, &c.; and $\sqrt[n]{\;}$ the nth root.

Proportion is commonly denoted by a colon between the antecedent and consequent of each ratio, and a double colon between the two ratios; thus, if a

be to b as c to d, we state it as follows, $a : b :: c : d$.

$=$ signifies equality, and is named *equal to*.

Henee $a + b$, denotes the sum of the quantities represented by a and b.

$a - b$ denotes that the quantity represented by b is to be subtracted from that represented by a.

$a \times b$, or $a : b$ or $a\,b$, represents the product of the quantity a multiplied into that of b.

$a \div b$ shows that the quantity represented by a is to be divided by that represented by b.

The sign of division is often omitted, and instead of it, the dividend is plaeed above the divisor with a line between them like a fraction.

Thus $\frac{a}{b}$ signifies that a is to be divided by b.

$x = a - b + c$ is an equation, showing that x is equal to the difference of a and b, added to the quantity c.

$\sqrt{a}$ or $a^{\frac{1}{2}}$, is the square root of a, $\sqrt[3]{a}$ or $a^{\frac{1}{3}}$ is the cube root of a, and $\sqrt[m]{a}$ or $a^{\frac{1}{m}}$, is the mth root of a.

a^2 is the square of a, a^3 is the eube of a, a^4 is the fourth power of a, and a^m is the mth power of a.

$\overline{a + b} \times c$, or $(a + b)\,c$ is the produet of the eompound quantity $a + b$ multiplied by the single quantity c, using the bar —— or the parenthesis ( ) as a vinculum to conneet several quantities into one.

$\overline{a + b} \div \overline{a - b}$ or $\frac{a + b}{a - b}$, expressed like a fraction is the quotient of $a + b$ divided by $a - b$.

$\sqrt{a\,b + c\,d}$ or $\overline{a\,b + c\,d}^{\frac{1}{2}}$, is the square root of the eômpound quantity $a\,b + c\,d$.

$c\,\sqrt{a\,b + b\,c}$ or $c\,(a\,b + b\,c)^{\frac{1}{2}}$ is the produet of c into the square root of the eompound quantity $a\,b + b\,c$.

4 a denotes that the quantity a is to be taken 4 times, and 6 $(a + b)$ is 6 times $a + b$, and these numbers 4 and 6 showing how often the quantities are to be taken or multiplied, are ealled the eo-efficients.

Like quantities, are those which consist of the same letters and powers; as a and $3a$, or $2ab$ and $4ab$, or $4a^2bc$ and $-6a^2bc$.

Unlike quantities are those which consist of different letters or different powers; as a and b, or $2a$ and a^2, or $3ab^2$ and $3abc$.

Simple quantities are those which consist of one term only; as a, or $3a$, or $5ab$, or $7abc^2$.

Compound quantities are those which consist of two or more terms; as $a+b$, or $a+2b-3c$.

A binomial is a compound quantity consisting of two terms, a trinomial of three, a quadrinomial of four, a multinomial of several terms.

Thus, $a+b$ is a binomial, $a+b+c$ is a trinomial, a quadrinomial is $a-b+c-d$, and a multinomial is $a+b-c+d+x-3y$, &c.

A residual quantity is a binomial, having one of its terms negative.

Thus $a-b$, $x-y$, and $a-z$, are residuals.

Positive or affirmative quantities, are those which are to be added, or have the sign $+$. As a or $+a$ or ab; for when a quantity is found without a sign, it is understood to be positive, or to have the sign $+$ prefixed.

Negative quantities are those which are to be subtracted. As $-a$, or $-2ab$, or $-3ab^2$.

The negative sign is never omitted in any case.

Like signs are either all positive ($+$), or all negative ($-$).

Unlike signs are when some are positive and some negative.

The power of a quantity (a) is its square (a^2), or cube (a^3), or biquadrate (a^4) &c. called also the second power, or third power, or fourth power, &c.

The index or exponent is the number which denotes the power or root of a quantity. So 2 is the exponent of the square or second power a^2, and 3 is the index of the cube or third power, and $\frac{1}{2}$ is the index of the square root, $a^{\frac{1}{2}}$, or $\sqrt{a}$; and $\frac{1}{3}$ is the index of the cube root $a^{\frac{1}{3}}$ or $\sqrt[3]{a}$.

A rational quantity is that which has no radical sign or index annexed to it. As a, or $3\,a\,b$.

An irrational quantity, or surd, is that which has not an exact root, or it is expressed by means of the radical sign $\sqrt{}$. As $\sqrt{2}$, or $\sqrt{a}$, or $\sqrt[3]{a^2}$, or $a\,b^{\frac{1}{2}}$.

The reciprocal of any quantity, is that quantity inverted, or unity divided by it. So the reciprocal of a or $\frac{a}{1}$ is $\frac{1}{a}$ and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

Having in the foregoing explanation of the signs, defined a vinculum to be a mark or character, either drawn over or including several letters, quantities, or terms, to connect them together as one quantity; we shall now give a few examples, showing the use of it in expressing the sums, differences, products, and quotients, of compound quantities. Suppose $A = 12$, $B = 5$, $C = 20$, and $D = 15$.

$$\overline{A+B}+\overline{C-D}=\overline{12+5}+\overline{20-15}=17+5=22$$

$$\overline{A-B}+\overline{C-D}=\overline{12-5}+\overline{20-15}=7+5=12$$

$$\overline{C+D}-\overline{A+B}=\overline{20+15}-\overline{12+5}=35-17=18$$

$$\overline{C+D}-\overline{A-B}=\overline{20+15}-\overline{12-5}=35-7=28$$

$$\overline{C+D}-A-B=\overline{20+15}-12-5=35-17=18$$

$$\overline{A+B}\times C=\overline{12+5}\times 20=17\times 20=340$$

$$A+B\times C=12+5\times 20=12+100=112$$

$$\overline{A+B}\times\overline{C-D}=\overline{12+5}\times\overline{20-15}=17\times 5=85$$

$$\overline{A+B}\times C-D=\overline{12+5}\times 20-15=340-15=325$$

$$\overline{C+D}\div\overline{A-B}=\overline{20+15}\div\overline{12-5}=35\div 7=5$$

or $$\frac{C+D}{A-B}=\frac{20+15}{12-5}=\frac{35}{7}=5$$

$$\overline{C+D}\div\overline{B+A}=\overline{20+15}\div\overline{5+12}=\tfrac{35}{17}=2\tfrac{1}{17}$$

$$\overline{C+D}\div B+A=\overline{20+15}\div 5+12=7+12=19$$

In the following examples the numeral value of each letter is given, from which the value of each example is to be computed, in order to learn perfectly the power or force of the algebraic characters.

Let $a=4$, $b=6$, $c=8$, $d=1$. Then will

1. $a+b-c+d=4+6-8+1=3$
2. $a+b-3c+d=4+6-24+1=-13$
3. $\frac{2a+3b+4c}{5a}=\frac{8+18+32}{20}=\frac{58}{20}=2\frac{18}{20}$
4. $\overline{a+b}\times\overline{a+c}=\overline{4+6}\times\overline{4+8}=10\times12=120$
5. $\frac{a+b}{c}+\frac{a+b}{d}=\frac{4+6}{8}+\frac{4+6}{1}=\frac{90}{8}=11\frac{1}{4}$

Required the numeral values of the following quantities, supposing a, b, c, d, e, to be 4, 6, 8, 1, *and* 0, *respectively.*

1. $5\sqrt{ab}+b^2-2ab-e^2=$
2. $\frac{a}{c}\times d-\frac{a-b}{d}+2a^2e=$
3. $3\sqrt{c}+2a\sqrt{\overline{2a+b-d}}=$
4. $3a^2b+\sqrt{c^2+\sqrt{2ac+c^2}}=$
5. $\frac{2b+c}{3-c}-\frac{\sqrt{5b+c}+\sqrt{d}}{2a+c}=$

ADDITION.

Addition in algebra is finding the sum of several algebraical quantities, and connecting those quantities together with their proper signs. And this is generally divided into the three following cases.

Case I. When all the indeterminate letters are the same, and have the same sign.

RULE.—*Add all the coefficients together, to which prefix the common sign, and place the common quantities or letters after.*

EXAMPLES.

$$\begin{array}{c} (1) \\ a \\ a \\ \hline 2a \end{array} \qquad \begin{array}{c} (2) \\ -a \\ -a \\ \hline -2a \end{array}$$

REMARK.—The reasons on which the rule is founded will readily appear from the following considerations; for let a represent any thing whatever, then, as in the first example, a and a added together will make $2a$, or two times the thing represented by a; if a represent 1 shilling, to which if 1 shilling be added, the sum will be 2 shillings or $2a$.

In like manner, if $-a$ represent the want or debt of 1 shilling, to which if we add the want or debt of another shilling, the sum will be the want or debt of 2 shillings, or $-2a$, as in the second example.

$$\begin{array}{c} (3) \\ 2a \\ 3a \\ 4a \\ 5a \\ \hline 14a \end{array} \qquad \begin{array}{c} (4) \\ -2a \\ -3a \\ -4a \\ -5a \\ \hline -14a \end{array} \qquad \begin{array}{r} (5) \\ 2a^2+1 \\ 3a^2+2 \\ 4a^2+3 \\ 5a^2+4 \\ \hline 14a^2+10 \end{array}$$

$$\begin{array}{r} (6) \\ 3ax-2y \\ 4ax-5y \\ 2ax-4y \\ ax-3y \\ \hline 10ax-14y \end{array} \qquad \begin{array}{r} (7) \\ x+y+z \\ 2x+3y+z \\ 3x+4y+z \\ 4x+5y+z \\ \hline 10x+13y+4z \end{array} \qquad \begin{array}{r} (8) \\ a^2-1y^2 \\ 2a^2-3y^2 \\ 3a^2-4y^2 \\ 4a^2-5y^2 \\ \hline 10a^2-13y^2 \end{array}$$

$$\begin{array}{r} (9) \\ 7x^2+3xy-5bc \\ 9x^2+2xy-7bc \\ 11x^2+5xy-4bc \\ x^2+4xy-bc \\ x^2+9xy-2bc \\ \hline 29x^2+23xy-19bc \end{array} \qquad \begin{array}{r} (10) \\ 4a^3-3a^2+1 \\ 2a^3-a^2+17 \\ 5a^3-2a^2+4 \\ 3a^3-7a^2+3 \\ a^3-a^2+10 \\ \hline 15a^3-14a^2+35 \end{array}$$

$$\begin{array}{ll}
(11) & (12) \\
30-13x^{\frac{1}{2}}-3xy & 5xy-3x+ab \\
23-10x^{\frac{1}{2}}-4xy & 8xy-4x+ab \\
14-14x^{\frac{1}{2}}-7xy & 3xy-5x+ab \\
10-16x^{\frac{1}{2}}-5xy & xy-2x+ab \\
\underline{16-20x^{\frac{1}{2}}-\ xy} & \underline{4xy-\ x+ab}
\end{array}$$

CASE II. To add quantities which are like, but have unlike signs.

RULE.—*Add all the affirmative coefficients into one sum and negative into another, and subtract the less of these sums from the greater, prefixing the sign of the greater; to which join the common letters or quantities.*

EXAMPLES.

$$\begin{array}{cc}
(1) & (2) \\
+5a & -5a \\
\underline{-3a} & \underline{+3a} \\
+2a & -2a
\end{array}$$

The reason of the rule will readily appear by considering the nature of the quantities which are to be added. The addition here consists in finding what will arise by taking or incorporating several quantities together, some of which are additive, and others subtractive. As such quantities are in their effects contrary, and destroy each other, in proportion as the lesser quantity prevails, their sum will be truly estimated by considering how much the greater quantity exceeds the less, that is, by taking the less from the greater, and prefixing the sign of the greater to what remains.

Suppose, in the first example above, that a represents 1 shilling, and that the quantities with the sign + are sums which I possess, then will those with the sign — be sums which I owe, being in effect contrary to the former; and let the example referred to be considered as an account which I have to settle, now it appears that I have $5a$ or five shillings in my possession, and that I owe $3a$ or 3 shillings; this debt will certainly reduce my stock instead of increasing it; therefore, instead of adding it to the 5 shillings, I must subtract it, and the remainder $2a$, or 2 shillings, will be what I am worth.

Again, in the second example, we see that the subtractive quantities prevail, or that my debts are more than my effects, that is —

$5a$ is greater than $+3a$ by $-2a$, or that I am worth 2 shillings less than nothing.

Here it ought to be remarked, that debts or subtractive quantities, or sums of money, are as much real sums of money, real numbers, as credits or additive sums, the mark — being a mark of the quality of such numbers, and not of their quantity, denoting that their quality is such that, in reasoning upon the worth of a person, they are to be subtracted.

$$\begin{array}{cccc}
(1) & (2) & (3) & (4) \\
+3a & -3a & +3a+1 & -3a-4 \\
-2a & +2a & -2a \quad 2 & +2a \quad 5 \\
+4a & -4a & +4a \quad 3 & -4a-2 \\
-2a & +2a & -2a-4 & +2a \quad 3 \\
\hline
+3a & -3a & +3a+2 & -3a+2
\end{array}$$

$$\begin{array}{ccc}
(5) & (6) & (7) \\
3z^2+\ ay & 3\sqrt{x}-\ y & -3\sqrt{y}-\ 1 \\
-2z^2-\ ay & -2\sqrt{x}+\ y & -4\sqrt{y}+\ 4 \\
4z^2+\ ay & 11\sqrt{x}-3y & +3\sqrt{y}-\ 8 \\
-2z^2+4ay & -\sqrt{x}-\ y & -2\sqrt{y}-10 \\
\hline
3z^2+5ay & 11\sqrt{x}-4y & -6\sqrt{y}-15
\end{array}$$

$$\begin{array}{cc}
(8) & (9) \\
-7ab+3bc-\ xy & +4x^2-3x+\ 4 \\
-\ ab+2bc+4xy & -2x^2+\ x-\ 5 \\
3ab-\ bc+2xy & 3x^2-5x+\ 1 \\
-2ab+4bc-3xy & 7x^2+2x-\ 4 \\
5ab-8bc+\ xy & -x^2-4x+13 \\
\hline
-2ab \qquad\qquad 3xy & 11x^2-9x+\ 9
\end{array}$$

$$\begin{array}{cc}
(10) & (11) \\
5a^3-3ab+\ b^2 & -2ab+7x \\
-a^3+\ ab-2b^2 & 3ab-4x \\
4a^3-3ab+\ b^2 & -6ab+2x \\
2a^3+4ab-4b^2 & 8ab-8x \\
\hline
 &
\end{array}$$

CASE III. To add quantities which are unlike, and have unlike signs.

RULE.—*Collect all the like quantities together, by taking their sums or differences, as in the two foregoing cases; set down those that are unlike one after another, with their proper signs.*

EXAMPLES.

First, Suppose it were required to add a and b together, which cannot be done otherwise than by connecting them together, with the sign $+$ wrote between them. That is done in this form, $a+b$.

For suppose a to represent 1 shilling, and b one yard, then the sum of a and b can be neither $2a$ nor $2b$, that is, neither 2 shillings nor 2 yards, but only 1 shilling plus 1 yard, that is, $a+b$.

(1)
$$\begin{array}{r} +3a \\ -4b \\ -2a \\ +6b \\ \hline a+2b \end{array}$$

(2)
$$\begin{array}{r} 4x \\ 5y \\ 3x \\ -2y \\ \hline 7x+3y \end{array}$$

(3)
$$\begin{array}{r} 2a^2-b \\ -3a+c \\ -2a-b \\ 3a+4c \\ \hline 2a^2-2a-2b+5c \end{array}$$

(4)
$$\begin{array}{l} 6xy-12x^2 \\ -4x^2+3xy \\ 4x^2-2xy \\ -3xy+4x^2 \\ \hline 4xy-8x^2 \end{array}$$

(5)
$$\begin{array}{l} 4ax-130+3x^{\frac{1}{2}} \\ 5x^2+3ax+9x^2 \\ 7xy-4x^{\frac{1}{2}}+90 \\ \sqrt{x}+40-6x^2 \\ \hline 7ax+8x^2+7xy \end{array}$$

(6)
$$\begin{array}{l} 9+10\sqrt{ax}-5y \\ 2x+7\sqrt{xy}+5y \\ 5y+3\sqrt{ax}-4y \\ 10-4\sqrt{ax}+4y \\ \hline \end{array}$$

(7)
$$\begin{array}{l} 3ab+x-y \\ 4c-2y+x \\ 5ab-3c+d \\ 4y+x^2-2y \\ \hline \end{array}$$

SUBTRACTION.

SUBTRACTION in algebra, is finding the difference between two algebraical quantities, and connecting those quantities together with their proper signs. Which is performed by the following general rule.

RULE.—*Set those quantities from which the subtraction is to be made in one line, and the quantities to be* *subtracted in a line below them, observing to place like quantities under each other when they occur.*

Conceive all the signs of the lower line to be changed from — into +, or + into —, and then add them together, as in addition, and the result will be the difference required.

Remark.—This rule being the reverse of addition, the method of operation must be so likewise. It depends upon this principle, that to subtract an affirmative quantity from an affirmative, is the same as to add a negative quantity to an affirmative.

That is, $+2a$ taken from $+3a$, is the same with $-2a$ added to $+3a$.

Consequently, to subtract a negative quantity from an affirmative, is the same as to add an affirmative quantity to an affirmative.

That is, $-2a$ taken from $+3a$, is the same with $+2a$ added to $+3a$.

EXAMPLES.

(1)

From $+6a$
Take $-3a$
Remains $+9a$

(2)

From $5z-7b$
Take $2z-3b$
Remains $3z-4b$

(3)

From $9x^2-4y+8$
Take $6x^2+5y-4$
Rem. $3x^2-9y+12$

(4)

From $4y^9-3y-4$
Take $2y^2+2y+4$
Rem. $2y^2-5y-8$

(5)

From $7xy-3y^2-7z$
Take $6xy+2y^2-3z$
Rem. $xy-5y^2-4z$

(6)

From $6x^2y-3\sqrt{xy}-6xy$
Take $3x^2y+3\sqrt{xy}-4xy$
Rem. $3x^2y-6\sqrt{xy}-2xy$

(7)

From $5\sqrt{xy}+2x+4z$
Take $7\sqrt{xy}-3x-z$
Rem.

(8)

From $-ax^2-8-x^{\frac{1}{2}}$
Take $-ax+8+3$
Rem.

(9)

From $a+b-c+4z$
Take $a-b+3c-z$
Rem.

(10)

From $2a\sqrt{x}-y$
Take $-a\sqrt{x}+40$
Rem.

MULTIPLICATION.

MULTIPLICATION in algebra, is the method of finding the product arising from the multiplication of any two or more indeterminate quantities, which may be divided into the three following cases.

CASE I. When the multiplier and the multiplicand are both simple quantities.

RULE.—*Multiply the coefficients of the two quantities together, and annex to the product all the letters in both factors, which will give the whole product required.*

NOTE.—*If the multiplier and multiplicand are both affected with the sign $+$, or both with the sign $-$, the sign to be prefixed to the product must be $+$. But if one of them be $+$, and the other $-$, the sign of the product must be $-$. This is commonly expressed by saying, like signs give plus, and unlike signs minus.*

REMARK.—The truth of this rule for the signs will appear evident from the following considerations.

1. When $+a$ is to be multiplied by $+b$, this implies that $+a$ is to be taken as many times as there are units in b; and since the sum of any number of affirmative quantities is affirmative, it is plain that $+a \times +b = +ab$.

2. If two or more quantities are to be multiplied together, the product will be exactly the same, in what order soever they are placed, for a times b is the same as b times a; and therefore when $-a$ is to be multiplied by $+b$, or $+b$ by $-a$, this is the same as taking $-a$ as many times as there are units in $+b$, and since the sum of any number of negative quantities is negative, it is evident that $-a \times +b = -ab$, or $+b \times -a = -ab$.

3. When $-a$ is to be multiplied by $-b$, here $a - a = o$, therefore $+a - a \times -b$ is likewise $= o$, because o multiplied by any quantity produces o; and since the first term of the product, or $+a \times -b$ is, by Case II. $= -ab$, the last term, or $-a \times -b$ must be $= +ab$, in order to make the sum $-ab + ab = o$; consequently, $-a \times -b = +ab$.

EXAMPLES.

	(1)	(2)	(3)	(4)
Multiply	a	$2a$	$-ab$	$-2ab$
By	a	$-2a$	$-ab$	2
Product,	a^2	$-4a^2$	$+a^2b^2$	$-4ab$

(5)	(6)	(7)	(8)
$ab\sqrt{x}$	xy	$-ay^2$	$-5a^2$
c	$-z$	$4x$	$-4ab$
$abc\sqrt{x}$	$-xyz$	$-4axy^2$	$20a^3b$

(9)	(10)	(11)	(12)
$-3ax$	$-ax$	$+3xy$	$-5xyz$
$4x$	$-6c$	-4	$-5ax$

CASE II. When the multiplicand is a compound quantity, and the multiplier a simple one.

RULE.—*Multiply every term of the multiplicand, or compound quantity, separately, into the simple one, or multiplier, as in Case I. and place the products one after another, with their proper signs.*

EXAMPLES.

(1)

$$\begin{array}{ll} \text{Mult.} & a+b \\ \text{By} & x \\ \hline & ax+bx \end{array}$$

(2)

$$\begin{array}{l} -a-b \\ -x \\ \hline +ax+bx \end{array}$$

(3)

$$\begin{array}{l} -a-b \\ x \\ \hline -ax-bx \end{array}$$

(4)

$$\begin{array}{l} a+b \\ -x \\ \hline -ax-bx \end{array}$$

(5)

$$\begin{array}{l} ab+4\sqrt{x+y} \\ 3c \\ \hline 3abc+12c\sqrt{x+y} \end{array}$$

(6)

$$\begin{array}{l} -2x^2+x \\ -3x \\ \hline 6x^3-3x^2 \end{array}$$

The first and second cases being nearly alike, we shall proceed to the third.

CASE III. When the multiplier and multiplicand are both compound quantities.

RULE.—*Multiply every term of the multiplicand into every term of the multiplier, add the products together, and their sum will be the answer.*

EXAMPLES.

(1)

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad\ +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

(2)

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad\ -ab-b^2 \\ \hline a^2 \qquad\ -b^2 \end{array}$$

(3)

$$\begin{array}{l} 3a-2b \\ 5a+4b \\ \hline 15a^2-10ab \\ \qquad\ -12ab-8b^2 \\ \hline 15a^2+2ab-8b^2 \end{array}$$

(4)

$$\begin{array}{l} 7a^2+5ab^2 \\ 3a^2+ab^2 \\ \hline 21a^4+15a^3b^2 \\ \qquad\ +7a^3b^2+5a^2b^4 \\ \hline 21a^4+22a^3b^2+5a^2b^4 \end{array}$$

(5)

$$\begin{array}{l} 3xy-2z^2 \\ 3xy+2z^2 \\ \hline 9x^2y^2-6xyz^2 \\ \qquad\ +6xyz^2-4z^2 \\ \hline 9x^2y^2 \qquad\qquad -4z^4 \end{array}$$

(6)

$$\begin{array}{l} x^2+xy+y^2 \\ x^2-xy+y^2 \\ \hline x^4+x^3y+x^2y^2 \\ \quad\ -x^3y-x^2y^2-xy^3 \\ \qquad\qquad +x^2y^2+xy^3+y^4 \\ \hline x^4 \quad + \quad x^2y^2 \quad + \quad y^4 \end{array}$$

(7)

$$\begin{array}{l} 1-x+x^2-x^3 \\ 1+x \\ \hline 1-x+x^2-x^3 \\ \quad +x-x^2+x^3-x^4 \\ \hline 1 \qquad\qquad\qquad -x^4 \end{array}$$

When we have three or more quantities to multiply together, it is indifferent which two we begin with, for the products will always be the same, as will fully appear from the following example.

Let it be proposed to find the value or product of the four following factors, viz.

(I.)	(II.)	(III.)	(IV.)
$(a+b)$	(a^2+ab+b^2)	$(a-b)$ and	(a^2-ab+b^2)

1st, *Multiply the factors I. and II.*

$$\begin{array}{l} a^2 + ab + b^2 \\ a + b \\ \hline a^3 + a^2b + ab^2 \\ \quad + a^2b + ab^2 + b^3 \\ \hline a^3 + 2a^2b + 2ab^2 + b^3 \end{array}$$

Next multiply the factors III. and IV.

$$\begin{array}{l} a^2 - ab + b^2 \\ a - b \\ \hline a^3 - a^2b + ab^2 \\ \quad - a^2b + ab^2 - b^3 \\ \hline a^3 - 2a^2b + 2ab^2 - b^3 \end{array}$$

It remains now to multiply the first product I. II. by the second product III. IV.

$$\begin{array}{l} a^3 + 2a^2b + 2ab^2 + b^3 \\ a^3 - 2a^2b + 2ab - b^3 \\ \hline a^6 + 2a^5b + 2a^4b^2 + a^3b^3 \\ \quad - 2a^5b - 4a^4b^2 - 4a^3b^3 - 2a^2b^4 \\ \qquad + 2a^4b^2 + 4a^3b^3 + 4a^2b^4 + 2ab^5 \\ \qquad\quad - a^3b^3 - 2a^2b^4 - 2ab^5 - b^6 \\ \hline a^6 \qquad\qquad\qquad\qquad\qquad\qquad - b^6 \end{array}$$

2d, *Change the order of the question, that is, multiply the factors I. and III. then II. and IV. together.*

$$\begin{array}{l} a + b \\ a - b \\ \hline a^2 + ab \\ \quad - ab - b^2 \\ \hline a^2 \qquad - b^2 \end{array}$$

Then
$$\begin{array}{l} a^2 + ab + b^2 \\ a^2 - ab + b^2 \\ \hline a^4 + a^3b + a^2b^2 \\ \quad - a^3b - a^2b^2 - ab^3 \\ \qquad + a^2b^2 + ab^3 + b^4 \\ \hline a^4 \qquad + a^2b^2 \qquad + b^4 \end{array}$$

Then multiply the products I. III. and the II. IV.

$$\begin{array}{l} a^4 + a^2b^2 + b^4 \\ a^2 - b^2 \\ \hline a^6 + a^4b^2 + a^2b^4 \\ \quad\; - a^4b^2 - a^2b^4 - b^6 \\ \hline a^6 \qquad\qquad\qquad\quad - b^6 \end{array}$$ *which is the product.*

3d, *Again, multiply the I. factor by the IV. and next the II. by the III.*

$$\begin{array}{l} a^2 - ab + b^2 \\ a + b \\ \hline a^3 - a^2b + ab^2 \\ \quad + a^2b - ab^2 + b^3 \\ \hline a^3 \qquad\qquad\qquad + b^3 \end{array}$$

Next $a^2 + ab + b^2$

$$\begin{array}{l} a - b \\ \hline a^3 + a^2b + ab^2 \\ \quad - a^2b - ab^2 - b^3 \\ \hline a^3 \qquad\qquad\qquad - b^3 \end{array}$$

It remains to multiply the product I. IV. and II. III.

$$\begin{array}{l} a^3 + b^3 \\ a^3 - b^3 \\ \hline a^6 + a^3b^3 \\ \quad - a^3b^3 - b^6 \\ \hline a^6 \qquad\quad - b^6 \end{array}$$ *as in the two foregoing cases.*

It will be proper to illustrate this example by a numerical application.

Suppose $a = 3$, and $b = 2$. We shall have $a + b = 5$, and $a - b = 1$, farther, $a^2 = 9$, $ab = 6$, and $b^2 = 4$. Therefore $a^2 + ab + b^2 = 19$, and $a^2 - ab + b^2 = 7$. So that the product required is that of $5 \times 19 \times 1 \times 7 = 665$.

Now $a^6 = 729$, and $b^6 = 64$, consequently the product is $a^6 - b^6 = 665$, as we have already seen.

EXAMPLES FOR PRACTICE.

1. Multiply $a^2 + ac - c^2$ by $a - c$.
   Ans. $a^3 - 2ac^2 + c^3$.

2. Multiply $x^2 + 2xy + y^2$ by $x - y$.
   Ans. $x^3 + 3x^2y + 3xy^2 + y^3$.

3. Multiply $c + y + z$ by $y + z$.
   Ans. $cy + y^2 + 2yz + cz + z^2$.

4. Multiply $x + z - y$ by $x + z - y$.
   Ans. $x^2 + 2xz + z^2 - 2xy - 2xz + y^2$.

5. Multiply $a^3 + a^2b + ab^2 + b^3$ by $a - b$.
   Ans. $a^4 - b^4$.

DIVISION.

Division in algebra, is the method of finding the quotient arising from the division of one indeterminate quantity by another, and consequently performed by direct contrary operations to multiplication, and admits of the three following cases.

Case I. When the divisor and dividend are both simple quantities.

Rule.—*Divide the coefficients, as in common arithmetic, and to the quotient annex those letters in the dividend, which are not found in the divisor.*

General rule for the signs in all the cases of division.

When the signs of the divisor and dividend are alike, (that is, both + or both —) the sign of the quotient will be +.

When they are unlike, (that is, the one + and the other —) the sign of the quotient will be —.

Remark.—It is evident from the nature of the rule, that the divisor and quotient multiplied together will produce the dividend, it follows that the signs of the divisor and quotient must be such as will, by the rule of multiplication, produce the sign of the dividend; consequently,

1. When both terms are +, the quotient is +; for + in the divisor must have + in the quotient to produce + in the dividend.

2. When the terms are both —, the quotient must be +, for — in the divisor must have + in the quotient to produce — in the dividend.

3. When one of the terms is +, and the other —, the quotient must be —; for + in the divisor must have — in the quotient to produce — in the dividend; and — in the divisor must have — in the quotient to produce + in the dividend.

These particulars may be briefly expressed in one view, as under.

	Divisor.	Dividend.	Quotient.
1.	+	+	+
2.	—	—	+
3.	—	+	—
	+	—	—

Therefore, in division as well as in multiplication, like signs produce +, and unlike —.

EXAMPLES.

(1) $a)ab(b$ $\quad ab$

(2) $-a)-ab(+b$ $\quad -ab$

(3) $-5a)+15ab(-b$ $\quad +15ab$

(4) $+3a)-15ab(-5b$ $\quad -15ab$

(5) $3b)18ab^2(6ab$ $\quad 18ab^2$

(6) $-5a)15a^2b^2(-3ab^2$ $\quad 15a^2b^2$

NOTE.—When the quantities and coefficients in the divisor and dividend are all the same, the quotient will be an unit or 1. As in the two following examples.

(1) $ab)ab(1$ $\quad ab$ $\quad 0$

(2) $8ab)8ab(1$ $\quad 8ab$ $\quad 0$

When the quantities in the divisor cannot be exactly found in the dividend, set the dividend above a line, and the divisor under it, like a vulgar fraction. Thus,

(1)	(2)	(3)
$\frac{5a}{3d}$	$\frac{8adc}{7bef}$	$\frac{a^2}{7b}$

EXAMPLES FOR PRACTICE.

1. Divide $16a^2$ by $8a$.
   Ans. $2a$.

2. Divide $12a^2b^2$ by $-3a^2b$.
   Ans. $-4b$.

3. Divide $-15ay^2$ by $3ay$.
   Ans. $5y$.

4. Divide abx by bxy, or $\frac{abx}{bxy}$.

   Ans. $\frac{a}{y}$.

CASE II. When the divisor is a simple quantity, and the dividend a compound one.

RULE.—*Divide every term of the dividend by the divisor, as in the first case.*

EXAMPLES.

1. Divide $4ab + 6a^2$ by $2a$.

$$\frac{4ab + 6a^2}{2a} = 2b + 3a.$$

2. Divide $10ab + 15ax$ by $5a$.

$$\begin{array}{l} 5a)10ab + 15ax(2b + 3x \\ \underline{\quad 10ab \qquad\qquad} \\ \qquad\quad + 15ax \\ \qquad\quad + 15ax. \end{array}$$

3. Divide $30ab - 48b$ by b.

$$\begin{array}{l} b)30ab - 48b(30a - 48 \\ \underline{\quad 30ab \qquad\qquad} \\ \qquad\quad - 48b \\ \qquad\quad - 48b. \end{array}$$

4. Divide $abc - acd$ by ac.

$$
\begin{array}{l}
ac)\,abc - acd\,(b - d \\
\quad abc \\
\hline
\qquad\quad - acd \\
\qquad\quad - acd.
\end{array}
$$

5. Divide $ax - ab$ by a.

$$
\begin{array}{l}
a)\,ax - ab\,(x - b \\
\quad ax \\
\hline
\qquad - ab \\
\qquad - ab
\end{array}
$$

6. Divide $ax - xy + x^2$ by $4x$.

$$
\begin{array}{l}
4x\,\Big)\,ax - xy + x^2\,\Big(\dfrac{a - y + x}{4} \\
\qquad ax \\
\hline
\qquad\quad - xy \\
\qquad\quad - xy \\
\hline
\qquad\qquad\quad x^2 \\
\qquad\qquad\quad x^2
\end{array}
$$

EXAMPLES FOR PRACTICE.

1. Divide $10\,xz + 15\,xy$ by $5\,x$.
   Ans. $2\,z + 3y$.
2. Divide $15\,ab - 27b$ by $-3b$.
   Ans. $5a + 9$.
3. Divide $18x^2 - 9x$ by $-9x$.
   Ans. $2x + 1$.
4. Divide $a^2 - 2ax + x^2$ by $-a$.
   Ans. $-a + 2x - \dfrac{x^2}{a}$.
5. Divide $14a^2 - 7ab + 21ax - 28$ by $7a$.
   Ans. $2a - b + 3x - \dfrac{4}{a}$.
6. Divide $6bcdz + 4bzd^2 - 2b^2z^2$ by $2bz$.
   Ans. $3cd + 2d^2 - bz$.

CASE III. When the divisor and dividend are both compound quantities.

RULE.—*Arrange both divisor and dividend according to the powers of the same letter, beginning with the* *highest, then divide the first term of the dividend by the first term of the divisor, and place the result in the quotient, multiply each term of the divisor by this quantity, and subtract the product from the dividend, and to the remainder bring down as many terms of the dividend as are requisite for the next operation; proceed in this manner till all the terms of the dividend are brought down, as in common arithmetic.*

EXAMPLES.

1. Divide $a^2 + 2ab + b^2$ by $a + b$.

$$\begin{array}{l} a+b)a^2+2ab+b^2(a+b \\ \quad\ \ \underline{a^2+\ ab} \\ \qquad\qquad ab+b^2 \\ \qquad\qquad ab+b^2 \end{array}$$

2. Divide $a^2 - b^2$ by $a - b$.

$$\begin{array}{l} a-b)a^2-b^2(a+b \\ \quad\ \ \underline{a^2-ab} \\ \qquad\ \ +ab-b^2 \\ \qquad\ \ +ab-b^2 \end{array}$$

3. Divide $x^2 + 2x + 1$ by $x + 1$.

$$\begin{array}{l} x+1)x^2+2x+1(x+1 \\ \quad\ \ \underline{x^2+\ x} \\ \qquad\qquad x+1 \\ \qquad\qquad x+1 \end{array}$$

4. Divide $a^3 + 5a^2x + 5ax^2 + x^3$ by $a + x$.

$$\begin{array}{l} a+x)a^3+5a^2x+5ax^2+x^3(a^2+4ax+x^2 \\ \quad\ \ \underline{a^3+\ a^2x} \\ \qquad\quad 4a^2x+5ax^2 \\ \qquad\quad \underline{4a^2x+4ax^2} \\ \qquad\qquad\qquad ax^2+x^3 \\ \qquad\qquad\qquad ax^2+x^3 \end{array}$$

5. Divide $6a^3 - 48$ by $2a^2 + 4a + 8$.

$$\begin{array}{l} 2a^2+4a+8)6a^3-48(3a-6 \\ \qquad\qquad\quad \underline{6a^3+12a^2+24a} \\ \qquad\qquad\qquad\ -12a^2-24a-48 \\ \qquad\qquad\qquad\ -12a^2-24a-48 \end{array}$$

6. Divide $a^2 - 2ab + b^2$ by $a - b$.

$$a - b)a^2 - 2ab + b^2(a - b$$
$$\underline{a^2 - ab}$$
$$-ab + b^2$$
$$-ab + b^2$$

7. Divide $x^4 - 16y^4$ by $x - 2y$.

$$x - 2y)x^4 - 16y^4\ (x^3 + 2x^2y + 4xy^2 + 8y^3$$
$$\underline{x^4 - 2x^3y}$$
$$+2x^3y - 16y^4$$
$$\underline{+2x^3y - 4x^2y^2}$$
$$+4x^2y^2 - 16y^4$$
$$\underline{4x^2y^2 - 8xy^3}$$
$$+8xy^3 - 16y^4$$
$$+8xy^3 - 16y^4$$

8. Divide $a^3 + x^3$ by $a + x$.

$$a + x)a^3 + x^3(a^2 - ax + x^2$$
$$\underline{a^3 + a^2x}$$
$$-a^2x + x^3$$
$$\underline{-a^2x + ax^2}$$
$$ax^2 + x^3$$
$$ax^2 + x^3$$

9. Divide $a^5 - x^5$ by $a - x$.

$$a - x)a^5 - x^5(a^4 + a^3x + a^2x^2 + ax^3 + x^4$$
$$\underline{a^5 - a^4x}$$
$$a^4x - x^5$$
$$\underline{a^4x - a^3x^2}$$
$$a^3x^2 - x^5$$
$$\underline{a^3x^2 - a^2x^3}$$
$$a^2x^3 - x^5$$
$$\underline{a^2x^3 - ax^4}$$
$$ax^4 - x^5$$
$$ax^4 - x^5$$

10. Divide x by $x - a$.

$$x - a)x \quad (1 + \frac{a}{x} + \frac{a^2}{x^2}, \text{ \&c.}$$
$$\begin{array}{l} x - a \\ \hline + a \\ a - \frac{a^2}{x} \\ \hline \frac{a^2}{x} \\ \frac{a^2}{x} - \frac{a^3}{x^2} \\ \hline \frac{a^3}{x^2} \end{array}$$

In this example, the quotient is a series whose terms consist of the powers of a, in their order, divided by the powers of x.

Hence, by observing the manner in which these terms arise, their law may sometimes be discovered, and the quotient may be carried to any length, without further division.

11. Divide x by $1 + x$.

$$1 + x)\,x \quad (x - x^2 + x^3 - x^4 + x^5 - x^6, \text{ \&c.}$$
$$\begin{array}{l} x + x^2 \\ \hline -x^2 \\ -x^2 - x^3 \\ \hline +x^3 \\ x^3 + x^4 \\ \hline -x^4 \\ -x^4 - x^5 \\ \hline +x^5 \\ x^5 + x^6 \\ \hline -x^6 \\ -x^6 - x^7 \\ \hline x^7 \end{array}$$

The law of the series in this example is evident, the signs are alternately + and —, the exponents increase by unity.

12. Divide x^2y by x^2-y^2.

$$x^2-y^2\big)\,x^2y\quad\left(y+\frac{y^3}{x^2}+\frac{y^5}{x^4}+\frac{y^7}{x^6}\right., \text{ \&c.}$$

$$\begin{array}{l}
\underline{x^2y-y^3} \\
\qquad y^3 \\
\qquad \underline{y^3-\frac{y^5}{x^2}} \\
\qquad\qquad \frac{y^5}{x^2} \\
\qquad\qquad \underline{\frac{y^5}{x^2}-\frac{y^7}{x^4}} \\
\qquad\qquad\qquad \frac{y^7}{x^4} \\
\qquad\qquad\qquad \underline{\frac{y^7}{x^4}-\frac{y^9}{x^6}} \\
\qquad\qquad\qquad\qquad \frac{y^9}{x^6}
\end{array}$$

In this series the powers of the numerators increase in the series of odd numbers, and those of the denominators in the series of even numbers.

13. Divide 1 by $a^2+2ab+b^2$.

$$a^2+2ab+b^2\big)\,1\qquad\left(\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}\right.$$

$$\begin{array}{l}
\underline{1+\frac{2ab}{a^2}+\frac{b^2}{a^2}} \\
-\frac{2ab}{a^2}-\frac{b^2}{a^2} \\
\underline{-\frac{2ab}{a^2}-\frac{4b^2}{a^2}-\frac{2b^3}{a^3}} \quad \times a/a \\
\qquad\qquad \frac{3b^2}{a^2}+\frac{2b^3}{a^3} \\
\qquad\qquad \underline{\frac{3b^2}{a^4}+\frac{6b^3}{a^3}+\frac{3b^4}{a^4}} \\
\qquad\qquad\qquad -\frac{4b^3}{a^3}+\frac{b^4}{a^4}, \text{ \&c.}
\end{array}$$

NOTE.—When there is a remainder after the division, it must be written over the divisor, and annexed as a fraction to the quotient.

EXAMPLES FOR PRACTICE.

1. Divide $a^2 + 2ab + b^2$ by $a + b$.
   Ans. $a + b$.
2. Divide $a^2 - 2ab + b^2$ by $a - b$.
   Ans. $a - b$.
3. Divide $a^2 - b^2$ by $a - b$.
   Ans. $a + b$.
4. Divide $x^3 + y^3$ by $x + y$.
   Ans. $x^2 - xy + y^2$.
5. Divide $x^4 - y^4$ by $x + y$.
   Ans. $x^3 - x^2y + xy^2 - y^3$.
6. Divide $x^4 - y^4$ by $x - y$.
   Ans. $x^3 + x^2y + xy + y^3$.
7. Divide $x^5 + y^5$ by $x + y$.
   Ans. $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.
8. Divide $x^5 - y^5$ by $x - y$.
   Ans. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$.
9. Divide $x^5 + 1$ by $x + 1$.
   Ans. $x^4 - x^3 + x^2 - x + 1$.
10. Divide $a^3 - x^3 - 1$ by $a - x$.
    Ans. $a^2 + ax + x^2 + \frac{-1}{a-x}$.
11. Divide $b^4 - 3y^4$ by $b - y$.
    Ans. $b^3 + b^2y + by^2 + y^3 - \frac{2y^4}{b-y}$.

ALGEBRAIC FRACTIONS.

THE rules for managing Algebraic Fractions are exactly the same as those for Vulgar Fractions in common arithmetic, as will appear from the following rules and cases.

CASE I. To reduce a mixed quantity to an improper fraction.

RULE.—*Multiply the integer by the denominator of the fraction, and to the product add the numerator, and the denominator being placed under this sum, will be the improper fraction required.*

EXAMPLES.

1. Reduce $x + \frac{x}{3}$ to an improper fraction.

Thus, $x \times 3 + x = 4x$,

and $\frac{4x}{3} =$ fraction required.

2. Reduce $x + \frac{x}{4}$ to an improper fraction.

Thus, $\frac{x \times 4 + x}{4} = \frac{5x}{4}$. The answer.

3. Reduce $5x + \frac{4x}{6a^2}$ to an improper fraction.

Thus, $\frac{5x \times 6a^2 + 4x}{6a^2} = \frac{30a^2x + 4x}{6a^2} =$ the fraction required.

4. Reduce $a - \frac{z^2 - a^2}{a}$ to an improper fraction.

Thus, $a \times a = a^2$. In adding the numerator $z^2 - a^2$, it is to be recollected that the sign $-$ affixed to the fraction $\frac{z^2 - a^2}{a}$, denotes that the whole of that fraction is to be subtracted, and consequently that the signs of each term of the numerator must be changed when it is combined with a^2, hence the improper fraction required is $\frac{a^2 - z^2 + a^2}{a}$ or $\frac{2a^2 - z^2}{a}$.

EXAMPLES FOR PRACTICE.

1. Reduce $z + \frac{d^2 - b^2}{x}$ to an improper fraction.

Ans. $\frac{zx + d^2 - b^2}{x}$.

2. Reduce $z - \frac{c^2 + d^2}{c - d}$ to an improper fraction.

Ans. $\frac{zc - zd - c^2 + d^2}{c - d}$.

CASE II. To reduce an improper fraction to a whole or mixed number or quantity.

RULE.—*Divide the numerator by the denominator, for the integral part, and place the remainder, if any, over the denominator, and it will be the mixed quantity required.*

EXAMPLES.

1. Reduce $\frac{10x}{2}$ to a whole or mixed quantity.

Thus, $\frac{10x}{2} = 5x$ the answer.

2. Reduce $\frac{5x}{4}$ to a whole or mixed quantity.

Thus, $\frac{5x}{4} = x + \frac{x}{4}$ the answer.

3. Reduce $\frac{ab + a^2}{b}$ to a whole or mixed quantity.

Thus, $\frac{ab + a^2}{b} = a + \frac{a^2}{b}$ the answer.

4. Reduce $\frac{3ax + 4x^2}{a + x}$ to a whole or mixed quantity.

Thus, $\frac{3ax + 4x^2}{a + x} = 3x + \frac{4x^2}{a + x}$ the answer.

EXAMPLES FOR PRACTICE.

1. Reduce $\frac{2ax - 3x^2}{a}$ to a whole or mixed quantity.

Ans. $2x - \frac{3x^2}{a}$.

2. Reduce $\frac{12a^2+4a-3c}{4a}$ to a whole or mixed quantity.

Ans. $3a+1-\frac{3c}{4a}$.

CASE III. To reduce fractions of different denominators to others of the same value, which shall have a common denominator.

RULE.—*Multiply every denominator separately, into all the denominators except its own, for the new numerators, and all the denominators together for the common denominator.*

EXAMPLES.

1. Reduce $\frac{x}{3}$ and $\frac{x}{4}$ to fractions of equal values that shall have a common denominator.

$$\text{Thus } \left.\begin{array}{r} x\times 4=4x \\ x\times 3=3x \end{array}\right\} \text{numerators.}$$
$$3\times 4=12 \text{ the common denominator.}$$

Whence $\frac{4x}{12}$ and $\frac{3x}{12}$ are the fractions required.

2. Reduce $\frac{a}{x}$ and $\frac{b}{z}$ to a common denominator.

$$\text{Thus } \left.\begin{array}{r} a\times z=az \\ b\times x=bx \end{array}\right\} \text{numerators.}$$
$$x\times z=xz \text{ common denominator.}$$

Hence $\frac{az}{xz}$ and $\frac{bx}{xz}$ are the fractions required.

3. Reduce $\frac{2x+1}{5}$, and $\frac{3x}{4}$ to a common denominator.

$$\left.\begin{array}{r} \overline{2x+1}\times 4=8x+4 \\ 3x\times 5=15x \end{array}\right\} \text{numerators.}$$
$$5\times 4=20 \text{ common denominator.}$$

Therefore $\frac{8x+4}{20}$, and $\frac{15x}{20}$ are the fractions required.

EXAMPLES FOR PRACTICE.

1. Reduce $\frac{2x+3}{x}$, and $\frac{5x+1}{3}$ to a common denominator.

Ans. $\frac{6x+9}{3x}$ and $\frac{5x^2+x}{3x}$.

2. Reduce $\frac{b+c}{a+b}$, and $\frac{d-c}{b-d}$ to a common denominator.

Ans. $\frac{b^2+bc-bd-dc}{ba+b^2-da-bd}$, and $\frac{ad-ac+bd-bc}{ba+b^2-da-bd}$.

3. Reduce $\frac{a}{2x}$, $\frac{x}{y}$, and $\frac{x-8}{z^2}$, to a common denominator.

Ans. $\frac{ayz^2}{2xyz^2}$, $\frac{2x^2z^2}{2xyz^2}$, and $\frac{2x^2y-16xy}{2xyz^2}$.

Case IV. To reduce a compound fraction to a simple one.

Rule.—*Multiply all the numerators together for a new numerator, and all the denominators for a new denominator.*

EXAMPLES.

1. Reduce $\frac{1}{4}$ of $\frac{25x}{16}$ to a simple fraction.

Thus $\frac{25x \times 1}{16 \times 4} = \frac{25x}{64}$ the simple fraction required.

2. Reduce $\frac{3}{4}$ of $\frac{5}{6}$ of $\frac{6x}{5}$ to a simple fraction.

Thus $\frac{6x \times 5 \times 3}{5 \times 6 \times 4} = \frac{90x}{120} = \frac{3x}{4}$ the answer.

Case V. To find the greatest common measure of two quantities.

RULE.—*Range the quantities according to the dimensions of some letter, as in division. Then divide the greater term by the less, and the last divisor by the remainder, and so on till nothing remains, and the last divisor used will be the common measure required.*

NOTE.—All the terms or figures which are common to each term of the divisors, must be thrown out of them before they are used in the operation.

EXAMPLES.

1. Required the greatest common measure of $\frac{b^2 - y^2}{b^3 - y^3}$.

$$\begin{array}{l} \text{Thus } b^2 - y^2)\,b^3 - y^3\,(b \\ \qquad\qquad\quad \underline{b^3 - by^2} \\ \qquad\qquad\qquad\quad by^2 - y^3)\,b^2 - y^2\,(\end{array}$$

Here y^2 is common to both terms of the divisor, therefore by throwing out y^2, we have

$$\begin{array}{r} b - y)\,b^2 - y^2\,(b + y \\ \underline{b^2 - by} \qquad \\ by - y^2 \\ \underline{by - y^2} \\ 0 \quad\ \ 0 \end{array}$$

Therefore $b - y$ is the greatest common measure.

2. Required the greatest common measure of $\frac{x^3 - b^2x}{x^2 + 2bx + b^2}$.

$$\begin{array}{r} x^2 + 2b^2x + b^2)\,x^3 - b^2x\,(x \\ \underline{x^3 + 2b^2x^2 + b^2x} \\ -\,2b^2x^2 - 2b^2x \end{array}$$

Now, dividing this remainder by $-\,2bx$, the quotient is $x + b$; and dividing the divisor $x^2 + 2bx + b^2$ by this quotient, we have as under,

$$\begin{array}{r} x + b)\,x^2 + 2bx + b^2\,(x + b \\ \underline{x^2 + bx} \qquad\qquad \\ bx + b^2 \qquad\quad \\ bx + b^2 \qquad\quad \end{array}$$

Hence $x + b$ is the greatest common measure.

3. Required the greatest common measure of $\frac{a^4 - x^4}{a^5 - x^2a^3}$.

$a^5 - x^2a^3)a^4 - x^4($

or $\frac{a^5 - x^2a^3}{a^3} = a^2 - x^2)a^4 - x^4(a^2 + x^2$

$$\begin{array}{r} a^4 - a^2x^2 \\ \hline a^2x^2 - x^4 \\ a^2x^2 - x^4 \end{array}$$

Therefore $a^2 - x^2$ is the greatest common measure.

EXAMPLES FOR PRACTICE.

1. Required the greatest common measure of $\frac{xy + y^2}{xz^2 + z^2y}$.

Ans. $x + y$.

2. Required the greatest common measure of $\frac{a^3 - ab^2}{a^2 + 2ab + b^2}$.

Ans. $a + b$.

3. Required the greatest common measure of $\frac{a^5 - a^3b^2}{a^4 - b^4}$.

Ans $a^2 - b^2$.

CASE VI. To reduce a fraction to its lowest terms.

RULE.—*Find the greatest common measure, as in the last case. Then divide both the numerator and the denominator of the fraction by the common measure thus found, and it will reduce it to its lowest terms.*

EXAMPLES.

1. Reduce $\frac{b^2 - y^2}{b^3 - y^3}$ to its lowest terms.

$$b^2 - y^2)\,b^3 - y^3\,(b$$
$$\underline{b^3 - by^2}$$
$$by^2 - y^3)\,b^2 - y^2.$$

or $\frac{by^2 - y^3}{y^2} = b - y)\; b^2 - y^2\,(b + y$

$$\underline{b^2 - by}$$
$$by - y^2$$
$$by - y^2$$

Therefore $b - y$ is the greatest common measure; now divide both terms of the fraction by it.

$$\text{Thus, } b - y\Big)\frac{b^2 - y^9}{b^3 - y^3} = \frac{b + y}{b^2 + by + y^2} =$$

the fraction in its lowest terms.

2. Reduce $\frac{x^3 - b^2x}{x^2 + 2bx + b^2}$ to its lowest terms.

$$x^2 + 2bx + b^2)\,x^3 - b^2x\,(x$$
$$\underline{x^3 + 2bx^2 + b^2x}$$
$$-2bx^2 - 2b^2x^2$$

now $\frac{-2bx^2 - 2b^2x^2}{-2bx} = x + b$

Then $x + b)\,x^2 + 2bx + b^2\,(x + b$

$$\underline{x^2 + bx}$$
$$bx + b^2$$
$$bx + b^2$$

Therefore $x + b$ is the greatest common measure, and dividing both terms of the fraction by it, we have,

$$x + b\Big)\frac{x^3 - b^2x}{x^2 + 2bx + x^2} = \frac{x^2 - bx}{x + b} = \text{the fraction}$$

in its lowest terms.

EXAMPLES FOR PRACTICE.

1. Reduce $\dfrac{a^4 - x^4}{a^5 - x^2a^3}$ to its lowest terms.

Ans. $\dfrac{a^2 + x^2}{a^3}$.

2. Reduce $\dfrac{a^2 - 1}{ab + b}$ to its lowest terms.

Ans. $\dfrac{a - 1}{b^2}$.

3. Reduce $\dfrac{a^2 - b^2}{a^4 - b^4}$ to its lowest terms.

Ans. $\dfrac{1}{a^2 - b^2}$.

CASE VII. To add fractional quantities together.

RULE.—*Reduce the fractions to a common denominator, add the numerators together, and under their sum write the common denominator.*

EXAMPLES.

1. Add $\dfrac{x}{2}$ and $\dfrac{x}{3}$ together.

Thus, $\left.\begin{array}{r} x \times 3 = 3x \\ x \times 2 = 2x \end{array}\right\}$ new numerator.

$2 \times 3 = 6$ common denominator.

Now $\dfrac{3x}{6} + \dfrac{2x}{6} = \dfrac{5x}{6}$ their sum.

2. Add $\dfrac{2x}{3}$ and $\dfrac{3x}{5}$ together.

Thus $\left.\begin{array}{r} 2x \times 5 = 10x \\ 3x \times 3 = 9x \end{array}\right\}$ new numerator.

$3 \times 5 = 15$ common denominator.

Now $\dfrac{10x}{15} + \dfrac{9x}{15} = \dfrac{19x}{15}$ their sum.

3. Add $a+\frac{b}{2}$, $x-\frac{b}{3}$, and $2a+\frac{3b}{4}$ together.

$$\left.\begin{aligned} \text{Thus, } b\times 3\times 4 &= 12\,b \\ -b\times 2\times 4 &= -8\,b \\ 3b\times 2\times 3 &= 18\,b \end{aligned}\right\}\text{ new numerator.}$$
$$2+3+4 = 24 \text{ common denominator.}$$

The sum of the numerators are $30b-8b=22b$.

Therefore $\frac{22b}{24}=\frac{11b}{12}$ is the sum of the fractions, to which annex the sum of the integers a, x, and $2a$, which is $3a+x$, the $3a+x+\frac{11b}{12}$.

4. Add $\frac{a}{b}$, $\frac{c}{d}$, $\frac{x}{y}$, and $\frac{4}{7}$ together.

$$\left.\begin{aligned} \text{Thus, } a\times d\times y\times 7 &= 7ady \\ c\times b\times y\times 7 &= 7bcy \\ x\times b\times d\times 7 &= 7bdx \\ 4\times b\times d\times y &= 4bdy \end{aligned}\right\}\text{ new numer.}$$
$$b\times d\times y\times 7 = 7bdy \text{ common denom.}$$

Hence $\frac{7ady+7bcy+7bdx+4bdy}{7bdy}$ their sum.

In the addition of mixed quantities, it is best to bring the fractional parts only to a common denominator, and annex their sum to the sum of the integers, with their proper sign; as in the third example above.

EXAMPLES FOR PRACTICE.

1. Add $a-\frac{3x^2}{b}$ and $b+\frac{2ax}{c}$ together.

Ans. $a+b+\frac{2abx-3cx^2}{bc}$.

2. Add $2a+\frac{a+3}{5}$ and $4a+\frac{2a-5}{4}$ together.

Ans. $6a+\frac{14a-13}{20}$.

CASE VIII. To subtract fractions.

RULE.—*Having reduced the fractions to a common denominator as in addition, find the difference of the numerators, under which write the common denominator.*

EXAMPLES.

1. Required the difference of $\frac{2x}{2}$, and $\frac{3x}{5}$.

$$\text{Thus, } \left.\begin{array}{l} 2x \times 5 = 10x \\ 3x \times 2 = \ \ 6x \end{array}\right\} \text{new numerator.}$$
$$5 \ \times 2 = 10 \text{ common denominator.}$$

Now $\frac{10x - 6x}{10} = \frac{4x}{10}$ or $\frac{2x}{5}$.

Hence $\frac{2x}{5}$ is the difference.

This example is done like those in case 7th, except that here we subtract the numerators instead of adding them.

2. Required the difference of $\frac{x}{a}$ and $\frac{y}{z}$.

$$\text{Thus } \left.\begin{array}{l} x \times z = xz \\ y \times a = ay \end{array}\right\} \text{new numerators.}$$
$$xz - ay \text{ their difference.}$$
$$a \times z = az \text{ common denominator.}$$

Therefore $\frac{xz - ay}{az}$ their difference.

3. Required the difference of $\frac{2a - b}{4c}$ and $\frac{3a - 4b}{3b}$.

$$\overline{2a - \ b} \times 3b = \ 6ab - \ 3b^2$$
$$\overline{3a - 4c} \times 4c = 12ac - 16bc$$
$$6ab - 3b^2 - 12ac + 16bc = \text{their}$$

difference.

$4c \times 3b = 12bc$, the common denominator.

Therefore $\frac{6ab - 3b^2 - 12ac + 16bc,}{12bc}$ the answer.

EXAMPLES FOR PRACTICE.

1. Required the difference of $\frac{x+4}{5}$ and $\frac{x-a}{y}$.

Ans. $\frac{xy+4y-5x+5a}{5y}$.

2. From $\frac{x-y}{2a}$ take $\frac{x+y}{3a}$.

Ans. $\frac{ax-5ay}{6a^2}$.

3. From $4+\frac{x}{3}$ take $3+\frac{2x}{9}$.

Ans. $1+\frac{3x}{27}$ or $1+\frac{x}{9}$.

CASE IX. To multiply fractional quantities.

RULE.—*Multiply all the numerators together for the numerator of the product, and all the denominators together for its denominator.*

NOTE.—In multiplication and division reduce integers and mixed numbers to improper fractions.

EXAMPLES.

1. Multiply $\frac{x}{2}$ by $\frac{x}{2}$.

Thus $\frac{x}{2}\times\frac{x}{2}=\frac{x^2}{4}$ the product.

2. Multiply $\frac{2x}{3}$ by $\frac{3x}{4}$.

Thus $\frac{2x}{3}\times\frac{3x}{4}=\frac{6x^2}{12}$ or $\frac{x^2}{2}$ the answer.

3. Multiply x by $\frac{x}{2}$.

First $\frac{x \times 2}{2} = \frac{2x}{2}$

Then $\frac{2x}{2} \times \frac{x}{2} = \frac{2x^2}{4}$ or $\frac{x^2}{2}$ the answer.

4. Multiply $\frac{12x}{x-12}$ by $\frac{20x}{x-20}$.

Thus $\frac{12x}{x-12} \times \frac{20x}{x-20} = \frac{240x}{x^2 - 32x + 240}$ Ans.

5. Multiply $\frac{3x+2}{x-2}$ by $\frac{3x+2}{x-2}$.

Thus $\frac{3x+2}{x-2} \times \frac{3x+2}{x-2} = \frac{9x^2 + 12x + 4}{x^2 - 4x + 4}$.

6. Multiply $x + \frac{2x}{3}$ by $x + \frac{2x}{3}$.

First $\frac{x \times 3 + 2x}{3} = \frac{5x}{3}$.

Then $\frac{5x}{3} \times \frac{5x}{3} = \frac{25x^2}{9}$ the answer.

7. Multiply $x + \frac{b}{2}$ by $x + \frac{b}{2}$.

First $\frac{x \times 2 + b}{2} = \frac{2x + b}{2}$.

Then $\frac{2x+b}{2} \times \frac{2x+b}{2} = \frac{4x^2 + 4xb + b^2}{4}$.

or $x^2 + xb + \frac{b}{4}$ the answer.

EXAMPLES FOR PRACTICE.

1. Multiply $\frac{4ax}{y}$ by $\frac{2}{y}$.

Ans. $\frac{8ax}{xy}$.

2. Multiply $-\frac{a}{x}$ by $-\frac{y}{z}$.

$$\text{Ans. } \frac{ay}{xz}.$$

3. Multiply $\frac{3y}{13}$ by $2x$.

$$\text{Ans. } \frac{3y \times 2x}{13} = \frac{6xy}{13}.$$

4. Multiply $\frac{2a+1}{a}$ by $\frac{2a-1}{2a+b}$.

$$\text{Ans. } \frac{4a^2-1}{2a^2+ab} = \frac{2-1}{ab} = \frac{1}{ab}.$$

CASE X. To divide fractional quantities.

RULE.—*Multiply the denominator of the divisor by the numerator of the dividend for a new numerator, and the numerator of the divisor by the denominator of the dividend for a new denominator. Or,*

Invert the divisor, and proceed as in multiplication.

EXAMPLES.

1. Divide $\frac{x}{2}$ by $\frac{x}{3}$.

$$\text{Thus } \frac{x}{3}\Big)\frac{x}{2}\Big(\frac{3x}{2x} = 1\tfrac{1}{2} \text{ the answer.}$$

Or by inverting the divisor,

$$\text{Thus } \frac{3}{x} \times \frac{x}{2} = \frac{3x}{2x} = 1\tfrac{1}{2} \text{ as before.}$$

We shall use this method in the solution of the following examples.

2. Divide $\frac{3x}{4}$ by $\frac{2x}{6}$

$$\text{Thus } \frac{6}{2x} \times \frac{3x}{4} = \frac{18x}{8x} = 2\tfrac{1}{4}. \text{ Ans.}$$

3. Divide $\frac{x^2}{2}$ by $\frac{x}{2}$.

$$\text{Thus } \frac{2}{x} \times \frac{x^2}{2} = \frac{2x^2}{2x} = x. \text{ Ans.}$$

4. Divide $\frac{3x+4}{3}$ by $\frac{2x+2}{5}$.

$$\text{Thus } \frac{5}{2x+2} \times \frac{3x+4}{3} = \frac{15x+20}{6x+6}. \text{ Ans.}$$

5. Divide $\frac{a+x}{a-y}$ by $\frac{a+y}{a+2x}$.

$$\text{Thus } \frac{a+2x}{a+y} \times \frac{a+x}{a-y} = \frac{a^2+3ax+2x^2}{a^2-y^2}. \text{ Ans.}$$

NOTE.—When the terms of the divisor will exactly divide those of the dividend, then divide numerator by numerator, and denominator by denominator, the result will be the quotient, as in the following example:

6. Divide $\frac{4ax}{9ab}$ by $\frac{2x}{3a}$.

$$\text{Thus } \frac{4ax}{9ab} \div \frac{2x}{3a} = \frac{2a}{3b}. \quad \text{Ans.}$$

Here $4ax$ divided by $2x$ gives $2a$ the numerator, then $9ab$ divided by $3a$ gives $3b$ the denominator.

7. Divide $\frac{x^2-y^2}{a^2+b^2}$ by $\frac{x-y}{a+b}$.

$$\text{Thus } \frac{x^2-y^2}{a^2+b^2} \div \frac{x-y}{a+b} = \frac{x+y}{a+b}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

1. Divide $\frac{x+z}{x^2+2xy+y^2}$ by $\frac{1}{x+y}$.

Ans. $\frac{x+z}{x+y}$.

2. Divide $x^2-2ax+a^2$ by $\frac{1}{x+y}$.

Ans. $x^3-3x^2a+3xa^2-a^2$.

3. Divide $\dfrac{x^2-9}{5}$ by $\dfrac{x+3}{4}$.

Ans. $\dfrac{4x-12}{5}$.

4. Divide $2z$ by $\dfrac{c+x}{c-y}$.

Ans. $\dfrac{2zc-2zy}{c+x}$.

5. Divide $\dfrac{x+z-y}{m+n}$ by $3d$.

Ans. $\dfrac{x+z-y}{3dm+3dn}$.

CASE XI. To change a fractional quantity into infinite series.

RULE.—*Divide the numerator by the denominator, and extend the quotient to as many terms as may be thought necessary.*

EXAMPLES.

1. Let the fraction $\dfrac{a}{a+x}$ be changed into a series.

Thus, $a+x\big)\,a\quad\Big(1-\dfrac{x}{a}+\dfrac{x^2}{a^2}-\dfrac{x^3}{a^3}+$, &c.

$$\begin{array}{l}
\underline{a+x} \\
-x \\
\underline{-x-\frac{x^2}{a}} \\
+\frac{x^2}{a} \\
\underline{+\frac{x^2}{a}+\frac{x^3}{a^2}} \\
-\frac{x^3}{a^2} \\
\underline{-\frac{x^3}{a^2}-\frac{x^4}{a^3}} \\
+\frac{x^4}{a^3}
\end{array}$$

Now it is easy to perceive that the next or 5th term of the quotient will be $+\frac{x^4}{a^4}$, and the 6th term $-\frac{x^5}{a^5}$, &c. and so on, alternately *plus* and *minus;* this is called the law of the *continuation* of the series. And the sum of all the terms when infinitely continued is said to be equal to the fraction $\frac{a}{a+x}$. Thus we say the *vulgar fraction* $\frac{2}{3}$ when reduced to a decimal is $=.6666$, &c. infinitely continued.

Note.—The terms in the quotient are found by dividing the remainders by (a) the first term of the divisor: thus, the first remainder $-x$ divided by a gives $-\frac{x}{a}$ the second term in the quotient; and the second remainder $+\frac{x^2}{a}$ divided by a gives $+\frac{x^2}{a^2}$ the third term, &c.

If the fraction is $\frac{a}{a-x}$ the series becomes wholly affirmative.

$$\text{Thus } a-x\,\Big)\,a\quad\Big(1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+,\ \&c.$$

$$\begin{array}{l}
\underline{a-x} \\
+x \\
\underline{+x-\frac{x^2}{a}} \\
\qquad +\frac{x^2}{a} \\
\qquad \underline{+\frac{x^2}{a}-\frac{x^3}{a^2}} \\
\qquad\qquad +\frac{x^3}{a^2} \\
\qquad\qquad \underline{+\frac{x^3}{a^2}-\frac{x^4}{a^3}} \\
\qquad\qquad\qquad +\frac{x^4}{a^3}
\end{array}$$

In this example, if x be less than a, the series is convergent, or the value of the terms continually diminish; but when x is greater than a, it is said to diverge.

To explain this by numbers:

Suppose $a = 3$, and $x = 2$.

Then, $1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3}$, &c.

The corresponding values are

$1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27}$, &c. where the fractions or terms of the series grow less and less, and therefore the farther they are extended, the more they converge or approximate to 0, which is supposed to be the last term or limit.

But if $a = 2$, and $x = 3$,

Then, $1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3}$, &c.

The corresponding values are

$1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8}$, &c. in which the terms become larger and larger. This is called a diverging series.

If $x = 1$, and $a = 1$ in example 1,

Then, $\frac{a}{a+x} = 1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3}$, &c.

will be $\frac{1}{1+1} = 1 - 1 + 1 - 1$, &c.

Now because $\frac{1}{1+1} = \frac{1}{2}$, it has been said that $1 - 1 + 1 - 1 + 1$, &c. infinitely continued, is $= \frac{1}{2}$: a singular conclusion, when it is perceivable, from the terms themselves, that their sum must necessarily be either 0, or $+ 1$, to whatever extent the division is supposed to be continued. The real question, however, results from the fractional parts, which (by the division) is always $+ \frac{1}{2}$ when the sum of the terms is 0, and $- \frac{1}{2}$ when the sum is $+ 1$: consequently $\frac{1}{2}$ is the true quotient in the former case, and $1 - \frac{1}{2}$ in the other.

3. Let the fraction $\frac{a}{c-x}$ be expanded into an infinite series.

$$\text{Thus, } c-x\,\big)\,a\,\left(\frac{a}{c}+\frac{ax}{c^2}+\frac{ax^2}{c^3}+\frac{ax^3}{c^4}+,\ \&c.\right.$$

$$\begin{array}{l}
a-\frac{ax}{c} \\
\hline
\quad +\frac{ax}{c} \\
\quad +\frac{ax}{c}-\frac{ax^2}{c^2} \\
\hline
\qquad\quad +\frac{ax^2}{c^2} \\
\qquad\quad +\frac{ax^2}{c^3}-\frac{ax^3}{c^2} \\
\hline
\qquad\qquad\quad +\frac{ax^3}{c^3} \\
\qquad\qquad\quad +\frac{ax^3}{c^3}-\frac{ax^4}{c^4} \\
\hline
\qquad\qquad\qquad\quad +\frac{ax^4}{c^4}
\end{array}$$

By substituting other quantities for a, c, and x in the quotient, $\frac{a}{c}+\frac{ax^2}{c^2}+\frac{ax^2}{c^3}+\frac{ax^3}{c^4}$, &c. different series may be obtained.

Suppose $a=3$, $c=10$, and $x=1$.

Then, $\frac{a}{c-x}=\frac{a}{c}+\frac{ax}{c^2}+\frac{ax^2}{c^3}+\frac{ax^3}{c^4}$, &c.

will be $\frac{3}{10-1}=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+$, &c. which is the same series as the decimal, answering to $\frac{1}{3}$ or $\frac{3}{10-1}$:

for $\frac{1}{3}=.3333$, &c. $=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+$, &c.

4. Reduce the fraction $\frac{a^2+x^2}{a^4+x^4}$ to an infinite series.

Thus, $a^4+x^4\Big)a^2+x^2\Big(\frac{1}{a^2}-\frac{x^4}{a^6}+\frac{x^8}{a^{10}}-$, &c.

$$\begin{array}{r}
a^2+\frac{x^4}{a^2}\qquad\qquad\qquad \\
\hline
-\frac{x^4}{a^2}+x^2\qquad\qquad \\
-\frac{x^4}{a^2}-\frac{x^8}{a^6}\qquad\qquad \\
\hline
+\frac{x^8}{a^6}+x^2\qquad \\
+\frac{x^8}{a^6}+\frac{x^{12}}{a^{10}}\qquad \\
\hline
-\frac{x^{12}}{a^{10}}+x^2
\end{array}$$

and the 4th term will be $-\frac{x^{12}}{a^{14}}$, the 5th $+\frac{x^{16}}{a^{18}}$, and so on.

It may be worth observing, that the same fraction will give different series, if the order of the terms in its denominator is changed.

Thus taking *Example* 1.

$\frac{a}{a+x}=1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+$, &c.

But $\frac{a}{x+a}=\frac{a}{x}-\frac{a^2}{x^2}+\frac{a^3}{x^3}-$, &c.

The two quotients, however, will always be equal, when the remainders are taken into the account.

EXAMPLES FOR PRACTICE.

1. Reduce the fraction $\frac{1+x}{1-x}$ into an infinite series.

   Ans. $1+2x+2x^2+2x^3+2x^4+$, &c.

2. Reduce $\frac{az}{a-z}$ into an infinite series.

   Ans. $z+\frac{z^2}{a}+\frac{z^3}{a^2}+\frac{z^4}{a^3}+$, &c.

3. Let $\frac{6}{10-1}$ be expanded into a series.

Ans. $\frac{6}{10}+\frac{6}{10^2}+\frac{6}{10^3}+\frac{6}{10^4}+$, &c.

4. Reduce $\frac{1}{1-a+a^2}$ to an infinite series.

Ans. $1+a-a^3-a^4+a^6+a^7-a^9$, &c.

5. Reduce $\frac{c^2}{(c+x)^2}$ to an infinite series.

Ans. $1-\frac{2x}{c}+\frac{3x^2}{c^2}-\frac{4x^3}{c^3}+$, &c.

INVOLUTION.

Involution is the raising of powers from any proposed root, or the method of finding the square, cube, biquadrate power, &c. of any given quantity.

Rule.—*Multiply the quantity into itself, as many times as there are units in the index, less one, and the last product will be the power required.* Or,

Multiply the index of the quantity by the index of the power, and the result will be the same as before.

Note.—When the sign of the root is $+$, all the powers of it will be $+$; and when the sign is $-$, all the even powers of it will be $+$, and the odd powers $-$.

Involve a to the second, third, and fourth powers,

By Rule I.

$a\times a=a^2$..............2d power.
$a\times a\times a=a^3$..............3d power.
$a\times a\times a\times a=a^4$.........4th power.

By Rule II.

$a^1\times 2=a^2$..................2d power.
$a^1\times 3=a^3$..................3d power.
$a^1\times 4=a^4$..................4th power.

A quantity composed of several factors, multiplied together, is involved by raising each factor to the power assigned.

A fraction is involved by raising both the numerator and denominator to the power proposed.

The nature of the powers of compound quantities will appear from the following examples.

1. Required the square, cube, and biquadrate powers of $a + b$, or any two quantities connected together by the sign $+$, called a *binomial*.

$$
\begin{array}{l}
a + b = \text{root} \\
\underline{a + b} \\
a^2 + ab \\
\underline{\quad + ab + b^2} \\
a^2 + 2ab + b^2 = \text{square or 2d power} \\
\underline{a + b} \\
a^3 + 2a^2b + ab^2 \\
\underline{\quad + a^2b + 2ab^2 + b^3} \\
a^3 + 3a^2b + 3ab^2 + b^3 = \text{cube or 3d power} \\
\underline{a + b} \\
a^4 + 3a^3b + 3a^2b^2 + ab^3 \\
\underline{\quad + a^3b + 3a^2b^2 + 3ab^3 + b^4} \\
a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = \text{4th power.}
\end{array}
$$

2. Required the square, cube, and biquadrate powers of $a - b$, or any two quantities connected by the sign $-$ called a *residual*.

$$
\begin{array}{l}
a - b = \text{root} \\
\underline{a - b} \\
a^2 - ab \\
\underline{\quad - ab + b^2} \\
a^2 - 2ab + b^2 = \text{square or 2d power} \\
\underline{a - b} \\
a^3 - 2a^2b + ab^2 \\
\underline{\quad - a^2b + 2ab^2 - b^3} \\
a^3 - 3a^2b + 3ab^2 - b^3 = \text{cube or 3d power}
\end{array}
$$

$$\begin{array}{l} a - b \\ \hline a^4 - 3a^3b + 3a^2b^2 - ab^3 \\ \quad - a^3b + 3a^2b^2 - 3ab^3 + b^4 \\ \hline a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 = \text{4th power.} \end{array}$$

By comparing these two examples together, we may make the following observations.

1. That the successive powers of a residual quantity (that is $a - b$) are the same as those of a binomial quantity, or $a + b$, except that the terms of a residual will be alternately $+$ and $-$.

2. That the number of terms is always one more than the index of the power, and is therefore even when that is odd, and odd when that is even.

3. That the index of the first or leading quantity is the same as that of the power, and in the succeeding terms it decreases always by unity or 1, while that of the second or following term increases by 1, and is equal to that of the power in the last. In all the terms the sum of the indices is equal to that of the power.

4. As for the coefficients, the first term is always unity, and the second the same as the index of the power; and for the rest, multiply the coefficient of each preceding term by the index of the leading quantity in that term, and divide the product by the number of terms to that place, and the quotient will be the coefficient of the next or following term.

Thus we are furnished with a general rule for involving the binomial $a + b$, or a residual $a - b$, to any power, without the trouble of those tedious multiplications which are required otherwise. For example, let it be required to involve $a + b$ to the fifth power, and the terms without their coefficients will be as under.

$$a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5.$$

Now to find the coefficients.

That of the first term is $= 1$

That of the second is 5, being according to the rule equal to the index of the power $\Big\} = 5$

That of the third $\dfrac{5 \times 4}{2}$ $= 10$

That of the fourth $\frac{10 \times 3}{3} = 10$

And those of the 5th and 6th terms will be 5 and 1, agreeable to the preceding rule.

Therefore the 5th power of $a + b$, with the coefficients, is

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

After the same manner we may find,

1. That $(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.

2. That $(a + b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$.

From these examples it may be further observed, with regard to the coefficients, that from either of the extreme terms they are the same, increasing from each end to the middle term, which is the greatest, when there is an odd number of terms, and to the two middle terms when there is an even number of terms which are equal to each other, but greater than any other of the coefficients.

Therefore, by attending to the law of the coefficients, we need only calculate them as far as the middle term, and set down the rest in an inverted order.

Thus in Example 2. $(a + b)^7$.

The first 4 coefficients 1, 7, 21, 35.
The last 4 35, 21, 7, 1.

The above rule, which is of such extensive utility in algebraical investigation, we are indebted to the inventive genius of *Sir Isaac Newton*, and it may be expressed in general terms as follows.

$$(a + x)^n = a^n + n.\, a^{n-1}x + n.\, \frac{n-1}{2}\, a^{n-2}x^2 + n.\, \frac{n-1}{2}.\, \frac{n-2}{3}\, a^{n-3}x^3, \&c.$$

$$(a - x)^n = a^n - n.a^{n-1}x + n.\, \frac{n-1}{2}\, a^{n-2}x^2 - n.\, \frac{n-1}{2}.\, \frac{n-2}{3}\, a^{n-3}x^3, \&c.$$

NOTE.— n is the index of the power, that is, $n = 5$ in the fifth power, 4 in the fourth, and so on for any other power.

Also the sum of the coefficients in every power, is equal to the number 2, raised to that power; thus, $1 + 1 = 2$ for the first power, $1 + 2 + 1 = 4 = 2^2$ for the squares, $1 + 3 + 3 + 1 = 8 = 2^3$ for the cube or 3d power, and so on in this manner with any other power.

EXAMPLE.

1. Involve $a - b$ to the 6th power. The terms without their coefficients $a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6$ the coefficients will be according to the rule 1,

$$\frac{1 \times 6}{1}, \frac{6 \times 5}{2}, \frac{15 \times 4}{3}, \frac{20 \times 3}{4}, \frac{15 \times 2}{5}, \frac{6 \times 1}{6}.$$

or 1, 6, 15, 20, 15, 6, 1.

Therefore the 6th power of $a - b$ is $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.

EXAMPLES FOR PRACTICE.

1. What is the 8th power of $x + y$?

Ans. $x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8$.

2. What is the 10th power of $x - y$?

Ans. $x^{10} - 10x^9y + 45x^8y^2 - 120x^7y^3 + 210x^6y^4 - 252x^5y^5 + 210x^4y^6 - 120x^3y^7 + 45x^2y^8 - 10xy^9 + y^{10}$.

If we assume the index $\frac{m}{n}$, in which form it may relate to roots as well as powers, it then becomes more general, viz.

$$(a+b)^{\frac{m}{n}} = a^{\frac{m}{n}} + \frac{m}{n} a^{\frac{m-n}{n}} b + \frac{m(m-n)}{n.2n} a^{\frac{m-2n}{n}} b^2 +, \text{\&c.}$$

$$\text{or } (a+b)^{\frac{m}{n}} = a^{\frac{m}{n}} \times \left\{ 1 + \frac{m}{n} \cdot \frac{b}{a} + \frac{m(m-n)}{n.2n} \cdot \frac{b^2}{a^2} + \frac{m(m-n)(m-2n)}{n.2n.3n} \cdot \frac{b^3}{a^3}, \text{\&c.} \right.$$

$$\text{or } (a+b)^{\frac{m}{n}} = a^{\frac{m}{n}} \times \left\{ 1 + \frac{m}{n} A + \frac{m-n}{2n} B + \frac{m-2n}{3n} C + \frac{m-3n}{4n} D +, \text{\&c.} \right.$$

Where A, B, C, D, &c. are the preceding terms, including their signs + or —, the terms of the series being all *plus* when b is positive, and alternately *plus* and *minus* when b is negative, independently, however, of the effect of the coefficients made up of m and n, which may be any number whatever, positive or negative.

Or this theorem may be otherwise expressed as follows, and which is indeed its most simple practical form; viz. Put $a = P$, and $b = PQ$, then we shall have

$$(P + PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} AQ + \frac{m-n}{2n} BQ + \frac{m-2n}{3n} CQ +, \&c.$$

Where $\frac{m}{n}$ is the index, P the first term, Q the second term divided by the first, and A, B, C, D, &c. the several foregoing terms, with their proper signs.

We shall illustrate this formula with an example or two.

1. What is the square root of $(a^2 + b^2)$ in an infinite series?

$$\text{Here } \left.\begin{aligned} P &= a^2 \\ Q &= \frac{b^2}{a^2} \\ \frac{m}{n} &= \frac{1}{2} \end{aligned}\right\}$$

That is, $m = 1$, and $n = 2$. Therefore

$$P^{\frac{m}{n}} = (a^2)^{\frac{m}{n}} = (a^2)^{\frac{1}{2}} = a = A.$$

$$\frac{m}{n} AQ = \frac{1}{2} \times a \times \frac{b^2}{a^2} = \frac{b^2}{2.a} = B.$$

$$\frac{m-n}{2n} BQ = \frac{1-2}{4} \times \frac{b^2}{2 \cdot a} \times \frac{b^2}{a^2} = \frac{-1 \cdot b^4}{4 \cdot 2 \cdot a^3} = C.$$

$$\frac{m-2n}{3n} CQ = \frac{1-4}{6} \times \frac{-1 \cdot b^4}{4 \cdot 2 \cdot a^3} \times \frac{b^2}{a^2} = \frac{3 \cdot 1 \cdot b^6}{6 \cdot 4 \cdot 2 \cdot a^5} =$$

D. Consequently

$$(a^2 + b^2)^{\frac{1}{2}} = a + \frac{b^2}{2 \cdot a} - \frac{1 \cdot b^4}{4 \cdot 2 \cdot a^3} + \frac{3 \cdot 1 \cdot b^6}{6 \cdot 4 \cdot 2 \cdot a^5} - \frac{5 \cdot 3 \cdot 1 \cdot b^8}{8 \cdot 6 \cdot 4 \cdot 2 \cdot a^7} +, \&c.$$

or $(a^2+b^2)^{\frac{1}{2}} = a + \frac{b^2}{2a} - \frac{b^4}{8a^3} + \frac{b^6}{16a^5} - \frac{5b^8}{128a^7}$ &c.

Where the terms may be continued at pleasure, the law of the series being evident.

When the quantity to be expanded into an infinite series comes under the form of a fraction, the denominator must be placed under a negative index, and brought up into the numerator, thus,

$$\frac{1}{\sqrt{(a^2+b)}} = \frac{1}{(a^2+b)}\,^{\frac{1}{2}} = (a+b)^{-\frac{1}{2}}.$$

Also, $\frac{a+b}{(a^2+x)^2} = (a+b)\,(a^2+x)^{-2}$.

And so on with others of the same form.

2. Required the value of $\frac{1}{(a-b)^2}$ in an infinite series.

First $\frac{1}{(a-b)^2} = (a-b)^{-\frac{2}{1}}$; therefore

$$\left.\begin{aligned} P &= a \\ Q &= \frac{-b}{a} \\ m &= -2 \\ n &= 1 \end{aligned}\right\}$$

Whence $P^{\frac{m}{n}} = (a)^{-\frac{2}{1}} = a^{-2} = \frac{1}{a^2} = A.$

$\frac{m}{n}\,AQ = \frac{-2}{1} \times \frac{1}{a^2} \times \frac{-b}{a} = \frac{2b}{a^3} = B.$

$\frac{m-n}{2n}\,BQ = \frac{-3}{2} \times \frac{2b}{a^3} \times \frac{-b}{a} = \frac{3b^2}{a^4} = C.$

$\frac{m-2n}{3n}\,CQ = \frac{-4}{3} \times \frac{3b^2}{a^4} \times \frac{-b}{a} = \frac{4b^3}{a^5} = D.$

Whence $\frac{1}{(a-b)^2} = \frac{1}{a^2} + \frac{2b}{a^3} + \frac{3b^2}{a^4} + \frac{4b^3}{a^5} + \frac{5b^4}{a^6} +$, &c. the value required

EXAMPLES FOR PRACTICE.

1. Extract the square root of $(a^2 - x^2)$ in an infinite series.

$$\text{Ans. } a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}, \text{ \&c.}$$

2. What is the value of $\frac{a}{b+c}$ in an infinite series?

$$\text{Ans. } a \times \left(\frac{1}{b} - \frac{c}{b^2} + \frac{c^2}{b^3} - \frac{c^3}{b^4} + \frac{c^4}{b^5} -, \text{ \&c.}\right)$$

3. Find the value of $\frac{1}{(x+y)^2}$ in an infinite series.

$$\text{Ans. } \frac{1}{x^2} - \frac{2y}{x^3} + \frac{3y^2}{x^4} - \frac{4y^3}{x^5} +, \text{ \&c.}$$

4. What is the square root of (a^2+b) in an infinite series?

$$\text{Ans. } a + \frac{b}{2a} - \frac{b^2}{8a^3} + \frac{b^3}{16a^5} - \frac{5b^4}{128a^7} +, \text{ \&c.}$$

EVOLUTION.

EVOLUTION is the reverse of involution, or it is the method of finding the square root, cube root, &c. of any given quantity, whether simple or compound.

CASE I. When the quantity is simple.

RULE.—*Divide the indices of the letters by the index of the root required, and prefix the root of the numeral coefficient.*

EXAMPLES.

1. Required the square root of x^4.

$$\text{Thus } x^4 = x^{\frac{4}{2}} = x^2. \text{ Ans.}$$

2. Required the square root of x^6.

$$\text{Thus } x^6 = x^{\frac{6}{2}} = x^3. \text{ Ans.}$$

3. Required the cube root of $8x^3$.

Thus $8x^3 = 8x^{\frac{3}{3}} = 2x$. Ans.

4. Required the cube root of $27x^6$.

Thus $27x^6 = 27x^{\frac{6}{3}} = 3x^2$. Ans.

5. Required the square root of $\frac{5a^2b^2}{9c^2}$

Thus $\sqrt{\frac{5a^2b^2}{9c^2}} = \frac{ab}{3c}\sqrt{5}$. Ans.

If the number which expresses the required root is not a divisor of the index of the given power, the root will in that case have a fractional exponent or index. Thus, the square root of a^3 is $a^{\frac{3}{2}}$; the cube root of a^5 is $a^{\frac{5}{3}}$, and the square root of a is $a^{\frac{1}{2}}$. In this manner arise powers, which are called imperfect powers or surds. They are multiplied and divided in the same manner as perfect powers. Thus, $a^{\frac{1}{2}} \times a^{\frac{5}{2}} = a^{\frac{1}{2}+\frac{5}{2}} = a^{\frac{6}{2}} = a^3$. Again, $a^{\frac{2}{3}} \times a^{\frac{3}{4}} = a^{\frac{2}{3}+\frac{3}{4}} = a^{\frac{17}{12}} = \sqrt[12]{a^{17}}$: Also, $\frac{a^{\frac{7}{2}}}{a^{\frac{3}{2}}} = a^{\frac{7-3}{2}} = a^{\frac{4}{2}} = a^2$.

They are involved, likewise, and evolved, as perfect powers. The square of $a^{\frac{3}{2}}$ is $a^{\frac{3}{2}\times 2} = a^3$; the cube of $a^{\frac{2}{5}}$ is $a^{\frac{2}{5}\times 3} = a^{\frac{6}{5}}$. The square root of $a^{\frac{2}{3}}$ is $a^{\frac{2}{3}\div 2} = a^{\frac{1}{3}}$; the cube root of $a^{\frac{3}{4}}$ is $a^{\frac{3}{4}\div 3} = a^{\frac{1}{4}}$.

REMARKS.—That any even root of an affirmative quantity, may be either + or —. Thus the square root of a^2 is either $+a$ or $-a$, for $+a\times+a=+a^2$ and $-a\times-a=+a^2$. And an odd root of any quantity will have the same sign as the quantity itself; thus the cube root of $+a^3$ is $+a$ and the cube root of $-a^3$ is $-a$ for $+a\times+a\times+a=+a^3$, and $-a\times-a\times-a=-a^3$. Any even root of a negative quantity is impossible, for neither $+a \times +a$, nor $-a\times-a$ can give $-a^2$.

The nth root of an affirmative product is equal to the nth root of each of the factors multiplied together; the same holds good of a negative product when n is odd. The nth root of a fraction is equal to the nth root of the numerator divided by the nth root of the denominator.

CASE II. When the quantity is compound.

RULE.—*To extract the square root of a compound quantity, arrange the quantities according to the power of some letter, as in division. Find the root of the first term, and set it in the quotient. Subtract the square of the root thus found, from the first term, and bring down the next two terms to the remainder for a dividend, and take double the root already found for a divisor. Divide the dividend by the divisor, and annex the result both to the quotient and divisor. Multiply the divisor so increased by the term last put in the quotient, subtract the product from the dividend, and bring down the next two terms to the remainder; procced as before until the work is finished.*

REMARK.—The method by which roots are extracted consists entirely in employing a process the reverse of involution, and which was evidently pointed out by it. Thus, if $a+b$ be a root of which the part a is known, and b unknown, then the square of this root being $a^2+2ab+b^2$, which is equal to the sum of the square of a and b, together with twice their product; therefore to find out the unknown part b, subtract the nearest square a^2 from the given quantity, there remains $2ab+b^2$, or $(2a+b)b$; therefore dividing this remainder by $2a$, double of the first part of the root, the quotient will be nearly b, the other part; then to $2a$ add this quotient b, and multiplying the sum $2a+b$ into b, the product will make up the remaining part $2ab+b^2$ of the given power, wherefore the root $a+b$ is found, and the method by which it is obtained agrees exactly with the above rule. Much in the same manner are the rules for extracting the cube root and other roots derived.

EXAMPLES.

1. Required the square root of $a^4+4a^3+6a^2+4a+1$.

$$
\begin{array}{rl}
 & a^4+4a^3+6a^2+4a+1\,(a^2+2a+1 \\
 & a^4 \\
\hline
2a^2+2a) & 4a^3+6a^2 \\
 & 4a^3+4a^2 \\
\hline
2a^2+4a+1) & 2a^2+4a+1 \\
 & 2a^2+4a+1
\end{array}
$$

2. Required the square root of $a^2 + 2ab + b^2 + 2ac + 2bc + c^2$.

$$\begin{array}{l} a^2 + 2ab + b^2 + 2ac + 2bc + c^2(a + b + c \\ a^2 \\ \hline 2a + b)2ab + b^2 \\ \quad 2ab + b^2 \\ \hline 2a + 2b + c)2ac + 2bc + c^2 \\ \quad 2ac + 2bc + c^2 \end{array}$$

3. Required the square root of $x^2 - ax + \frac{1}{4}a^2$.

$$\begin{array}{l} x^2 - ax + \frac{1}{4}a^2(x - \frac{1}{2}a \\ x^2 \\ \hline 2x - \frac{1}{2}a) - ax + \frac{1}{4}a^2 \\ \quad - ax + \frac{1}{4}a^2 \end{array}$$

EXAMPLES FOR PRACTICE.

1. Required the square root of $x^2 + 2xy + y^2$.
Ans. $x + y$.

2. Required the square root of $x^2 - 2x + 1$.
Ans. $x - 1$.

3. Required the square root of $4x^4 - 12x^3y + 13x^2y^2 - 6xy^3 + y^4$.
Ans. $2x^2 - 3xy + y^2$.

4. Required the square root of $x^2 - \frac{x}{2} + \frac{1}{16}$.

Ans. $x - \frac{1}{4}$.

Case III. To extract any root of any compound algebraical quantity.

Rule.—*Find the root of the leading term, which place in the quotient, and bring down the second term for a dividend. Involve the root last found to the power next below that whose root is to be found, and multiply it by the index of the given power for a divisor. Then divide the dividend by the divisor for a new term in the root. Again, involve this new root to the given power, and subtract the result from the proposed quantity, and always divide the first term of the remainder by the divisor first found, for a new term, and thus proceed till the root is completely obtained.*

EXAMPLES.

1. Required the square root of $a^4 - 2a^3 b + 3a^2 b^2 - 2ab^3 + b^4$.

$$a^4 - 2a^3 b + 3a^2 b^2 - 2ab^3 + b^4 \; (a^2 - ab + b^2$$
$$a^4$$
$$2a^2) - 2a^3 b$$
$$a^4 - 2a^3 b + a^2 b^2 = (a^2 - ab)^2$$
$$2a^2) 2a^2 b^2$$
$$a^4 - 2a^3 b + 3a^2 b^2 - 2ab^3 + b^4 = (a^2 - ab + b)^2$$

Therefore $a^2 - ab + b^2$ is the root required.

It may be observed, that in all such cases as the above, the proposed quantity must be arranged according to the powers of the unknown letter or letters.

2. Required the cube root of $x^6 + 6x^5 - 40x^3 + 96x - 64$.

$$x^6 + 6x^5 - 40x^3 + 96x - 64 \; (x^2 + 2x - 4$$
$$x^6$$
$$3x^4) \; 6x^5$$
$$x^6 + 6x^5 + 12x^4 + 8x^3 = (x^2 + 2x)^3$$
$$3x^4) - 12x^4$$
$$x^6 + 6x^5 - 40x^3 + 96x - 64 = (x^2 + 2x - 4)^3$$

Therefore $x^2 + 2x - 4$ is the root required.

3. Required the 4th root of $x^4 - 3x^3 y + 6x^2 y^2 - 4xy^3 + y^4$

$$x^4 - 3x^3 y + 6x^2 y^2 - 4xy^3 + y^4 \; (x - y$$
$$x^4$$
$$4x^3) - 3x^3 y$$
$$x^4 - 3x^3 y + 6x^2 y^2 - 4xy^3 + y^4 . = (x - y)^4$$

In extracting roots, it will often happen that the exact root cannot be found in finite terms, as will appear from the following example.

4. Required the square root of a^2+x^2.

$$
\begin{array}{l}
a^2+x^2\left(a+\dfrac{x^2}{2a}-\dfrac{x^4}{8a^3}+\dfrac{x^6}{16a^5}\text{, \&c.}\right.\\
\underline{a^2}\\
2a+\dfrac{x^2}{2a}\Big)\ x^2\\
\qquad \underline{x^2+\dfrac{x^4}{4a^2}}\\
2a+\dfrac{x}{a}-\dfrac{x^4}{8a^3}\Big)-\dfrac{x^4}{4a^2}\\
\qquad\qquad \underline{-\dfrac{x^4}{4a^2}-\dfrac{x^6}{8a^4}+\dfrac{x^8}{64a^6}}\\
\qquad\qquad\qquad +\dfrac{x^6}{8a^4}-\dfrac{x^8}{64a^6}\text{, \&c.}
\end{array}
$$

EXAMPLES FOR PRACTICE.

1. Required the square root of $a^4+4a^3+6a^2+4a+1$.
   Ans. a^2+2a+1.

2. Required the square root of $a^4-2a^3+2a^2-a+\frac{1}{4}$.
   Ans. $a^2-a+\frac{1}{2}$.

3. Required the square root of a^2-ab.
   Ans. $a-\dfrac{b}{2}-\dfrac{b^3}{8a}-\dfrac{b^3}{16a^2}$, &c.

4. Required the cube root of $a^3-3a^2x+3ax^2-x^3$
   Ans. $a-x$.

5. Required the 4th root of $16x-32x^3y+24x^2y^2-8xy^2+y^4$.
   Ans. $2x-y$.

6. Required the 5th root of $32x^5-80x^4+80x^3-40x^2+10x-1$.
   Ans. $2x-1$.

SURDS.

Surds are quantities under the radical sign, or expressed by means of fractional indices, which have no exact roots; they are likewise named irrational quantities, to distinguish them from rational ones, the exact roots of which can be found.

Thus $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{8}$, &c. or $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$, $8^{\frac{1}{4}}$, &c. are surds having no exact roots.

And $\sqrt{4}$, $\sqrt[3]{27}$, $\sqrt[4]{256}$, &c. or $4^{\frac{1}{2}}$, $27^{\frac{1}{3}}$, $256^{\frac{1}{4}}$, &c. are rational quantities, whose exact roots are 2, 3, and 4 respectively.

Reduction of Surds.

The reduction of surds consists in changing them from one form to another without altering their values, for the greater ease in adding, subtracting, multiplying, dividing, or approximating to their values.

The reasons on which the several operations are founded, will be sufficiently plain to those who are well acquainted with Fractions, Involution, and Evolution.

Case I. To reduce a rational quantity to the form of a surd.

Rule.—*Involve the quantity to a power equivalent to that denoted by the index of the surd, and over this power place the radical sign, and it will be of the form required.*

EXAMPLES.

1. Reduce 5 to the form of the square root.
   Thus $5 \times 5 = 5^2 = 25$.
   Whence $\sqrt{25}$ the answer.

2. Reduce 3 to the form of the cube root.
   Here $\sqrt[3]{3 \times 3 \times 3} = \sqrt[3]{27}$, the answer.

3. Reduce $x^{\frac{1}{4}}$ to the form of the square root.

Here $\sqrt{x^{\frac{1}{4}} \times x^{\frac{1}{4}}} = \sqrt{x^{\frac{2}{4}}} = \sqrt{x^{\frac{1}{2}}}$, the answer.

EXAMPLES FOR PRACTICE.

1. Reduce 4 to the form of the square root.
   Ans. $\sqrt{16}$.
2. Reduce $\frac{1}{2}x$ to the form of the cube root.
   Ans. $(\frac{1}{8}x^3)^{\frac{1}{3}}$ or $\sqrt[3]{\frac{1}{8}x^3}$.
3. Reduce $a + z$ to the form of the square root.
   Ans. $(a^2 + 2az + z^2)^{\frac{1}{2}}$.

Case II. To reduce quantities to other equivalent ones having a common index.

Rule.—*Reduce the indices to fractions having a common denominator, and place the new numerators, each over its respective quantity, for a new index. Place 1 over the common denominator, and then write this fraction as an index over the given quantities with their new indices.*

EXAMPLES.

1. Reduce $7^{\frac{3}{4}}$ and $8^{\frac{1}{2}}$ to equivalent quantities having a common index.

$$\left.\begin{array}{r} \text{First } 3 \times 2 = 6 \\ 1 \times 4 = 4 \end{array}\right\} \text{ new numerators,}$$

$$\text{And } 4 \times 2 = 8 \text{ com. denominator.}$$

Therefore $(7^6)^{\frac{1}{8}}$ and $(8^4)^{\frac{1}{8}}$ are the quantities required. But $7^6 = 117649$ and $8^4 = 4096$.

Therefore $(117649)^{\frac{1}{8}}$ and $(4096)^{\frac{1}{8}}$ are the answer.

2. Reduce $x^{\frac{1}{4}}$ and $x^{\frac{1}{5}}$ to a common index.

$$\left.\begin{array}{r} \text{Thus } 1 \times 5 = 5 \\ 1 \times 4 = 4 \end{array}\right\} \text{ new num.}$$

$$\text{And } 4 \times 5 = 20 \text{ com. denom.}$$

Therefore $(x^5)^{\frac{1}{20}}$ and $(x^4)^{\frac{1}{20}}$ the answer.

3. Reduce $(a + x)^{\frac{1}{2}}$ and $(a - x)^{\frac{1}{3}}$ to a common index.

$$\text{Thus } \left.\begin{array}{l} 1 \times 3 = 3 \\ 1 \times 2 = 2 \end{array}\right\} \text{new num.}$$
$$2 \times 3 = 6 \text{ com. denom.}$$

Therefore $(a + x^3)^{\frac{1}{6}}$ and $(a - x^2)^{\frac{1}{6}}$ being actually involved are $(a^3 + 3a^2x + 3ax^2 + x^3)^{\frac{1}{6}}$ and $(a^2 - 2ax + x^2)^{\frac{1}{6}}$, the answer.

EXAMPLES FOR PRACTICE.

1. Reduce $3^{\frac{1}{2}}$ and $4^{\frac{1}{3}}$ to a common index.
   Ans. $27^{\frac{1}{6}}$ and $16^{\frac{1}{6}}$.

2. Reduce $a^{\frac{1}{4}}$ and $b^{\frac{1}{6}}$ to a common index.
   Ans. $(a^6)^{\frac{1}{24}}$ and $(b^4)^{\frac{1}{24}}$.

3. Reduce $(x + y)^{\frac{1}{3}}$ and $(x - y)^{\frac{1}{2}}$ to a common index.
   Ans. $(x^2 + 2xy + y^2)^{\frac{1}{6}}$ and $(x^3 - 3x^2y + 3xy^2 - y^3)^{\frac{1}{6}}$.

Case III. To reduce quantities to equivalent ones having a given index.

Rule.—*Divide the indices of the quantities by the given index, and place the quotients over the said quantities for new indices. Then over the quantities, with their new indices, place the given index, and the result will be the equivalent quantities required.*

EXAMPLES.

1. Reduce $4^{\frac{1}{3}}$ and $5^{\frac{1}{2}}$ to equivalent quantities having the given index $\frac{1}{4}$.

Thus $\frac{1}{3} \div \frac{1}{4} = \frac{1}{3} \times \frac{4}{1} = \frac{4}{3}$, the 1st index.
$\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = \frac{4}{2}$, the 2d index.

Therefore $(4^{\frac{4}{3}})^{\frac{1}{4}}$ and $(5^2)^{\frac{1}{4}}$ are the quantities required.

2. Reduce $a^{\frac{1}{m}}$ and $b^{\frac{1}{n}}$ to equivalent quantities having the common index $\frac{r}{s}$.

$$\frac{1}{m} \div \frac{r}{s} = \frac{1}{m} \times \frac{s}{r} = \frac{s}{mr}, \text{ first index.}$$

$$\frac{1}{n} \div \frac{r}{s} = \frac{1}{n} \times \frac{s}{r} = \frac{s}{nr}, \text{ second index.}$$

Therefore $\left(a^{\frac{s}{mr}}\right)^{\frac{r}{s}}$ and $\left(b^{\frac{s}{nr}}\right)^{\frac{r}{s}}$, the answer.

EXAMPLES FOR PRACTICE.

1. Reduce $2^{\frac{1}{3}}$ and $3^{\frac{1}{2}}$ to the common index $\frac{1}{6}$.

Ans. $4^{\frac{1}{6}}$ and $27^{\frac{1}{6}}$.

2. Reduce $x^{\frac{1}{2}}$ and $y^{\frac{1}{4}}$ to the common index $\frac{1}{8}$.

Ans. $(x^4)^{\frac{1}{8}}$ and $(y^2)^{\frac{1}{8}}$.

Case IV. To reduce surds to their simplest form.

Rule.—*Divide the given surd by the greatest power that will divide it without a remainder, and place the said power and quotient, connected by the sign* $\times$ *of multiplication, under the radical sign* —.

Extract the root of the forementioned power, and place its root before the said quotient, with the proper radical sign between them.

EXAMPLES.

1. Reduce $\sqrt{18}$ to its most simple terms.

Thus $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3 \times \sqrt{2} = 3\sqrt{2}$, the answer.

2. Reduce $\sqrt[3]{54a^4}$ to its simplest terms.

Here $\sqrt[3]{54a^4} = \sqrt[3]{27a^3 \times 2a} = 3a\sqrt[3]{2a}$, the answer.

3. Reduce $\sqrt{12x^3y - 16x^5}$ to its most simple terms.

$\sqrt{12x^3y - 16x^5} = \sqrt{4x^2 \times (3xy - 4x^3)} = 2x\sqrt{3xy - 4x^3}$. Ans.

EXAMPLES FOR PRACTICE.

1. Reduce $\sqrt{50}$ to its simplest terms.

Ans. $5\sqrt{2}$.

2. Reduce $\sqrt[3]{128a^3x}$ and $\sqrt[4]{2a^4x^5}$ to their simplest terms.

Ans. $4a\sqrt[3]{2x}$ and $ax\sqrt[4]{2x}$.

3. Reduce $(64a^5x^2-32a^6z)^{\frac{1}{5}}$ to its simplest terms.

Ans. $2a\sqrt[5]{2x^2-az}$.

CASE V. The quantity under the radical sign being a fraction, to reduce it to a whole number.

RULE.—*Multiply both terms of the fraction under the radical sign, into that power of its denominator whose index is 1 less than the index of the surd.*

Then the denominator of the resulting fraction may be taken away, provided you divide the quantity without the radical sign by its root, so shall the surd part be a whole number.

EXAMPLES.

1. Given $\frac{3}{4}\sqrt{\frac{4}{5}}$ to reduce the radical part to a whole number.

First, both terms of the fraction $\sqrt{\frac{4}{5}}$ being multiplied into 5, the denominator will be 25, a complete square, that is the fraction $\sqrt{\frac{4}{5}}=\sqrt{\frac{4\times5}{5\times5}}=\sqrt{\frac{20}{25}}$. Now, taking away this denominator, and dividing $\frac{3}{4}$ (the rational part) by 5 its root, the result is $\frac{3}{4\times5}\sqrt{20}$ or $\frac{3}{20}\sqrt{20}$. This, reduced to its simplest terms, will be $\frac{3}{20}\sqrt{4\times5}=\frac{3\times2}{20}\sqrt{5}=\frac{6}{20}\sqrt{5}=\frac{3}{10}\sqrt{5}$, the answer.

2. Reduce $\sqrt[4]{\frac{1}{3}}$ to an equal quantity whose surd part shall be a whole number.

$$\sqrt[4]{\tfrac{1}{3}}=\sqrt[4]{\frac{1\times3\times3\times3}{3\times3\times3\times3}}=\sqrt[4]{\frac{1\times27}{3\times27}}=\sqrt[4]{\tfrac{27}{81}}.$$

Whence $\frac{1}{3}\sqrt[4]{27}$, the answer.

EXAMPLES FOR PRACTICE.

1. Reduce the radical part of $\frac{2}{3}\sqrt{\frac{1}{2}}$ to a whole number.

Ans. $\frac{1}{3}\sqrt{2}$.

2. Reduce the radical part of $3\sqrt[3]{\frac{4}{7}}$ to a whole number.

Ans. $\frac{3}{7}\sqrt[3]{196}$.

3. Given $\sqrt[3]{\frac{32}{250}}$ and $4\sqrt{\frac{3}{7}}$ to find their simplest terms, having their surd parts whole numbers.

Ans. $\frac{2}{5}\sqrt[3]{2}$ and $\frac{4}{7}\sqrt{21}$.

CASE VI. In fractional forms, having compound surds in the denominator.

RULE.—*Multiply both numerator and denominator by the terms of the denominator, but connected with a different sign, which renders the denominator rational, and reduces the whole expression to its simplest form.*

EXAMPLES.

1. Reduce $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}$, multiply by $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$, and it becomes $\frac{16+2\sqrt{15}}{2}=8+\sqrt{15}$. Answer.

2. Reduce $\frac{3\sqrt{15}-4\sqrt{5}}{\sqrt{15}+\sqrt{5}}$.

Multiply by $\frac{\sqrt{15}-\sqrt{5}}{\sqrt{15}-\sqrt{5}}$ and we have $\frac{65-7\sqrt{75}}{15-5}$ $=\frac{65-35\sqrt{3}}{10}=\frac{13-7\sqrt{3}}{2}$, the answer.

CASE VII. To add surd quantities together.

RULE.—*Reduce the quantities to a common index, the fractions to a common denominator, and the surds to their simplest terms. If the surd part be the same in all the quantities, add the rational parts (or coefficients) toge-*

ther, and annex their sum to the common surd. But if the surd part be not the same, the quantities can be added by means of their signs only.

EXAMPLES.

Required the sum of $\sqrt{8}$, $\sqrt{18}$, and $\sqrt{32}$.

These reduced to their simplest terms, are,

$$\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$$
$$\sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}$$
$$\sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2}$$

Their sum $9\sqrt{2}$. Ans.

2. Add $\sqrt{4a^2x}$ and $\sqrt{16a^2x}$.

$$\sqrt{4a^2x} = 2a\sqrt{x}$$
$$\sqrt{16a^2x} = 4a\sqrt{x}$$

$6a\sqrt{x}$, the answer.

3. Add $\sqrt[3]{8a^3x^2}$ and $\sqrt[3]{16a^3x^2}$ together.

$\sqrt[3]{8a^3x^2} = 2a\sqrt[3]{x^2}$ and $\sqrt[3]{16a^3x^2} = 2a\sqrt[3]{2x^2}$.

These surds being unlike their sum can be denoted only by connecting them by their proper signs, therefore $2a\sqrt[3]{x^2} + 2a\sqrt[3]{2x^2}$ is the sum required.

4. Add $\sqrt{\frac{1}{6}}$ and $\sqrt{\frac{8}{27}}$ together.

These reduced first to a common denominator, and then to their simplest terms, become

$$\sqrt{\tfrac{1}{6}} = \sqrt{\tfrac{27}{162}} = \sqrt{\tfrac{9 \times 3}{81 \times 2}} = \tfrac{3}{9}\sqrt{\tfrac{3}{2}}$$
$$\sqrt{\tfrac{8}{27}} = \sqrt{\tfrac{48}{162}} = \sqrt{\tfrac{16 \times 3}{81 \times 2}} = \tfrac{4}{9}\sqrt{\tfrac{3}{2}}$$

$\frac{7}{9}\sqrt{\frac{3}{2}} = \frac{7}{18}\sqrt{6}$, the sum required.

EXAMPLES FOR PRACTICE.

1. Add $\sqrt{27}$ and $\sqrt{48}$ together.
Ans. $7\sqrt{3}$.

2. Required the sum of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{1}{15}}$.
Ans. $4\sqrt{\frac{1}{15}}$ or $\frac{4}{15}\sqrt{15}$.

3. Required the sum of $\sqrt[3]{\frac{1}{4}}$ and $\sqrt[5]{\frac{1}{32}}$.

Ans. $\frac{3}{4}\sqrt[3]{2}$.

Case VIII. To subtract surd quantities.

Rule.—*Prepare the quantities as in addition, and if the surd parts are alike, subtract their coefficients, annexing the remainder to the common surd. But if the surd parts are not alike, change the sign of the quantity to be subtracted, and then connect it with the other quantity.*

EXAMPLES.

1. Required the difference of $\sqrt{24}$ and $\sqrt{6}$.

First, reducing $\sqrt{24}$ to its simplest terms, we have $\sqrt{24}=\sqrt{4\times 6}=2\sqrt{6}$.

Then subtracting 1 from 2 the remainder is 1 (understood).

Therefore $\sqrt{24}-\sqrt{6}=2\sqrt{6}-\sqrt{6}=1\sqrt{6}$, or $\sqrt{6}$, the answer.

2. Required the difference of $\sqrt[3]{128}$ and $\sqrt[3]{16}$.

$$\sqrt[3]{128}=\sqrt[3]{64\times 2}=4\sqrt[3]{2}$$
$$\sqrt[3]{16}=\sqrt[3]{8\times 2}=2\sqrt[3]{2}$$

Therefore $4\sqrt[3]{2}-2\sqrt[3]{2}=2\sqrt[3]{2}$ the difference required.

3. Required the difference of $\sqrt[4]{32a^5x}$ and $\sqrt[4]{2a^5x}$.

$$\sqrt[4]{32a^5x}=\sqrt[4]{16a^4\times 2ax}=2a\sqrt[4]{2ax}$$
$$\sqrt[4]{2a^5x}=\sqrt[4]{a^4\times 2ax}=a\sqrt[4]{2ax}$$

Whence $2a\sqrt[4]{2ax}-a\sqrt[4]{2ax}=a\sqrt[4]{2ax}$, the answer.

4. What is the difference of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{27}{50}}$?

These reduced to a common denominator become,

$$\sqrt{\tfrac{2}{3}}=\sqrt{\tfrac{100}{150}} \text{ and } \tfrac{27}{150}=\sqrt{\tfrac{81}{150}}.$$

Now, reducing these to their simplest terms, we have

$$\sqrt{\tfrac{100}{150}} = \sqrt{\tfrac{100 \times 1}{25 \times 6}} = \tfrac{10}{5}\sqrt{\tfrac{1}{6}}$$

$$\sqrt{\tfrac{81}{150}} = \sqrt{\tfrac{81 \times 1}{25 \times 6}} = \tfrac{9}{5}\sqrt{\tfrac{1}{6}}.$$

Therefore $\frac{10}{5}\sqrt{\frac{1}{6}} - \frac{9}{5}\sqrt{\frac{1}{6}} = \frac{1}{5}\sqrt{\frac{1}{6}}$;

Which, by making the surd part a whole number,

$\frac{1}{5}\sqrt{\frac{1 \times 6}{6 \times 6}} = \frac{1}{5}\sqrt{\frac{6}{36}} = \frac{1}{5 \times 6}\sqrt{6} = \frac{1}{30}\sqrt{6}$, the answer.

EXAMPLES FOR PRACTICE.

1. What is the difference of $\sqrt{128}$ and $\sqrt{8}$?

   Ans. $6\sqrt{2}$.

2. From $\sqrt[3]{192}$ take $\sqrt[3]{24}$.

   Ans. $2\sqrt[3]{3}$.

3. From $\sqrt{6a^2}$ take $\sqrt{2a^2}$.

   Ans. $a\sqrt{6} - a\sqrt{2}$.

4. From $\sqrt[4]{162a^5x^2}$ take $\sqrt[4]{2a^5x^2}$.

   Ans. $2a\sqrt[4]{2ax^2}$.

5. From $3\sqrt[5]{64a^5z - 32a^6z^2}$ take $\sqrt[5]{64a^5z - 32a^6z^2}$.

   Ans. $4a\sqrt[5]{2z - az^2}$.

Case IX. To multiply surd quantities together.

Rule.—*Reduce the surds to the same index, then multiply the coefficients (or rational parts) together for the rational part of the product, and the surd parts together for the surd part.*

Annex the former product to the latter, and reduce the surd to its simplest terms.

EXAMPLES.

1. Multiply $4\sqrt{3}$ by $5\sqrt{6}$.

The product of the rational part is $4 \times 5 = 20$,

And of the surd parts is $\sqrt{3} \times \sqrt{6} = \sqrt{18}$.

These connected give $20\sqrt{18}$, which, reduced to its simplest terms, becomes $20\sqrt{9 \times 2} = 20 \times 3\sqrt{2} = 60\sqrt{2}$, the answer.

2. Multiply $\sqrt{4ax}$ by $\sqrt{3ab}$.

$\sqrt{4ax} \times \sqrt{3ab} = \sqrt{12a^2bx} = \sqrt{4a^2 \times 3bx} = 2a\sqrt{3bx}$, the answer.

3. Multiply $\sqrt{\frac{4}{7}}$ into $\sqrt{\frac{5}{8}}$.

$\sqrt{\frac{4}{7}} \times \sqrt{\frac{5}{8}} = \sqrt{\frac{20}{56}} = \sqrt{\frac{4 \times 5}{4 \times 14}} = \sqrt{\frac{5}{14}} = \sqrt{\frac{5 \times 14}{14 \times 14}} = \frac{5}{196}\sqrt{70}$. Ans.

EXAMPLES FOR PRACTICE.

1. Multiply $5\sqrt{5}$ by $3\sqrt{8}$.
Ans. $30\sqrt{10}$.

2. Multiply $\sqrt[3]{18}$ by $5\sqrt[3]{4}$.
Ans. $10\sqrt[3]{9}$.

3. Multiply $\frac{2}{3}\sqrt{\frac{1}{2}}$ by $\frac{1}{2}\sqrt{\frac{5}{6}}$.
Ans. $\frac{1}{18}\sqrt{15}$.

CASE X. To divide one surd by another.

RULE.—*Reduce the surds to the same index, and divide the rational parts by the rational, and the surd by the surd; the former quotient annexed to the latter will be the quotient required, which may be reduced to its simplest terms as before.*

EXAMPLES.

1. Divide $12\sqrt{15}$ by $4\sqrt{3}$.

First $\frac{12}{4} = 3$ the rational part.

And $\frac{\sqrt{15}}{\sqrt{3}} = \sqrt{5}$ the surd part.

Therefore $3\sqrt{5}$ is the quotient required.

2. Divide $8\sqrt[3]{48}$ by $2\sqrt[3]{2}$.

$\frac{8\sqrt[3]{48}}{2\sqrt[3]{2}} = 4\sqrt[3]{24} = 4\sqrt[3]{8 \times 3} = 8\sqrt[3]{3}$, the answer.

3. Divide $\frac{3}{4}\sqrt{\frac{5}{6}}$ by $\frac{1}{2}\sqrt{\frac{1}{3}}$.

$\frac{3}{4} \div \frac{1}{2} = \frac{3}{2}$ the rational part,

$\sqrt{\frac{5}{6}} \div \sqrt{\frac{1}{3}} = \sqrt{\frac{5}{2}}$ surd part;

Wherefore $\frac{3}{2}\sqrt{\frac{5}{2}}=\frac{3}{2}\sqrt{\frac{5\times 2}{2\times 2}}=\frac{3}{2}\sqrt{\frac{10}{4}}=\frac{3}{4}\sqrt{10}$, the answer.

EXAMPLES FOR PRACTICE.

1. Divide $8\sqrt{108}$ by $2\sqrt{6}$.
 Ans. $12\sqrt{2}$.

2. Divide $10\sqrt[3]{108}$ by $5\sqrt[3]{4}$.
 Ans. 6.

3. Divide $\sqrt{\frac{10}{11}}$ by $\sqrt{\frac{2}{3}}$.
 Ans. $\frac{1}{11}\sqrt{165}$.

4. Divide $\sqrt[3]{\frac{3az}{4x}}$ by $\sqrt[3]{\frac{ay}{5x}}$.

 Ans. $\frac{1}{2y}\sqrt[3]{30y^2z}$.

5. Divide $x^2-x^2z\sqrt{z}$ by $x-x\sqrt{z}$.
 Ans. $x+x\sqrt{z}+xz$.

CASE XI. To involve surds to any power.

RULE.—*Multiply the index of the surd into the index of the power to which it is to be raised, and to the result annex the power of the rational parts, and it will give the power required.*

EXAMPLES.

1. Involve $4\sqrt[3]{x}$ to the third power.
First $4\times 4\times 4=64$ the rational part.

$$\sqrt[3]{x}=x^{\frac{1}{3}}$$

Therefore $x^{\frac{1\times 3}{3}}=x^{\frac{3}{3}}=x$ the third power of the surd part.
Whence $64x=$ the answer required.

2. Involve $\frac{2}{3}\sqrt[5]{x}$ to the square, $\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$ the rational part, $x^{\frac{1\times 2}{5}}=x^{\frac{2}{5}}=(x^2)^{\frac{1}{5}}$ the surd part.

Therefore $\frac{4}{9}\sqrt[5]{x^2}$ the answer.

3. Involve $\frac{2}{3}\sqrt{\frac{5}{7}}$ to the square.

$\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$ and $\sqrt{\frac{5}{7}}\times\sqrt{\frac{5}{7}}=\sqrt{\frac{25}{49}}=\frac{5}{7}$.

Whence $\frac{4}{9}\times\frac{5}{7}=\frac{20}{63}=$ answer.

EXAMPLES FOR PRACTICE.

1. Involve $2\sqrt[3]{2}$ to the square.

Ans. $4\sqrt[3]{4}$.

2. What is the cube of $\sqrt{2}$, and the 4th power $\frac{1}{6}\sqrt{6}$?

Ans. $2\sqrt{2}$ and $\frac{1}{36}$.

3. What is the cube of $2a-x\sqrt{y}$?

Ans. $8a^3-12a^2x\sqrt{y}+6ax^2y-x^3y\sqrt{y}$.

Case XII. To extract any root of surd quantities.

Rule.—*Multiply the index of the surd into the index of the root to be extracted, annex the root of the rational part to the result, and it will give the root required.*

EXAMPLES.

1. Extract the square root of $16a^{\frac{1}{3}}$.

First $\sqrt{16}=4=$ root of the rational part,

And $a^{\frac{1}{3}\times\frac{1}{2}}=a^{\frac{1}{6}}=$ root of the surd part;

Therefore $4a^{\frac{1}{6}}$ is the root required.

2. Required the cube root of $\frac{8}{27}\sqrt[3]{\frac{3}{4}}$.

First $\sqrt[3]{\frac{8}{27}}=\frac{2}{3}$,

and $(\frac{3}{4})^{\frac{1}{3}\times\frac{1}{3}}=(\frac{3}{4})^{\frac{1}{9}}$.

Whence $\frac{2}{3}\sqrt[9]{\frac{3}{4}}$ is the answer.

3. What is the 4th root of $\frac{1}{16}a^{\frac{1}{2}}$?

$\sqrt[4]{\frac{1}{16}}=\frac{1}{2}$ and $a^{\frac{1}{2}\times\frac{1}{4}}=a^{\frac{1}{8}}$;

Wherefore $\frac{1}{2}\sqrt[8]{a}$ the answer.

4. What is the square root of $1-2\sqrt{2}+2$?

$$\begin{array}{l} 1-2\sqrt{2}+2\ (1-\sqrt{2} \\ 1 \\ 2-\sqrt{2})\ -2\sqrt{2}+2 \\ \quad\quad\quad -2\sqrt{2}+2 \end{array}$$

EXAMPLES FOR PRACTICE.

1. What is the square root of $9\sqrt[3]{3}$?

   Ans. $3\sqrt[6]{3}$.

2. What is the cube root of $\frac{1}{8}a^3b$?

   Ans. $\frac{1}{2}a\sqrt[3]{b}$.

3. Required the 4th root of $\frac{1}{81}a^3$.

   Ans. $\frac{1}{3}a^{\frac{3}{4}}$.

4. Required the 5th root of $\frac{2}{3}\sqrt{\frac{x}{a}}$.

   Ans. $\left(\frac{2}{3}\right)^{\frac{1}{5}}\left(\frac{x}{a}\right)^{\frac{1}{10}}$.

5. What is the square root of $a^2-4a\sqrt{x}+4x$?

   Ans. $a-2\sqrt{x}$.

CASE XIII. To find the square root of a binomial or residual surd.

Let it be proposed to find the square root of $a\pm\sqrt{b}$.

Assume $\sqrt{a\pm\sqrt{b}}=\sqrt{x}\pm\sqrt{y}$; then by squaring, we have $a\pm\sqrt{b}=x+y\pm4xy$.

$$\text{this gives us}\left\{\begin{array}{l} x+y=a \\ 4xy=b; \end{array}\right.$$

Whence we find $x=\dfrac{a+\sqrt{a^2-b}}{2}$,

$$\text{and } y=\frac{a-\sqrt{a^2-b}}{2}$$

which may therefore be considered as a general formula for this purpose.

EXAMPLES.

1 Required the square root of $12 + 2\sqrt{35}$.

$$\text{Here } \left.\begin{aligned} a &= 12 \\ b &= 140 \end{aligned}\right\}$$

$$x = \frac{12 + \sqrt{144 - 140}}{2} = \frac{12 + \sqrt{4}}{2} = 7,$$

$$y = \frac{12 - \sqrt{144 - 140}}{2} = \frac{12 - \sqrt{4}}{2} = 5;$$

Therefore the root is $\sqrt{7} + \sqrt{5}$.

EXAMPLES FOR PRACTICE.

1. What is the square root of $4 + 2\sqrt{3}$?
   Ans. $1 + \sqrt{3}$.
2. Required the square root of $6 - 2\sqrt{5}$.
   Ans. $\sqrt{5} - 1$.
3. What is the square root of $7 + 4\sqrt{3}$?
   Ans. $2 + \sqrt{3}$.
4. What is the square root of $32 - 10\sqrt{7}$?
   Ans. $5 - \sqrt{7}$.

CASE XIV. In a given fraction to remove any quantity from the denominator into the numerator, or from the numerator into the denominator.

RULE.—*Any quantity may be removed from one term of a fraction to the other, provided you change the sign of its index.*

REMARK.—If the power of any quantity be divided by that quantity, its index will be decreased by 1, as is evident from the nature of division. Therefore, $a^3 \div a = (a^{3-1} =) a^2$; $a^2 \div a = (a^{2-1} =) a^1$; $a^1 \div a = (a^{1-1} =) a^0$, $a^0 \div a = a^{-1}$; $a^{-1} \div a$; $= a^{-2}$, and so on. Hence it follows that $(a \div a =) a^0 = 1$; $(a^0 \div a =) a^{-1} = \frac{1}{a}$; $(a^{-1} \div a =) a^{-2} = \frac{1}{a^2}$ $(a^{-2} \div a =) a^{-3} = \frac{1}{a^3}$ &c. from whence the rule will be manifest.

EXAMPLES.

1. In the fraction $\frac{1}{xy}$ the quantity y is required to be put in the numerator.

Ans. $\frac{y^{-1}}{x}$.

2. Change a^{-2}, b^{-3}, x^{-4}, and y^{-5}, into fractions.

Ans. $\frac{1}{a^2}$, $\frac{1}{b^3}$, $\frac{1}{x^4}$, and $\frac{1}{y^5}$.

3. What is the value of $a \times (x - y)^{-\frac{1}{3}}$ and $xy \times (x + y)^{-2}$ in fractions having affirmative exponents.

Ans. $\frac{a}{(x-y)^{\frac{1}{3}}}$ and $\frac{xy}{x^2 + 2xy + y^2}$.

SIMPLE EQUATIONS.

An equation is when two equal quantities, differently expressed, are compared together by placing the sign $=$ between them.

Thus $4 + 2 = 6$ is an equation expressing the equality of the quantities $4 + 2$ and 6.

A simple equation is that which contains only one unknown quantity, without including its power.

Thus $x + 3 = 6$ is a simple equation, containing only the unknown quantity x.

Reduction of equations is the method of finding the value of the unknown quantity, which is shewn in the following rules:

Rule 1.—*Any term of an equation may be transposed from one side to the other by changing its sign.*

Remark.—If equal quantities be added to equals, the sums will be equal; and if equals be subtracted from equals, the remainders will be equal. These self-evident truths being admitted, the reason of the above rule for transposition will readily appear.

EXAMPLES.

1. Given $x + 2 = 6$ to find x.
   Then $x + 2 = 6$ is given,
   and $x = 6 - 2$ by transposition;
   Therefore $x = 4$.

From which it appears, that $+ 2$ may be transposed to the other side of the equation by changing it to $- 2$, and by this means we obtain the value of x, which is $6 - 2$, that is 4.

2. Given $x - 2 = 2$ to find x.
   Thus $x - 2 = 2$ is given;
   Then $x = 2 + 2 = 4$ by transposition;
   Therefore $x = 4$.

Here $- 2$ has been transposed to the other side of the equation by changing it to $+ 2$.

In the same manner may the value of x be found in the following examples:

3. Given $3x - 5 = 2x + 9$ to find x.
   Here $3x - 5 = 2x + 9$ is given,
   $3x = 2x + 9 + 5$ by transposing $- 5$,
   $3x - 2x = 9 + 5$ by transposing $+ 2x$;
   But $3x - 2x = x$,
   and $9 + 5 = 14$;
   Therefore $x = 14$.

4. Given $5x + 50 = 4x + 56$ to find x.
   $5x - 4x = 56 - 50$ by transposition;
   Therefore $x = 6$.

EXAMPLES FOR PRACTICE.

1. Given $2x + 3 = x + 17$ to find x.
   Ans. $x = 14$.

2. Given $4x - 8 = 3x + 20$ to find x.
   Ans. $x = 28$.

3. Given $2x + a = x + b$ to find x.
   Ans. $x = b - a$.

4. Given $x + 8 - 2 = 3$ to find x.
   Ans. $x = -3$.

RULE 2.—*If the unknown term be multiplied by any quantity, it may be taken away by dividing all the other terms by it.*

REMARK.—If equals be divided by equals, the quotient will be equal, which is the foundation of the above rule.

EXAMPLES.

1. Given $2x = 8$ to find x.
Then $x = \frac{8}{2} = 4$ by dividing by 2;
Therefore $x = 4$.

2. Given $4x - 2 = 2x + 6$ to find x.
Then $4x - 2x = 6 + 2$ by transposition,
or $2x = 8$;
Therefore $x = 4$ by dividing by 2.

3. Given $6x + 10 = 3x + 22$ to find x.
Then $6x = 3x + 22 - 10$ by transposing 10,
and $6x - 3x = 22 - 10$ by transposing $3x$,
or $3x = 12$;
Therefore $x = 4$ by dividing by 3.

EXAMPLES FOR PRACTICE.

1. Given $15x + 4 = 34$ to find x.
Ans. $x = 2$.

2. Given $4x - 8 = 3x + 20$ to find x.
Ans. $x = 28$.

3. Given $9x - 15 = x + 6$ to find x.
Ans. $x = 2\frac{5}{8}$.

4. Given $ax + 3ab = 4c^2$ to find x.

$$\text{Ans } x = \frac{4c^2}{a} - 3b.$$

RULE 3.—*If the unknown term be divided by any quantity, it may be taken away by multiplying all the other terms of the equation by it; but if there be more than one fraction, multiply all the terms by each denominator successively: Or multiply all the terms by the product of the denominators, or by the least multiple of the denominators.*

REMARK.—Like the foregoing rules, this agrees with a self-evident principle, namely, that if equals be multiplied by equals, the products will be equal.

EXAMPLES.

1. Given $\frac{x}{4} = 3$ to find x.

Then $x = 12$ by mult. by 4.

2. Given $\frac{x}{4} = x - 9$ to find x.

Then $x = 4x - 36$ by mult. by 4,
$4x - x = 36$, by transp.
or $3x = 36$;
Therefore $x = 12$ by dividing by 3.

3. Given $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 13$ to find x.

Then $x + \frac{2x}{3} + \frac{2x}{4} = 26$ by mult. by 2.

and $3x + 2x + \frac{6x}{4} = 78$ by mult. by 3.

Also $12x + 8x + 6x = 312$ by mult. by 4,
or $26x = 312$;
Therefore $x = 12$ by dividing by 26.

4. Given $\frac{56}{5x+3} = \frac{63}{14x-5}$ to find x.

Then $56 = \frac{315x + 189}{14x - 5}$ by mult. by $5x + 3$,

and $784x - 280 = 315x + 189$ by mult. by $14x - 5$.
$784x - 315x = 189 + 280$ by transp.
or $469x = 469$;
Therefore $x = 1$ by dividing by 469.

EXAMPLES FOR PRACTICE.

1. Given $\frac{2x}{3} + \frac{x}{4} = 22$ to find x.

Ans. $x = 24$.

2. Given $\frac{x}{3}+\frac{x}{5}+\frac{x}{2}=62$ to find x.
Ans. $x=60$.

3. Given $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x-19}{2}$ to find x.
Ans. $x=23\frac{1}{4}$.

4. Given $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{x+3}{4}$ to find x.
Ans. $x=13$.

5. Given $x+\frac{x}{3}-\frac{x}{7}+\frac{2x}{5}=7x$ to find x.
Ans. $x=\frac{167}{735}$.

RULE 4.—*Any analogy or proportion may be converted into an equation, by making the product of the two mean terms equal to that of the two extremes.*

REMARK.—The foundation of this rule is, that, when four quantities are proportional, the product of the two means are equal to that of the extremes. Thus,

Let $2:4::6:12$ be proportionals. Then $2\times 12=6\times 4=24$. Also $a:b::ma:mb$. Then is $a\times mb=b\times ma=mab$, as is evident.

EXAMPLES.

1. Given $x:12::6:18$ to find x.
Then $18x=72$ by mult. ext. and means,
and $x=4$ by division.

2. Given $x:6-x::2:4$ to find x.
Then $4x=12-2x$ by mult. ext. and means,
and $6x=12$ by transposition;
Therefore $x=2$ by dividing by 6.

3. Given $x:y::1:2$ to find x.
Then $2x=y$ by mult. ext. and means;
Therefore $x=\frac{y}{2}$ by dividing by 2.

4. Given $\frac{3}{4}x : a :: 5b : 2c$ to find x.
Then $\frac{3}{2}xc = 5ab$ by mult. ext. and means,
and $3xc = 10ab$ by mult. by 2;
Therefore $x = \frac{10ab}{3c}$ by dividing by $3c$.

EXAMPLES FOR PRACTICE.

1. Given $10 - x : \frac{2}{3}x :: 3 : 1$ to find x.
Ans. $x = 3\frac{1}{3}$.

2. Given $\frac{3x}{5} : 4 :: 2 : 1$ to find x.
Ans. $x = 13\frac{1}{3}$.

3. Given $\frac{x}{2} : \frac{2}{3} :: 3 : 2$ to find x.
Ans. $x = 1$.

Rule 5.—*If the same quantity be found on both sides of the equation with the same sign, it may be taken away from each; and if every term in an equation be multiplied by the same quantity, it may be struck out of them all.*

Remark.—The reason of this will appear from the Remarks on Rules 1. and 2.

EXAMPLES.

1. Given $2x + y = a + y$ to find x.
Then $2x = a$ by taking away y from both sides.
and $x = \frac{a}{2}$.

2. Given $x^2 + 2x = x^2 + 100$ to find x.
Then $2x = 100$ by exter. x^2 from both sides of the equation,
and $x = 50$ dividing by 2.

3. Given $4xy + 8xz = 9xv$ to find x.
Then $4y + 8z = 9v$ by exter. x,
and $4y = 9v - 8z$ by transposition;
Therefore $y = \frac{9v - 8z}{4}$ by division.

EXAMPLES FOR PRACTICE.

1. Given $2x+a=b+a$ to find x.

   Ans. $x=\frac{b}{2}$.

2. Given $ax=ab-ac$ to find x.

   Ans. $x=b-c$.

3. Given $4ax-4ab=4ac$ to find x.

   Ans. $x=c+b$.

4. Given $\frac{2}{3}x-\frac{7}{3}=\frac{10}{3}-\frac{7}{3}$ to find x.

   Ans. $x=5$.

RULE 6.—*The unknown quantity in any equation may be made free from surds, by transposing the rest of the terms by Rule* 1. *and then involving each side to such a power as is denoted by the index of the surd.*

EXAMPLES.

1. Given $\sqrt{x}-2=3$ to find x.

   Then $\sqrt{x}=3+2=5$ by transposition,

   and $x=25$ by involution.

2. Given $\sqrt{2x+14}=8$ to find x.

   Then $2x+14=8^2=64$ by involution,

   and $2x=64-14$ by transposition,

   or $2x=50$;

   Therefore $x=25$ by dividing by 2.

3. Given $2\sqrt{x}+3=9$ to find x.

   Then $2\sqrt{x}=9-3=6$ by transposition,

   and $\sqrt{x}=3$ by dividing by 2;

   also $x=3^2=9$ by involution.

   Therefore $x=9$.

4. Given $\frac{3\sqrt[3]{x}}{4}+5=8$ to find x.

   Then $\frac{3\sqrt[3]{x}}{4}=8-5=3$ by transposition,

   and $3\sqrt[3]{x}=12$ by multiplying by 4;

also $\sqrt[3]{x}=4$ by dividing by 3.
Therefore $x=4^3=64$ by involution,
Or $x=64$.

EXAMPLES FOR PRACTICE.

1. Given $\sqrt{x}-9=1$ to find x.
   Ans. $x=100$.
2. Given $\sqrt{x+1}-2=3$ to find x.
   Ans. $x=24$.
3. Given $\sqrt[3]{3x+4}+3=6$ to find x.
   Ans. $x=7\frac{2}{3}$.
4. Given $\sqrt{4+x}=4-\sqrt{x}$ to find x.
   Ans. $x=2\frac{1}{4}$.
5. Given $\sqrt{4a^2+x^2}=\sqrt[4]{4b^4+4x^4}$ to find x.
   Ans. $x=\sqrt{\dfrac{b^4-4a^4}{2a^2}}$.

RULE 7.—*If that side of the equation which contains the unknown quantity be a complete power, it may be reduced, by extracting the root of the said power from both sides of the equation.*

EXAMPLES.

1. Given $x^2=36$ to find x.
   Then $x=\sqrt{36}=6$ by extraction;
   Therefore $x=6$.
2. Given $x^2+10=46$ to find x.
   Then $x^2=46-10$ by transposition,
   and $x^2=36$;
   also $x=\sqrt{36}=6$ by extraction;
   Therefore $x=6$.
3. Given $x^2+2x+1=25$ to find x.
   Then $x+1=\sqrt{25}=5$ by evolution,
   and $x=5-1=4$ by transposition;
   Therefore $x=4$.

4. Given $\frac{3}{4}x^2 - 6 = 24$ to find x.
Then $\frac{3}{4}x^2 = 24 + 6 = 30$ by transp.
and $3x^2 = 120$ by mult. by 4.
Then $x^2 = 40$ by dividing by 3,
and $x = \sqrt{40} = 6.324555$ by evolution.

EXAMPLES FOR PRACTICE.

1. Given $9x^2 - 3 = 22$ to find x.
Ans. $x = \frac{5}{3} = 1\frac{2}{3}$.
2. Given $x^3 + 8 = 35$ to find x.
Ans. $x = 3$.
3. Given $x^2 - 5 + 2 = 97$ to find x.
Ans. $x = 10$.
4. Given $x^2 + ax + \frac{a^2}{4} = b^2$ to find x.
Ans. $x = b - \frac{a}{2}$.
5. Given $ax^2 - b = c$ to find x.
Ans. $x = \sqrt{\frac{c+b}{a}}$.
6. Given $x^2 + 14x + 49 = 121$ to find x.
Ans. $x = 4$.

PROMISCUOUS EXAMPLES FOR PRACTICE.

1. Given $2x - 5 + 16 = 21$ to find x.
Ans. $x = 5$.
2. Given $ax = ab - a$ to find x.
Ans. $x = b - 1$.
3. Given $8 - 3x + 12 = 30 - 5x + 4$ to find x.
Ans. $x = 7$.
4. Given $\frac{3x}{4} + 5 = \frac{5x}{6} + 2$ to find x.
Ans. $x = 36$.

5. Given $\frac{5x}{9} - 8 = 74 - \frac{7x}{12}$ to find x.
Ans. $x = 72$.

6. Given $56 - \frac{3x}{4} = 48 - \frac{5x}{8}$ to find x.
Ans. $x = 64$.

7. Given $\frac{x-2}{4} + \frac{x-1}{5} + \frac{x+1}{3} + \frac{x+2}{7} = 24$ to find x.
Ans. $x = 26$.

8. Given $\frac{x-2}{3} - \frac{x-4}{8} = \frac{x}{4} + \frac{x}{5} - 6$ to find x.
Ans. $x = 20$.

9. Given $100 - 3x = 50 - \frac{x}{3}$ to find x.
Ans. $x = 18\frac{3}{4}$.

10. Given $\sqrt{40x + \frac{x}{3}} = 3x + \frac{2x}{3}$ to find x.
Ans. $x = 3$.

11. Given $\sqrt{a^2 + ax} = a - \sqrt{a^2 - ax}$ to find x.
Ans $x = \frac{a}{2}\sqrt{3}$.

12. Given $x^2 + 42 = 285 - 2x^2$ to find x.
Ans. $x = 9$.

13. Given $615 - 7x^3 = 48x$ to find x.
Ans. $x = 9$.

14. Given $ax + b^2 = \frac{ax^2 + ac^2}{a + x}$ to find x.
Ans. $x = \frac{ac^2 - ab^2}{a^2 + b^2}$.

Before we proceed to the management of equations involving two or more unknown quantities, we shall show the method of applying the foregoing Rules to the solution of a few easy questions containing only one unknown quantity.

1. What number is that, to which 5 being added, the sum will be 40?

Let $x =$ the number required;

Then $x + 5 =$ the number, when 5 is added.

But $x + 5 = 40$ by the quest.

And $x = 40 - 5 = 35$. by transp.

Therefore $x = 35 =$ the number required.

2. What number is that from which 8 being subtracted, the remainder is 45?

Let $x =$ the number required.

Then $x - 8 =$ the number when 8 is subtracted.

But $x - 8 = 45$ by the quest.

And $x = 45 + 8 = 53$ by transp.

Therefore $x = 53 =$ the number required.

3. What number is that, which being multiplied by 6, the product increased by 18, and that sum divided by 9, the quotient shall be 20?

Let $x =$ the number required;

Then $6x =$ the number multiplied by 6.

and $6x + 18 =$ the product increased by 18,

and $\dfrac{6x + 18}{9} =$ the sum divided by 9.

Hence $\dfrac{6x + 18}{9} = 20$ by the quest.

And $6x + 18 = 180$ by mult. by 9,

or $6x = 180 - 18 = 162$ by transp.

Then $x = 27$, by dividing by 6;

Therefore $x = 27 =$ the number required.

4. What number is that, from which 6 being subtracted, and the remainder multiplied by 11, the product is 121?

Let $x =$ the number required;

Then $x - 6 =$ the number when 6 is subtracted,

and $\overline{x - 6} \times 11 = 11x - 66 =$ the number multiplied by 11.

Then $11x - 66 = 121$ by the question,

and $11x = 121 + 66$ by transp.

or $11x = 187$;

Therefore $x = 17$ by dividing by 11.

5. A mercer having cut 12 yards off from each of three equal pieces of silk, found that the remnants taken together were 126 yards; required the length of each piece.

Let $x =$ length of each piece;
Then $x - 12 =$ length of each remnant,
and $\overline{x - 12} \times 3 = 3x - 36 = 126$ by the question;
also $3x = 126 + 36 = 162$ by transp.
or $3x = 162$;
Therefore $x = 54$ by dividing by 3.
Hence the length of each piece was 54 yards.

6. A person whose ability was known to be greater than his industry, was hired to work for a year at 7*l*. per day, but with this condition, that for every day he played he should forfeit 3*l*. Now at the year's end he had neither money to receive nor to pay: How many days did he work?

Suppose $x =$ days he wrought;
Then $365 - x =$ the days he played,
and $7x =$ sum he earned.
Also $\overline{365 - x} \times 3 = 1095 - 3x =$ the sum forfeited.
But $7x = 1095 - 3x$ by quest.
and $10x = 1095$ by transp.
Therefore $x = 109\frac{1}{2}$ by division.

7. A and B began trade with equal stocks; A gained £.200, and B lost £.100, after which A's stock was to B's as 3 to 2. What did each begin with?

Let $x =$ pounds each began with;
Then $x + 200 =$ A's stock increased by 200,
and $x - 100 =$ B's diminished by 100.
Now $x + 200 : x - 100 :: 3 : 2$ by quest.
Then $2x + 400 = 3x - 300$ by mult. extremes and means.
And $400 = x - 300$ by transposing $2x$;
also $700 = x$ by transp. -300;
Therefore each began with £.700.

QUESTIONS FOR PRACTICE.

1. What two numbers are those whose difference is 10, and if 15 be added to their sum, the whole will be 43?

Ans. 9 and 19.

2. What two numbers are those, whose difference is 14, and if 9 times the lesser be subtracted from 6 times the greater, the remainder will be 33?

Ans. 17 and 31.

3. What number is that which being divided by 6, and 2 subtracted from the quotient, the remainder is 2?

Ans. 24.

4. What two numbers are those whose difference is 14, and the quotient of the greater divided by the less is 3?

Ans. 21 and 7.

5. What two numbers are those whose sum is 60, and the greater is to the lesser as 9 to 3?

Ans. 45 and 15.

6. What two numbers are those whose difference is 10, and the difference of their squares 120?

Ans. 1 and 11.

7. Divide 560 into two such parts, that one part may be to the other as 5 to 2.

Ans. 400 and 160.

8. Two persons, A and B, engage to play. A has 72 guineas, and B 52. After a certain number of games won and lost between them, A rises with three times as many guineas as B: Required how many guineas A won of B?

Ans. 21.

To exterminate two unknown quantities, or to reduce the two simple equations containing them to a single one.

RULE 1.—*Observe which of the unknown quantities is the least involved, and find its value in each of the equations by the methods already explained. Let the two values thus found be compared, by making them equal to each other, from whence there will arise a new equation involving only one unknown quantity whose value may be found as before.*

REMARK.—This rule is founded on the obvious principle, that two quantities which are equal to the same quantity, are equal to each other.

EXAMPLES.

Given $\left\{\begin{array}{l} x + 3y = 100 \\ 2x + y = 100 \end{array}\right\}$ to find x and y.

From 1st $x = 100 - 3y$

From 2d $x = \dfrac{100 - y}{2}$.

But $100 - 3y = \dfrac{100 - y}{2}$ both being $= x$.

$200 - 6y = 100 - y$ by mult. by 2,

$6y - y = 200 - 100$ by transp.

or $5y = 100$;

Therefore $y = 20$ by dividing by 5,

and $x = 100 - 3y = 40$.

2. Given $\left\{\begin{array}{l} x + y = 360 \\ x : y :: 5 : 4 \end{array}\right\}$ to find x and y.

From 1st $x = 360 - y$

From 2d $x = \dfrac{5y}{4}$.

But $\dfrac{5y}{4} = 360 - y$ both $= x$.

Then $5y = 1440 - 4y$ by mult. by 4,

and $5y + 4y = 1440$ by transp.

or $9y = 1440$;

Therefore $y = 160$ by divid. by 9,

and $x = 360 - y$;

That is, $x = 360 - 160 = 200$,

Therefore $x = 200$,

and $y = 160$.

EXAMPLES FOR PRACTICE.

1. Given $\left\{\begin{array}{l}\frac{1}{2}x+2y=a\\ \frac{1}{2}x-2y=b\end{array}\right\}$ to find x and y.

Ans. $x=a+b$, and
$y=\frac{1}{4}a-\frac{1}{4}b$.

2. Given $\left\{\begin{array}{l}\frac{x}{2}+\frac{y}{3}=8\\ \frac{x}{3}-\frac{y}{2}=1\end{array}\right\}$ to find x and y.

Ans. $x=12$, and
$y=6$.

3. Given $\left\{\begin{array}{l}\frac{x}{2}+\frac{y}{3}=9\\ x:y::4:3\end{array}\right\}$ to find x and y.

Ans. $x=12$, and
$y=9$.

RULE 2.—*Consider which of the unknown quantities you would first exterminate, and let its value be found in that equation where it is least involved. Then substitute the value thus found for its equal in the other equation, and there will arise a new equation, with only one unknown quantity, whose value may be found as before.*

EXAMPLES.

1. Given $\left\{\begin{array}{l}x+2y=80\\ x+\ \ y=60\end{array}\right\}$ to find x and y.

From 2d. $x=60-y$.
From 1st. $60-y+2y=80$ by subst.
Therefore $y=80-60=20$ by transp.
and $x=60-y=40$.
That is, $x=40$,
and $y=20$.

2. Given $\left\{\begin{array}{l}2x+3y=17\\ 5x-2y=14\end{array}\right\}$ to find x and y.

From 1st. $x=\frac{17-3y}{2}$.

From 2d. $\frac{85-15y}{2}-2y=14$ by subst.

Then $85-15y-4y=28$ by mult. by 2.

and $57=19y$ by transp.

Therefore $3=y$ by dividing by 19.

Then $x=\frac{17-3y}{2}=4$.

That is, $x=4$

and $y=3$.

EXAMPLES FOR PRACTICE.

1. Given $\left\{\begin{array}{l} 5x-3y=150 \\ 10x+15y=825 \end{array}\right\}$ to find x and y.

Ans. $x=45$

and $y=25$.

2. Given $\left\{\begin{array}{l} x+y=16 \\ x:y::3:1 \end{array}\right\}$ to find x and y.

Ans. $x=12$

$y=4$.

3. Given $\left\{\begin{array}{l} x+\frac{y}{2}=12 \\ y+\frac{x}{2}=9 \end{array}\right\}$ to find x and y.

Ans. $x=10$

$y=4$.

4. Given $\left\{\begin{array}{l} x:y::3:2 \\ x^2-y^2=20 \end{array}\right\}$ to find x and y.

Ans. $x=6$

$y=4$.

Rule 3.—*Let the given equations be multiplied or divided by such numbers or quantities as will make the term which contains one of the unknown quantities the same in both equations, then, by adding or subtracting the equations, according as is required, there will arise a new equation with only one unknown quantity.*

EXAMPLES.

1. Given $\left\{\begin{array}{r} 3x+5y=40 \\ x+2y=14 \end{array}\right\}$ to find x and y.

$3x+6y=42$ by mult. 2d by 3.

$y=2$ by sub. 1st from 3d.

From 2d equa. $x=14-2y$ by transp.

Therefore $x=14-4=10$ by subst.

That is, $x=10$ and $y=2$.

2. Given $\left\{\begin{array}{r} 3x+2y=23 \\ 4x-7y=21 \end{array}\right\}$ to find x and y.

Then $12x+8y=92$ by mult. 1st by 4.

$12x-21y=63$ by mult. 2d by 3.

$29y=29$ subtraction.

Therefore $y=1$ by dividing by 29,

and $x=\dfrac{23-2y}{3}$ from 1st equa.

Therefore $x=\dfrac{23-2}{3}=\dfrac{21}{3}=7$ by divid.

EXAMPLES FOR PRACTICE.

1. Given $\left\{\begin{array}{r} 3x+8y=14 \\ 2x-y=3 \end{array}\right\}$ to find x and y.

Ans. $x=2$
$y=1$.

2. Given $\left\{\begin{array}{r} \dfrac{x}{2}+\dfrac{y}{3}=8 \\ \dfrac{x}{3}-\dfrac{y}{2}=1 \end{array}\right\}$ to find x and y.

Ans. $x=12$
$y=6$.

3. Given $\dfrac{x+8}{4}+6y=21$, and $\dfrac{y+6}{3}+5x=23$, to find x and y.

Ans. $x=4$
$y=3$.

The first of these three methods is the most commonly used; but the last of them is, for the general part, the most easy and expeditious in practice.

To exterminate three unknown quantities, as x, y, and z, or to reduce the three simple equations containing them to a single one.

RULE.—*Find the value of* x *from each of the three given equations. Then compare the first value of* x *with the second, and an equation will arise involving only* y *and* z. *In like manner compare the first value of* x *with the third, and you will have another equation involving only* y *and* z. *Then find the value of* y *and* z *from these two equations, according to the former rules, and* x, y, *and* z, *will be exterminated, as required.*

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+y+z=6 \\ x+3y+2z=13 \\ 2x+4y+5z=25 \end{array}\right\}$ to find x, y, and z.

From 1st. $x = 6 - y - z$.
From 2d. $x = 13 - 3y - 2z$.
From 3d. $x = \dfrac{25 - 4y - 5z}{2}$.

Now, *let two values of* y *be found from these, by comparing the first with the second, and the second with the third.*

Thus $6 - y - z = 13 - 3y - 2z$, both being $= x$.
$2y = 7 - z$ by transposition;

Therefore $y = \dfrac{7 - z}{2}$ by division.

Also $13 - 3y - 2z = \dfrac{25 - 4y - 5z}{2}$ being $= x$,

and $26 - 6y - 4z = 25 - 4y - 5z$ by mult. by 2,
$2y = 1 + z$ by transp.

Therefore $y = \dfrac{1 + z}{2}$ by division.

But $\dfrac{7 - z}{2} = \dfrac{1 - z}{2}$ both being $= y$.

$$\text{or } 7 - z = 1 + z,$$
$$6 = 2z \text{ by transp.}$$
Therefore $3 = z$,
$$\text{and } y = \frac{1 + z}{2} = 2;$$
$$\text{also } x = 6 - y - z = 1.$$

$$2.\ \text{Given} \left\{\begin{array}{l} x + y + z = 29 \\ x + 2y + 3z = 62 \\ \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 10 \end{array}\right\} \text{to find } x, y, \text{and } z.$$

From 1st. $x = 29 - y - z$.
From 2d. $x = 62 - 2y - 3z$.
From 3d. $x = 20 - \frac{2y}{3} - \frac{z}{2}$.

Then by comparing the first with the second, and the second with the third, we have

$$29 - y - z = 62 - 2y - 3z \text{ being} = x,$$
$$\text{and } y = 33 - 2z \text{ by transp.}$$
$$\text{also } 62 - 2y - 3z = 20 - \frac{2y}{3} - \frac{z}{2} \text{ both} = x,$$
$$\text{and } y = \frac{252 + 15z}{8}.$$
$$\text{But } \frac{252 + 15z}{8} = 33 - 2z \text{ both} = y,$$
$$252 + 15z = 254 - 16z \times \text{by } 8;$$
Therefore $z = 12$ by transp.
$$\text{and } y = 33 - 2z = 9,$$
$$\text{also } x = 29 - y - z = 8.$$

EXAMPLES FOR PRACTICE.

$$1.\ \text{Given} \left\{\begin{array}{l} x + \frac{y}{2} + \frac{z}{3} = 27 \\ x + \frac{y}{3} + \frac{z}{4} = 20 \\ x + \frac{y}{4} = \frac{z}{5} = 16 \end{array}\right\} \text{to find } x, y, \text{and } z.$$

Ans. $x = 1$, $y = 20$, and $z = 60$.

2. Given $\left\{\begin{array}{l} x - y = 2 \\ x - z = 3 \\ y - z = 1 \end{array}\right\}$ to find x, y, and z.

$$\text{Ans. } x = 7$$
$$y = 5$$
$$z = 4.$$

If there be four or more unknown quantities to exterminate, and as many equations to resolve, we should proceed in the same manner, but the calculation would often prove very tedious.

It is proper, therefore, to remark, that, in each particular case, means may always be discovered of greatly facilitating its resolution. These means consist in introducing into the calculation, beside the principal unknown quantities, a new unknown quantity arbitrarily assumed, such as, for example, the sum of all the rest: and when a little practised in such calculations, they become easy.

We shall give one example as a specimen of this sort of ingenuity.

Thus, let it be required to find four numbers, such, that the sum of the three first is 13, the sum of the two first and the last is 17, the sum of the first and two last is 18, the sum of the three last is 21.

Put a, b, c, d, for the four numbers respectively, and the equations will stand thus,

$$a + b + c = 13$$
$$a + b + d = 17$$
$$a + c + d = 18$$
$$b + c + d = 21.$$

Substitute S for the sum of all the four numbers; that is, put S for $a + b + c + d$, *and the above equations will be transformed into the following ones.*

$$S - d = 13$$
$$S - c = 17$$
$$S - b = 18$$
$$S - a = 21.$$

Add all these equations together, and we have

$4S - a - b - c - d = 69$, that is
$4S - (a + b + c + d) = 69$.
But $a + b + c + d$ is equal S,
Therefore $4S - S = 69$;
That is, $3S = 69$,
and $S = 23$ by div. by 3.

For S put its value in the four transformed equations, and we shall have

First $23 - a = 21$, and $a = 2$.
Then $23 - b = 18$, and $b = 5$.
Again $23 - c = 17$, and $c = 6$.
Lastly, $23 - d = 13$, and $d = 10$.

Therefore 2, 5, 6, *and* 10, *are the four numbers required.*

We shall now give a few examples for practice, and then proceed to the solution of questions producing simple equations.

1. Given $\begin{cases} 3x + 2y = 40 \\ 2x + 3y = 35 \end{cases}$ to find x and y.

Ans. $x = 10$
$y = 5$.

2. Given $\begin{cases} 5x - 4y = 19 \\ 4x + 2y = 36 \end{cases}$ to find x and y.

Ans. $x = 7$
$y = 4$.

3. Given $\begin{cases} 3x + 7y = 79 \\ 2y - \frac{x}{2} = 9 \end{cases}$ to find x and y.

Ans. $x = 10$
$y = 7$.

4. Given $\begin{cases} \frac{3x - 5y}{2} + 3 = \frac{2x + y}{5} \\ 8 - \frac{x - 2y}{4} = \frac{x}{2} + \frac{y}{3} \end{cases}$ to find x and y.

Ans. $x = 12$
$y = 6$.

5. Given $\left\{\begin{matrix} ax + by = c \\ dx + cy = f \end{matrix}\right\}$ to find x and y.

$$\text{Ans. } x = \frac{ce - bf}{ac - bd}$$

$$y = \frac{af - dc}{ac - bd}.$$

6. Given $\left\{\begin{matrix} x + y + z = 9 \\ x + 2y + 3z = 16 \\ x + y - 2z = 3 \end{matrix}\right\}$ to find x, y, and z.

Ans. $x = 4$
$y = 3$
$z = 2$.

7. Given $\left\{\begin{matrix} x + y + z = 7 \\ 2x - y - 3z = 3 \\ 5x - 3y + 5z = 19 \end{matrix}\right\}$ to find x, y, and z.

Ans. $x = 4$
$y = 2$
$z = 1$.

8. Given $\left\{\begin{matrix} x + y + z = 12 \\ x + 2y + 3z = 20 \\ \frac{x}{3} + \frac{y}{2} + z = 6 \end{matrix}\right\}$ to find x, y, and z.

Ans. $x = 6$
$y = 4$
$z = 2$.

10. Given $\left\{\begin{matrix} x + y + z = 26 \\ x - y \quad = 4 \\ x - z \quad = 6 \end{matrix}\right\}$ to find x, y, and z.

Ans. $x = 12$
$y = 8$
$z = 6$.

QUESTIONS PRODUCING SIMPLE EQUATIONS.

1. A person being asked his age, answered that $\frac{3}{4}$ of his age multiplied by $\frac{1}{12}$ of his age gives a product equal to his age: Required what was his age?

Suppose $x =$ his age.

Then $\frac{3x}{4} \times \frac{x}{12} = x$, by the quest.

or $\frac{3x^2}{48} = x$, by multiplication,

and $3x^2 = 48x$ by mult. by 48

$3x = 48$ by dividing by x

$x = 16$ by dividing by 3.

Therefore his age was 16.

2. Divide 64 into two such parts that $\frac{1}{6}$ of the one may be equal to $\frac{1}{4}$ of the other.

Suppose $x =$ the one part,

Then $64 - x =$ the other,

and $\frac{x}{4} = \frac{64 - x}{6}$ by the question,

$x = \frac{256 - 4x}{6}$ by mult. by 4,

$6x = 256 - 4x$ by mult. by 6,

$10x = 256$ by transposition;

Therefore $x = 25\frac{3}{5}$ by division,

and $64 - x = 38\frac{2}{5}$.

Hence the two parts required are $38\frac{2}{5}$, and $25\frac{3}{5}$.

3. Divide 60 into two such parts that the greater may be triple the lesser.

Assume $x =$ the lesser,

and $y =$ the greater.

$$\left.\begin{array}{l} \text{First condition } x + y = 60 \\ \text{Second condition } \underline{3x = y} \end{array}\right\} \text{per quest.}$$

From 1st. $x = 60 - y$,

From 2d. $x = \frac{y}{3}$.

But $\frac{y}{3} = 60 - y$ both being $= x$,

and $y = 180 - 3y$ by mult. by 3,

$4y = 180$ by transp.

Th. $y = 46$ by division,

also $x = 60 - y = 15$;

Therefore the two parts required are 45 and 15.

4. Divide 80 into two such parts that $\frac{2}{3}$ of the one will just be equal to $\frac{3}{4}$ of the other.

Assume $x =$ the one part,
and $y =$ the other.

$$\left.\begin{array}{l}\text{First cond. } x + y = 80 \\ \text{Second cond. } \dfrac{2x}{3} = \dfrac{3y}{4}\end{array}\right\} \text{per question.}$$

From 1st. $x = 80 - y$,

From 2d. $2x = \dfrac{9y}{4}$ by mult. by 3,

and $8x = 9y$ by mult. by 4;

Therefore $x = \dfrac{9y}{8}$ by division.

But $\dfrac{9y}{8} = 80 - y$ both $= x$,

$9y = 640 - 8y$ by mult. by 8,
$17y = 640$ by transp.
Therefore $y = 37\frac{11}{17}$ by division,
and $x = 80 - y = 42\frac{6}{17}$.
So that the required parts are $37\frac{11}{17}$, and $42\frac{6}{17}$.

5. A and B had a certain number of crowns, and said A to B, If you give me one of your crowns, I shall then have five times as many as you behind; but said B to A, If you give me one of your crowns, then shall each of us have an equal number: how many crowns had each person?

Suppose $x =$ A's crowns,
And $y =$ B's;

$$\left.\begin{array}{l}\text{Then } x + 1 = 5y - 5 \\ \text{And } x - 1 = y + 1\end{array}\right\} \text{per quest.}$$

From 1st. $x = 5y - 6$,
From 2d. $x = y + 2$.
But $5y - 6 = y + 2$ both being $= x$,
And $5y = y + 8$ by transp.
Also $5y - y = 8$ by subtraction,
or $4y = 8$;
Therefore $y = 2$ by div. by 4,
And $x = y + 2 = 4$.
Therefore A had 4 crowns, and B 2.

6. A gentleman caught a fish whose head was 6 inches long, the tail as long as the head and half the body, the body was just the length of the head and tail. What was the length of the fish. what the length of the body, and what the length of the tail?

Suppose $x =$ length of the body,
And $y =$ length of the tail,
Then $6 + x + y =$ whole length of the fish.

$$\left.\begin{array}{l} \text{Now } y = \frac{x}{2} + 6 \\ \text{And } x = y + 6 \end{array}\right\} \text{by the quest.}$$

From 2d. $y + 6 = x$.
$y = x - 6$ by transp.

But $\frac{x}{2} + 6 = x - 6$ both $= y$.

And $x + 12 = 2x - 12$ by mult. by 2.
Also $2x - x = 24$ by transp.
or $x = 24$ body.

Therefore $\frac{x}{2} + 6 = 18$ tail.

and $x + \frac{x}{2} + 12 = 48$ length of the fish.

7. When first the marriage knot was tied
Betwixt my wife and me,
My age to her's we found agreed
As nine doth unto three.
But after ten and half ten years
We man and wife had been,
Her age came up as near to mine
As eight is to sixteen.
Now tell me, if you can, I pray,
What was our age o' th' marriage day.

Assume $x =$ man's age.
And $y =$ woman's.

$$\left.\begin{array}{l} \text{Then } x : y :: 9 : 3, \text{ or as } 3 : 1 \\ \text{And } x + 15 : y + 15 :: 16 : 8, \text{ or as } 2 : 1 \end{array}\right\} \text{per quest.}$$

From 1st. $x = 3y$,
From 2d. $x + 15 = 2y + 30$.

For x in the second equation, substitute its value found in the first, and we have,

$3y + 15 = 2y + 30$,
and $3y - 2y = 15$ by transp.
or $y = 15$,
and $x = 3y = 45$.

Whence the man's age was 45, and the woman's 15.

8. A, B, and C, bought a ship for 200 guineas. A said, with $\frac{1}{2}$ of B's money he could pay for the ship; B replied, that with $\frac{1}{3}$ of C's money he could pay for the ship; and C rejoined, that with $\frac{1}{4}$ of A's money he could pay the ship: Required the particulars.

Assume x, y, and z, for their respective shares.

$$\left.\begin{array}{l} \text{1st cond. } x + \frac{y}{2} = 200 \\ \text{2d cond. } y + \frac{z}{3} = 200 \\ \text{3d cond. } z + \frac{x}{4} = 200 \end{array}\right\} \text{by the quest.}$$

From 1st. $x = 200 - \frac{y}{2}$.

From 3d. $x = 800 - 4z$.

But $200 - \frac{y}{2} = 800 - 4z$ both $= x$,

or $4z - \frac{y}{2} = 600$ by transp.

Therefore $\frac{y}{2} = 4z - 600$,

and $y = 8z - 1200$ by mult. by 2.

Then from 2d. $y = 200 - \frac{z}{3}$;

Therefore $200 - \frac{z}{3} = 8z - 1200$ both $= y$.

And $600 - z = 24z - 3600$ by mult. by 3.
Also $25z = 4200$ by transp.

Therefore $\quad z = 168 =$ C's money.

$y = 200 - \frac{z}{3} = 144 =$ B's money.

$x = 200 - \frac{y}{2} = 128 =$ A's money.

9. A man, his wife, and son's years make 96, of which the father and son's equal the wife's and 15 years over, and the wife's and son's equal the man's, and 2 years over, what was the age of each?

Suppose x, y, and $z =$ their respective ages.

$$\left.\begin{array}{ll} \text{1st condition} & x + y + z = 96 \\ \text{2d condition} & x + z = y + 15 \\ \text{3d condition} & y + z = x + 2 \end{array}\right\} \text{ per quest.}$$

From 1st. $2y + 15 = 96$ by subt.
and $2y = 81$ by transp.
Therefore $y = 40\frac{1}{2}$ by division.

From 1st. $2x + 2 = 96$ by subt.
$2x = 94$ by transp.
$x = 47$ by division,

And $z = 96 - \overline{x + y} = 8\frac{1}{2}$.

Therefore their ages are 47, $40\frac{1}{2}$, and $8\frac{1}{2}$ respectively.

10. Says a pert little pedant, In all this vast nation
There's none who, like me, can resolve an equation;
I care not who hears, what I say I'll make good,
And he cast a sly leer towards the place where I stood.
To be sure, sir, said I, none can doubt what you say,
Here's a thing I wrote down,* that pos'd me th' other day;
But, no doubt, you'll solve it with ease, and quite clever,
Without any affected equation whatever.
He took it, and tried, but it silenc'd his prating;
Pray say was his skill worth the stir he'd been making?

* $\left\{\begin{array}{l} x^2y + xy^2 = a \\ x^3 + y = b \end{array}\right\}$ to find x and y.

First, add 3 times the 1st equation to the 2d, we have

$x^3 + 3x^2y + 3xy^2 + y^3 = 3a + b,$

Then by extracting the cube root

$x + y = \sqrt[3]{3a + b},$

But $x^2y + xy^2 = (x + y)\,xy = a,$

Therefore by substitution

$xy \times \sqrt[3]{3a + b} = a,$

or $xy = \dfrac{a}{\sqrt[3]{3a+b}}$ by division.

Then putting $x + y = s$, and $xy = p$ we have

$(x + y = s)^2 = x^2 + 2xy + y^2 = s^2.$

$(xy = p)\,4 = \qquad 4xy \qquad = 4p.$

By subtr. $x^2 - 2xy + y^2 = s^2 - 4p.$

By evol. $x - y = \sqrt{s^2 - 4p}.$

and $x + y = s.$

By add. $2x = s + \sqrt{s^2 - 4p},$

or $x = \dfrac{s + \sqrt{s^2 - 4p}}{2}.$

By subtr. $2y = s - \sqrt{s^2 - 4p},$

or $y = \dfrac{s - \sqrt{s^2 - 4p}}{2}.$

The same, by assuming $a = 30$, *and* $b = 35$. The equations become

$\left.\begin{array}{r} x^2y + xy^2 = 30 \\ x^3 + \;\; y^3 = 35 \end{array}\right\}$ to find x and y.

Adding 3 times the 1st equation to the 2d gives

$x^3 + 3x^2y + 3xy^2 + y^3 = 125.$

Extracting the cube root we have $x + y = 5.$

$x^2y + xy^2 = (x + y)\,xy = 30$

By subst. $5xy = 30$

or $xy = \dfrac{30}{5} = 6$

$(x + y = 5)^2 = x^2 + 2xy + y^2 = 25$

$(xy = 6)^4 = \qquad 4xy \qquad = 24$

By subt. $x^2 - 2xy + y^2 = 1$

By evol. $x - y = \sqrt{1} = \pm 1$
and $x + y \quad = 5$

By addition $2x = 6$.
or $x = 3$.
and $x + y = 5$
That is $y = 5 - x = 2$.
Hence $x = 3$, and $y = 2$, the numbers required.

QUESTIONS FOR PRACTICE.

1. Four men A, B, C, and D, built a ship, which cost £.2607, whereof B paid twice as much as A; C paid as much as A, and B and D paid as much as C and B. What did each pay?

Ans. £.237 = A.

2. A charitable lady, relieving four poor persons, gave among them 6s. 8d. To the second she gave twice as much as to the first, to the third thrice, and to the fourth four times as much as to the first. What did she give to each?

Ans. 8d. to the first.

3. A cask, which held 126 gallons, was filled with a mixture of brandy, wine, and cyder; in it there were 13 gallons of wine more than there were of brandy; and as much cyder as of both wine and brandy: What quantity was there of each?

Ans. 25 Brandy, 38 Wine, and 63 Cyder.

4. At a certain election 375 persons voted, and the candidate chosen had a majority of 91: How many voted for each?

Ans. 233 and 142.

5. Two men, who had between them 35 guineas, played together till one of them had won 4 guineas of the other, and then the winner had twice as many guineas as the loser had at first. How many had each?

Ans. 13 and 22.

6. A gentleman being asked the age of his two sons, replied, that if to the sum of their ages 25 be added, the number arising will be double the age of the eldest, but if 8 be taken from the difference of their ages, the remainder will be the youngest's age: How old was each?

Ans. 17 and 42.

7. The paving of a square court, at 2s. a-yard, cost as much as the inclosing it at 5s. a-yard: What was the side of the square?

Ans. 10.

8. Upon measuring the corn produced by a field (being 8 quarters), it appeared that it had yielded but $\frac{1}{3}$ part more than was sown: How much was that?

Ans. 6 quarters.

9. From each of 16 pieces of gold, an artist filed the worth of half a crown, and then offered them in payment for their original value; but being detected, and the pieces weighed, they were found to be worth no more than 8 guineas: Their original worth is required.

Ans. 13s.

10. One being asked how old he was, answered, that the product of $\frac{1}{20}$ of the years he had lived, being multiplied by $\frac{5}{8}$ of the same, would be his age: What was his age?

Ans. 32.

11. One being asked the hour of the day, replied, that the time then passed from noon, was equal to $\frac{29}{43}$ of the time remaining until midnight: What was the time?

Ans. 4 h. 50 m.

12. A son asking his father how old he was, the father replied, my age 7 years ago was just 4 times as great as your age at that time; but 7 years hence, if you and I live, my age will be only double of yours: What is the age of each person?

Ans. 35 and 14.

13. A company of 18 persons, men and women, sharing a reckoning of £.9. 18s., paid each as many shillings as there were men in company: How many were there?

Ans. 11 men, and 7 women.

14. Bought 8 yards of cloth for 62s.; for part of it I gave 9s. a-yard, and for the rest 7s. a-yard: How much was bought of each?

Ans. 3 at 9s. and 5 at 7s.

15. A person paid a bill of £.50 with half guineas and crowns, using 101 pieces in all: How many of each sort did he pay?

Ans. 90 and 11.

16. Having laid out 37s. in brandy at 2s. and rum at 3s. a-pint, I find that I could have bought as many pints of rum as I now have of brandy, and as many of brandy as now of rum, for 4s. less: How much was bought?

Ans. 5 and 9.

17. There is a cask containing 120 gallons, filled with brandy, white wine, cyder, and water; the brandy and wine (taken together) make $\frac{1}{2}$ the content of the cask; the brandy and cyder make $\frac{2}{3}$ of the content; and the brandy and water $\frac{3}{4}$ thereof. The quantities of each are required?

Ans. 55, 5, 25, and 35.

18. A hare, 50 of her leaps before a greyhound, takes 4 leaps to the greyhound's 3; but 2 of the greyhound's leaps are as much as 3 of the hare's: How many leaps must the greyhound take to catch the hare?

Ans. 300.

19. It is required to find two numbers, the greater whereof shall be to the lesser as their sum is to 54, and as their difference is to 9.

Ans. $40\frac{1}{2}$ and 27.

20. There are three numbers whose differences are equal, (that is, the second exceeds the first as much as the third exceeds the second), and the first is to the

third as 5 to 7 ; also the sum of the 3 numbers 324. What are those numbers?

Ans. 90, 108, and 126.

21. A and B severally cut the cards, so as to cut off less than they left; now what A left, added to what B cut off, make 50, also the cards left by both exceed those cut off by 64. What number did each cut off?

Ans. 11 and 9.

22. Three persons A, B, and C, being at play, A won $\frac{1}{2}$ the money that B and C had, and carried off £.153 ; now if B had won $\frac{1}{3}$ of what A and C had, or if C had won $\frac{1}{4}$ of what A and B had, they would have carried off the same sum. How much had each?

Ans. 45, 99, and 117.

QUADRATIC EQUATIONS.

Quadratic equations are either pure or adfected.

A pure quadratic equation is that which involves the square of the unknown quantity only. As $ax^2 = b$, or $ax^2 - b = c$.

The solution of such quadratics has been already given under simple equations.

An adfected quadratic equation is that which contains the square of the unknown quantity in one term, and the first power in the other. As $ax^2 + bx = c$, or $x^2 - 3x = 10$.

All quadratic equations may be solved by the following

Rule.—1. *Transpose all the terms which contain the unknown quantity to one side of the equation, and the known terms to the other; and so that the term containing the square of the unknown quantity may be positive.*

2. *Divide all the terms of the equation by the coefficient of the square of the unknown quantity,* if it is multiplied by any, *so that the coefficient of the square of the unknown quantity may be* 1.

3. *Add the square of half the coefficient of the unknown quantity in the second term to both sides of the equation, and that side which contains the unknown quantity will be a complete square.*

4. *Extract the square root of both sides of the equation, making that of the known side both + and —, which will give two roots of the equation, or two values of the unknown quantity.*

5. *Transpose the known part from the first side, and the value of the unknown quantity will be found.*

REMARK.—The reason of this rule is manifest from the composition of the square of a binomial, for it consists of the squares of the two parts, with twice their product.

The different forms of quadratic equations, expressed in general terms, being reduced by the first and second parts of the rule, are these:

1st. $x^2 + ax = b.$
2d. $x^2 - ax = b.$
3d. $x^2 - ax = -b.$

And the roots, or values of the unknown quantity, in each of these forms, will be expressed by the following formula:

$$\text{1st. } x = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b}.$$

$$\text{2d. } x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b}.$$

$$\text{3d. } x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}.$$

Every quadratic equation will have two roots, except such as are of the third form, whose roots become impossible. For the square root of an affirmative quantity may be either + or —. Therefore every quadratic equation admits of two solutions. Thus, the square root of a^2 is either $+a$ or $-a$; for $+a \times + a = a^2$, and $-a \times -a = a^2$. And in extracting the square root of the known side of the equation, both the affirmative and negative roots must be taken into the account, and the square root marked $\pm$. But the square root of a negative quantity is impossible or imaginary, as neither $+a$ into $+a$, gives $-a^2$, nor $-a$ into $-a$, gives $-a^2$. In the two first forms, one of the roots must be affirmative and the other negative.

In the third form, if $\frac{a^2}{4}$, or the square of half the coefficient of the unknown quantity in the second term be greater than b, the known quantity, the two roots will be affirmative. And if $\frac{a^2}{4}$ be equal to b, the two roots become equal; but if $\frac{a^2}{4}$ is less than b, (so in that case the known side of the equation becomes a negative quantity), both roots will be impossible, and the problem incapable of an answer.

This will fully appear from the following examples:

1. Let it be required to divide the number 20 into two such parts, that their product shall be 96.

Let $x =$ the one part,
and $20 - x =$ the other.
Then $\overline{20 - x} \times x = 96$ by the quest.
and $20x - x^2 = 96$ by multiplying.
$x^2 - 20x = -96$ by transp.
$x^2 - 20x + 100 = 100 - 96 = 4$ by completing the square.
Then $x - 10 = \pm\sqrt{4} = \pm 2$, by extracting the square root.
and $x = 10 \pm 2 = 12$, or 8 by transposition.

Consequently 12 and 8 are the two roots or numbers required, being both affirmative. As $\frac{a^2}{4}$ is greater than b, or their equals $(10)^2 = 100$, is greater than -96 by 4, so that the known side of the equation is affirmative, and the two roots are so in like manner, being composed of two affirmative terms.

2. Let it be required to divide the same number 20, into two such parts, so that their product shall be 100.

Let $x =$ the one part,

and $20 - x =$ the other.

Then $\overline{20 - x} \times x = 100$, per quest.

and $20x - x^2 = 100$ by mult.

$x^2 - 20x = -100$ by transp.

$x^2 - 20x + (10)^2 = 100 - 100 = 0$, by comp. square.

and $x - 10 = \pm \sqrt{0}$ by evolution.

Therefore $x = 10$ by transp.

Here we find that $\frac{a^2}{4}$ is equal to b, so that the two roots are equal to each other, both of them being equal to 10.

3. In this case, let their product be 104.

Here, by proceeding in the same manner as in the two foregoing examples, we come to the following equation.

That $x^2 - 20x + 100 = -104 + 100$, by comp. square.

and $x - 10 = \pm \sqrt{-4}$, by evolution.

Whence $x = 10 \pm \sqrt{-4}$, by transp.

or $x = 10 + \sqrt{-4}$, or $x = 10 - \sqrt{-4}$.

Here $\frac{a^2}{4}$ is less than b, or their equals $(10)^2 = 100$, is less than -104 by -4. But -4 being an imaginary quantity, has no square root: Hence we see that both the values of x are impossible; for the question itself is impracticable, it being very well known that the greatest product which can arise from the multiplication of the two parts into which any number may be divided is, when those parts are equal to each other, as will appear from what has been already said.

We shall now give a few examples illustrative of the rule.

1. Given $x^2+8x=33$ to find x.

Then $x^2+8x+16=33+16=49$ by comp. square.

and $x+4=\sqrt{49}=\pm 7$ by evolution.

Therefore $x=-4\pm 7=3$ or -11 by transp.

$x=3$ or -11.

2. Given $x^2+x=42$ to find x.

Then $x^2+x+\frac{1}{4}=42\frac{1}{4}$ by comp. square.

and $x+\frac{1}{2}=\pm\sqrt{42\frac{1}{4}}=6.5$ by evol.

or $x=6.5\pm\frac{1}{2}=6$ or -7 by transp.

3. Given $x+y=a$, and $xy=b$, to find x and y.

$$\left.\begin{array}{l}\text{1st. } x+y=a\\ \text{2d. } xy=b\end{array}\right\}\begin{array}{l}\text{or } x=a-y\\ \text{or } x=\frac{b}{y}\end{array}\Bigg\}$$

But $a-y=\frac{b}{y}$ both being $=x$,

and $ay-y^2=b$ by mult. by y,

$y^2-ay=-b$ by transp.

$y^2-ay+\frac{a^2}{4}=-b+\frac{a^2}{4}$ by completing the square.

and $y-\frac{a}{2}=\pm\sqrt{-b+\frac{a^2}{4}}$ by evol.

Therefore $y=\frac{a}{2}\pm\sqrt{-b+\frac{a^2}{4}}$ by transp.

and $x=(a-y)=\frac{a}{2}\mp\sqrt{-b+\frac{a^2}{4}}$.

4. Given $x+y=a$, and $x^2+y^2=b$, to find x and y.

$$\left.\begin{array}{l}\text{1st. } x+y=a\\ \text{2d. } x^2+y^2=b\end{array}\right\}\begin{array}{l}\text{or } x=a-y\\ \text{or } x^2=b-y^2\end{array}\Big\}$$

$x^2=a^2-2ay+y2$ by squaring 1st.

But $a^2-2ay+y^2=b-y^2$ both $=x$.

and $2y^2-2ay=b-a^2$ by transp.

$y^2 - ay = \frac{b - a^2}{2}$ by div. by 2.

$y^2 - ay + \frac{a^2}{4} = \frac{b - a^2}{2} + \frac{a^2}{4} = \frac{2b - a^2}{4}$ by completing the square.

$y - \frac{a}{2} = \pm\sqrt{\frac{2b - a^2}{4}}$ by evolution.

Therefore $y = \frac{a}{2} \pm \sqrt{\frac{2b - a^2}{4}}$ by transp.

and $x = (a - y) = \frac{a}{2} \pm \sqrt{\frac{2b - a^2}{4}}$.

or $y = \frac{a \pm \sqrt{2b - a^2}}{2}$;

and $x = \frac{a \pm \sqrt{2b - a^2}}{2}$.

5. Given $x - y = 2$, and $xy = 80$ to find x and y.

$\left.\begin{array}{l}\text{1st. } x - y = 2 \\ \text{2d. } xy = 80\end{array}\right\}$

From 1st. $x = y + 2$ by transp.

Then $y^2 + 2y = 80$ by subt.

$y^2 + 2y + 1 = 81$ by com. sq.

$y + 1 = \pm\sqrt{81} = 9$ by evol.

and $y = -1 + 9 = 8$ by transp.

also $x = y + 2 = 10$.

EXAMPLES FOR PRACTICE.

1. Given $x^2 + 4x = 32$ to find x.
   Ans. $x = 4$ or -8.

2. Given $x^2 - 6x + 8 = 80$ to find x.
   Ans. 12 or -6.

3. Given $4x^2 + 12x + 8 = 80$ to find x.
   Ans. $x = 3$ or -6.

4. Given $\frac{2x^2}{3} + \frac{3x}{4} = \frac{4}{5}$ to find x.
   Ans. $x = \cdot6689$ or -1.793.

5. Given $x^2+4x=a^2+2$ to find x.

Ans. $x=\sqrt{a^2+6}-2$.

6. Given $2x^2+15=3x$ to find x.

Ans. $x=\dfrac{3\pm\sqrt{-111.}}{4}$

7. Given $x^2+xy=40$, and $2xy-3y^2=3$, to find x and y.

Ans. $x=\pm5$ or $\pm\dfrac{24}{\sqrt{15}}$.

$y=\pm3$ or $\pm\dfrac{1}{\sqrt{15}}$.

All equations whatever, in which there are only two different dimensions of the unknown quantity, if the index of the one be double that of the other, may be solved like quadratics by the following

Rule.—*Complete the square, as directed by the foregoing rule, extract the square root, transpose the quantities which are on the same side with the unknown quantity, and then extract that root from both sides of the equation, which the index of the unknown quantity denotes.*

EXAMPLES.

1. Given $x^4+2x^2=24$ to find x.
Then $x^4+2x^2+1=24+1=25$ by comp. square.
and $x^2+1=\pm\sqrt{25}=5$ by evol.
Also $x^2=\pm5-1=4$ or -6 by transposition.
Th. $x=\pm\sqrt{4}=2$ or $=\pm\sqrt{-6}$ by evolution.

2. Given $x^6-4x^3=32$ to find x.
Then $x^6-4x^3+4=36$ by comp. square.
and $x^3-2=\sqrt{36}=6$ by ext. square root.
also $x^3=6+2=8$ by transp.
Therefore $x=\sqrt[3]{8}=2$. by evolution.

3. Given $x^4 - 6x^2 = 27$ to find x.
Then $x^4 - 6x^2 + 9 = 36$ by comp. square,
and $x^2 - 3 = \pm\sqrt{36} = \pm 6$ by evol.
Also $x^2 = \pm 6 + 3 = 9$ or -3 by transp.
Th. $x = \pm\sqrt{9} = \pm 3$ or $\pm\sqrt{-3}$ by evol.
That is, $x = 3$ or $\pm\sqrt{-3}$.

Equations of the above form may be reduced to a quadratic equation, by substituting a letter for the lower power of the unknown quantity. Thus, in the last example above,

Given $x^4 - 6x^2 = 27$ to find x.
Let $x^2 = y$, and $x^4 = y^2$, and the equation becomse
$y^2 - 6y = 27$,
Then $y^2 - 6y + 9 = 36$ by comp. square.
and $y - 3 = \pm\sqrt{36} = \pm 6$ by evol.
Also $y = \pm 6 + 3 = 9$ or -3 by transp.
But $x^2 = y$.
Therefore $x = \pm\sqrt{y} = \pm\sqrt{9} = 3$ or $\pm\sqrt{-3}$.

EXAMPLES FOR PRACTICE.

1. Given $x^4 + 4x^2 = 12$ to find x.
Ans. $x = \pm\sqrt{2}$ or $\pm\sqrt{-6}$.

2. Given $x^6 - 8x^3 = 513$ to find x.
Ans. $x = 3$ or $\sqrt[3]{-19}$.

3. Given $x^6 - 4x^3 + 4 = 1$ to find x.
Ans. $x = 1$.

4. Given $2x^4 - x^2 = 496$ to find x.
Ans. $x = \pm 4$, or $\sqrt{\frac{-31}{2}}$.

5. Given $x^{2n} - 4x^n = 10$ to find x.
Ans. $x = \sqrt[n]{2 \pm \sqrt{14}}$.

6. Given $x^{2n} - x^n = a$ to find x.
Ans. $x = \sqrt[n]{\tfrac{1}{2} \pm \sqrt{a + \tfrac{1}{4}}}$.

7. Given $x^6 - x^3 = a$ to find x.

$$\text{Ans. } x = \sqrt[3]{\frac{1 \pm \sqrt{4a+1}}{2}}.$$

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. The difference of two numbers is 4, and their product is 96; what are the numbers?

Assume $x =$ the greater,
and $x - 4$ the less.
Then $x \times \overline{x - 4} = 96$ per quest.
or $x^2 - 4x = 96$ by mult. by x.
Then $x^2 - 4x + 4 = 100$ by comp. square,
and $x - 2 = \pm \sqrt{100} = \pm 10$ by evol.
Also $x = 10 + 2 = 12$ by transp.
Therefore $x - 4 = 8$.
Whence the numbers required are 12 and 8.

2. The difference of two numbers is 6, and the sum of their square 50; what are the numbers?

Assume $x =$ the greater,
and $x - 6 =$ the lesser.
First $x^2 =$ the square of the greater and $(x-6)^2 = x^2 - 12x + 36 =$ the square of the lesser.
Then $x^2 + x^2 - 12x + 36 = 50$ per quest.
or $2x^2 - 12x + 36 = 50$,
and $2x^2 - 12x = 50 - 36 = 14$ by transp.
also $x^2 - 6x = 7$ by dividing by 2,
$x^2 - 6x + 9 = 7 + 9 = 16$ by comp. square,
$x - 3 = \pm \sqrt{16} = 4$ by evol.
$x = \pm 4 + 3 = 7$ by transp.
Whence $x - 6 = 1$.

3. What number is that which being tripled, and 8 added to the triple, the sum shall be to the square of the number sought as 5 to 4?

Suppose $x =$ the number required.
Then $3x =$ the number tripled,
and $3x + 8 : x^2 :: 5 : 4$ per question.

$12x + 32 = 5x^2$ by mult. ext. and means,
$5x^2 - 12x = 32$ by transp.
$x^2 - 2{\cdot}4x = 6{\cdot}4$ by dividing by 5,
$x^2 - 2{\cdot}4x + 1{\cdot}44 = 6{\cdot}4 + 1{\cdot}44 = 7{\cdot}84$ by comp. square.
$x - 1{\cdot}2 = \pm\sqrt{7{\cdot}84} = \pm 2{\cdot}8$ by evol.
$x = \pm 2{\cdot}8 + 1{\cdot}2 = 4$ by transp.
That is, $x = 4$ the number required.

4. What number is that whose product by 12 exceeds its square by 32?

Assume $x =$ the number.
Then $x^2 = 12x - 32$ by the quest.
$x^2 - 12x = -32$ by transp.
$x^2 - 12x + 6^2 = 36 - 32$ by comp. square,
and $x - 6 = \pm\sqrt{4} = \pm 2$ by evol.
$x = \pm 2 + 6 = 8$ or 4 by transp.

5. What two numbers are those, whose difference is 6, and the difference of their cubes 2232?

Assume $x =$ the greater,
and $y =$ the lesser.

$$\left.\begin{array}{l}\text{1st condition } x - y = 6 \\ \text{2d condition } x^3 - y^3 = 2232\end{array}\right\}$$

From 1st. $x = 6 + y$,
From 2d. $x^3 = 2232 + y^3$,
$x^3 = y^3 + 18y^2 + 108y + 216$ by cubing the third.
But $y^3 + 2232 = y^3 + 18y^2 + 108y + 216$ both being $= x^3$,
$18y^2 + 108y = 2016$ by transp.
$y^2 + 6y = 112$ by div. by 18,
$y^2 + 6y + 3^2 = 121$ by comp. square,
$y + 3 = \pm\sqrt{121} = 11$ by evol.
$y = 11 - 3 = 8$ by transp.
$x = y + 6 = 14$.
Therefore the two numbers required are 14 and 8.

6. It is required to divide the number 14 into two such parts that their product may be 48.

Suppose x and y to denote the number required.

$$\left.\begin{array}{l} \text{1st condition } x+y=14 \\ \text{2d condition } \overline{xy=48} \end{array}\right\} \text{ per quest.}$$

From 1st. $x=14-y$ by transp.

$14y-y^2=48$ by subt.

$y^2-14y=-48$ by transp.

$y^2-14y+49=49-48=1$ by comp. sq.

$y-7=\pm\sqrt{1}=1$ by evol.

$y=1+7=8$,

$x=14-y=6$.

7. To find a number, from the cube of which, if 19 be subtracted, and the remainder multiplied by that cube, the product shall be 216.

Assume $x=$ the number required.

Then $\overline{x^3-19}\times x^3=216$ by quest.

or $x^6-19x^3=216$.

Now put $x^3=y$, then $x^6=y^2$, and we will have the following equation:

$y^2-19y=216$,

$y^2-19y+\left(\frac{19}{2}\right)^2=216+\frac{361}{4}=\frac{1225}{4}$ by comp. square.

$$y-\frac{19}{2}=\pm\sqrt{\frac{1225}{4}}=\pm\frac{35}{2} \text{ by evol.}$$

Th. $y=\frac{19\pm35}{2}=27$ or -8 by transp.

But $x=\sqrt[3]{y}=\sqrt[3]{27}=3$ or $\sqrt[3]{-8}=-2$.

Wherefore $x=3$ or -2.

8. In a rectangular pavement consisting of 1000 square stones, there are 15 more in the length than in the breadth. How many stones are there in the length and breadth?

Suppose $x=$ the length,

And $x-15=$ the breadth.

Then $x\times\overline{x-15}=1000$ per quest.

or $x^2-15x=1000$,

$$x^2 - 15x + (7\cdot5)^2 = 1000 + (7\cdot5)^2 = 1056\cdot25$$

by comp. square,

$x - 7\cdot5 = \pm\sqrt{1056\cdot25} = \pm 32\cdot5$ by evolution.

Th. $x = \pm 32\cdot5 + 7\cdot5 = 40$,
and $x - 15 = 25$.

Hence 40 the number in length, and 25 the number in breadth.

9. One bought a horse and sold him again for £.56, and gained as much per cent as the horse cost him; required the price of the horse?

Assume $x =$ the price,
And $56 - x =$ the gain.

Then $x : 56 - x :: 100 : \dfrac{5600 - 100x}{x} =$ the gain per cent,

and $x = \dfrac{5600 - 100x}{x}$ per quest.

Also $x^2 = 5600 - 100x$ by mult.
$x^2 + 100x = 5600$ by transp.
$x^2 + 100x + (50)^2 = 8100$ by comp. square,
$x + 50 = \pm\sqrt{8100} = 90$ by evol.
Th. $x = 90 - 50 =$ £.40 the sum required.

10. A company at a tavern had £.7. 4s. to pay; but before the reckoning was settled two of them sneaked off, which obliged the rest to pay a shilling a-piece more than they should have done. How many persons were there?

Assume $x =$ the number of persons.
Then $144 =$ £.7. 4s. reduced to shillings,

Also $\dfrac{144}{x} =$ what each should have paid,

And $\dfrac{144}{x-2} =$ what each did pay;

But $\dfrac{144}{x-2} = \dfrac{144}{x} + 1$ per quest.

This equation, cleared of fractions, and the known quantities brought to one side, Gives $x^2 - 2x = 288$,

Then $x^2 - 2x + 1 = 289$ by comp. square,

$x - 1 = \pm \sqrt{289} = \pm 17$ by evol.

Th. $x = \pm 17 + 1 = 18$ or -16.

That the number of persons was 18 is evident, for then they pay 8 shillings a-piece. But when their number was reduced to 16, they must pay 9 shillings a-piece, or 1 shilling more than they should have done.

11. A sets out from London to Darlington, and travels 8 miles per day, and after he has travelled 5 days, B sets out from Darlington for London, and travels $\frac{1}{22}$ part of the distance per day, and after travelling as many days as he travels miles per day he meets A: Required the distance between the places?

Suppose x days $=$ the time of their meeting,

And y miles $=$ the distance.

Now $8x + 40 =$ the miles travelled by A,

And $\frac{y}{22} =$ miles travelled per day by B.

But $\frac{y}{22} = x$ per question.

Then $x^2 =$ the miles travelled by B,

also $y = 22x =$ the distance.

Now $x^2 + 8x + 40 = 22x$,

$x^2 - 14x = -40$ by transp.

$x^2 - 14x + 7^2 = 49 - 40 = 9$ by comp. square.

$x - 7 = \pm \sqrt{9} = \pm 3$ by evol.

and $x = 7 \pm 3 = 10$ or 4 by trans.

also $y = 22x = 220$ or 88.

12. If to Aminta's age exact
Its square you add, and 18 more;
And from her age one third † subtract,
And to the difference add three score;
The latter sum to the former then
Will just the same proportion bear
As 18 does to nine times ten:
Aminta's age I pray declare.

† Of her age.

Assume $x =$ Aminta's age.

Then will $x + x^2 + 18 =$ the first sum,

And $\left(x - \frac{x}{3} + 60 =\right) \frac{2x}{3} + 60 =$ 2d sum;

Then $x + x^2 + 18 : \frac{2x}{3} + 60 :: 90 : 18$ by the question,

or $x + x^2 + 18 : \frac{2x}{3} + 60 :: 5 : 1$,

$x + x^2 + 18 = 3\frac{1}{3}x + 300$ by multiplying extremes and means together,

$x^2 - 2\frac{1}{3}x = 282$ by transposition,

or $x^2 - \frac{7x}{3} = 282$,

$x^2 - \frac{7x}{3} + \frac{49}{36} = 282 + \frac{49}{36}$ by comp. the square,

or $x^2 - \frac{7x}{3} + \frac{49}{36} = \frac{10201}{36}$,

$x - \frac{7}{6} = \sqrt{\frac{10201}{36}} = \frac{101}{6}$ by evol.

Therefore $x = \frac{101}{6} + \frac{7}{6} = \frac{108}{6} = 18$ by transposition;

That is, $x = 18$ the age required.

13. The sum (s) and product (p) of any two numbers being given, to find the sum of their squares, cubes, biquadrates, &c.

Let the two numbers be x and y.

$$\left.\begin{array}{r} \text{Then } x + y = s \\ \text{and } xy = p \end{array}\right\} \text{by the question.}$$

The first equa. squared gives

$$x^2 + 2xy + y^2 = s^2.$$

Whence $x^2 + y^2 = s^2 - 2xy$ by transp.

But $2xy = 2p$.

Therefore $x^2 + y^2 = s^2 - 2p =$ the sum of the squares.

Now, multiply the last equation by the first, which gives $(x^2 + y^2)(x + y) = (s^2 - 2p)\, s$,

or $x^3 + xy(x + y) + y^3 = s^3 - 2sp$.

But $xy(x+y)=sp$.

Which substituted in the last equation,

gives $x^3+sp+y^3=s^3-2sp$,

or $x^3+y^3=s^3-3sp=$ the sum of the cubes.

Again, let this last equation be multiplied by the first;

Then $(x^3+y^3)(x+y)=(s^3-3sp)s$,

or $x^4+xy(x^2+y^2)+y^4=s^4-3s^2p$.

But $xy(x^2+y^2)=p(s^2-2p)$.

Therefore by substitution

We have $x^4+p(s^2-2p)+y^4=s^4-3s^2p$,

or $x^4+y^4=s^4-3s^2p-p(s^2-2p)=s^4-4s^2p+2p^2=$ the sum of the fourth powers;

And the sum of the fifth powers will be

$$(s^4-4s^2p+2p^2)s-(s^3-3sp)p.$$

That is, the sum of the next superior powers is constantly obtained by multiplying the sum of the powers last found by s, and subtracting from that product the sum of the next preceding ones multiplied by p.

And the sum of the nth powers, expressed in a general manner will be

$$s^n-ns^{n-2}p+n\times\frac{n-3}{2}\times s^{n-4}p^2-n\times\frac{n-4}{2}\times\frac{n-5}{3}\times s^{n-6}p^3+n\times\frac{n-5}{2}\times\frac{n-6}{3}\times\frac{n-7}{4}\times s^{n-8}p^4,$$
&c.

When it is evident the series, or expression for the sum of the powers, will terminate, when the last index of s becomes $=0$.

14. What two numbers are those, whose sum, product, and difference of their squares, are all equal to each other?

Suppose $x=$ the greater number, and $y=$ the less.

Then $\left\{\begin{array}{l}x+y=xy\\x+y=x^2-y^2\end{array}\right\}$ by the question.

The second equation divided by $x+y$, gives $1=x-y$.

Therefore $y+1=x$ by transposition.

This value substituted in the first equation

gives $2y+1=y^2+y$,

or $1=y^2-y$ by transposition.

Then $y^2 - y + \frac{1}{4} = 1\frac{1}{4}$ by comp. square,
and $y - \frac{1}{2} = \frac{1}{2}\sqrt{5}$ by evolution.
Therefore $y = \frac{1}{2}\sqrt{5} + \frac{1}{2}$ by transposition,
and $x = y + 1 = \frac{1}{2}\sqrt{5} + \frac{3}{2}$.

And if these expressions be turned into numbers, we shall have $x = 2{\cdot}6180$ and $y = 1{\cdot}6180$.

15. To find two numbers such, that their sum, product, and sum of their squares shall, if possible, be equal to each other.

Suppose x and y the two numbers.

$$\left.\begin{array}{l}\text{Then } x + y = xy \\ \text{and } x^2 + y^2 = xy\end{array}\right\} \text{by the question.}$$

Now, if $2xy$ be added to each side of this last equation,

We have $x^2 + 2xy + y^2 = 3xy$,
and $x + y = \sqrt{3xy}$ by evolution;
But $x + y = xy$.
Therefore by equality we have
$\sqrt{3xy} = xy$.
Whence $3xy = x^2y^2$.
Then $3 = xy$ by dividing by xy;
But $x + y = xy$.
Therefore $x + y = 3$,
And $x + y = x^2 + y^2$.
Whence $x^2 + y^2 = 3$.
Now $x + y$ being $= 3$,
We have $x = 3 - y$ by transposition.
Therefore $(3 - y)^2 + y^2 = 3$.

From this equation $y - 1\frac{1}{2} = \sqrt{-\frac{3}{4}}$, which being an impossible quantity, it follows that no two numbers can be found to answer the condition of the question.

It will appear from these and the foregoing examples, that quadratic equations involving two unknown quantities, may (in most cases) be reduced to an equation involving only one unknown quantity, by the rules given for simple equations; but as no general rule can be given, we shall here give a few problems related to each other, which afford a great variety, and will be found to be of great service in the solution of most questions.

Let x and y be any two numbers, whereof x is the greater, and y the less.

$$\text{Let}\left\{\begin{array}{r} x+y=s \\ x-y=d \\ xy=p \\ \dfrac{x}{y}=q \\ x^2+y^2=a \\ x^2-y^2=b \end{array}\right\}\text{Their}\left\{\begin{array}{l} \text{Sum.} \\ \text{Difference.} \\ \text{Product.} \\ \text{Quotient.} \\ \text{Sum of their squares.} \\ \text{Difference of their squares.} \end{array}\right.$$

Any two of these, viz. (s, d, p, q, a, b,) being given, to find the rest, which admit of 15 variations or questions, and are those which *Mr Ward* has so excellently answered, in all their particulars, in his *Mathematician's Guide*; but we shall, after the first, only find the values of x and y, leaving the rest for the *learner's practice.*

1. Given $\left\{\begin{array}{l} x+y=s \\ x-y=d \end{array}\right\}$ to find the rest.

$2x=s+d$ by addition,

Therefore $x=\dfrac{s+d}{2}$,

$2y=s-d$ by subt.

and $y=\dfrac{s-d}{2}$.

Now, as x *and* y *are found, the rest may very easily be found as follows:*

$$xy=\frac{s^2-d^2}{4}=p,$$

$$\text{and }\frac{x}{y}=\frac{s+d}{s-d}=q.$$

Also $x^2+y^2=\dfrac{s^2+d^2}{2}=a$.

Likewise $x^2-y^2=sd=b$.

2. Given $\left\{\begin{array}{r} x+y=s \\ xy=p \end{array}\right\}$ to find the rest.

$$x^2 + 2xy + y^2 = s^2 \text{ by squaring,}$$
$$4xy = 4p \text{ by mult. by 4,}$$
$$x^2 - 2xy + y^2 = s^2 - 4p \text{ subt.}$$
$$x - y = \sqrt{s^2 - 4p} \text{ by extract.}$$
$$x + y = s \text{ 1st equation,}$$
$$2x = s + \sqrt{s^2 - 4p} \text{ by addition,}$$
$$x = \frac{s + \sqrt{s^2 - 4p}}{2},$$
$$2y = s - \sqrt{s^2 - 4p} \text{ subt.}$$
$$y = \frac{s - \sqrt{s^2 - 4p}}{2}.$$

The rest may be very easily found by proceeding as in the last example.

3. Given $\left\{\begin{array}{l} x + y = s \\ \dfrac{x}{y} \quad = q \end{array}\right\}$ to find the rest.

$$x = qy \text{ mult. by } y,$$
$$y = s - qy \text{ subt.}$$
$$qy + y = s \text{ transp.}$$
$$y = \frac{s}{q+1},$$
$$x + y = s \text{ 1st equation,}$$
$$x = s - \frac{s}{q+1} \text{ by subt.}$$

4. Given $\left\{\begin{array}{l} x + y = s \\ x^2 + y^2 = a \end{array}\right\}$ to find the rest.

$$x^2 + 2xy + y^2 = s^2 \text{ by squaring,}$$
$$2xy = s^2 - a \text{ by subt.}$$
$$x^2 - 2xy + y^2 = 2a - s^2 = 2d - 4\text{th},$$
$$x - y = \sqrt{2a - s^2} = d \text{ by evol.}$$
$$x + y = s \text{ 1st equation,}$$
$$2x = s + \sqrt{2a - s^2} \text{ by addition,}$$
$$x = \frac{s + \sqrt{2a - s^2}}{2},$$
$$2y = s - \sqrt{2a - s^2} \text{ by subt.}$$
$$y = \frac{s - \sqrt{2a - s^2}}{2}.$$

5. Given $\left\{\begin{array}{l} x+y=s \\ x^2-y^2=b \end{array}\right\}$ to find the rest.

$x-y=\frac{b}{s}=d$ 2d divid. by 1st.

$x+y=s$ 1st equation,

$2x=s+\frac{b}{s}=\frac{s^2+b}{s}$ by add.

$x=\frac{s^2+b}{2s}$,

$x+y=s$ 1st equation,

$x-y=\frac{b}{s}$ 3d equation,

$2y=s-\frac{b}{s}=\frac{s^2-b}{s}$ by subt.

$y=\frac{s^2-b}{2s}$.

6. Given $\left\{\begin{array}{r} x-y=d \\ xy=p \end{array}\right\}$ to find the rest.

$x^2-2xy+y^2=d^2$ 1st squared,

$4xy \quad =4p$ 2d mult. by 4,

$x^2+2xy+y^2=d^2+4p$ by add.

$x+y=\sqrt{d^2+4p}=s$ by extr.

$x-y=d$ 1st equation,

$2x=d+\sqrt{d^2+4p}$ by add.

$x=\frac{d+\sqrt{d^2+4p}}{2}$,

$2y=\sqrt{d^2+4p}-d$ by subt.

$y=\frac{\sqrt{d^2+4p}-d}{2}$.

7. Given $\left\{\begin{array}{r} x-y=d \\ \frac{x}{y}=q \end{array}\right\}$ to find the rest.

$x=qy$ 2d mult. by y,

$x=d+y$ 1st transp.

$qy=d+y$ both $=x$,

$y=\frac{d}{q-1}$ for $\overline{q-1}\times y=qy-y$.

And, by adding this last equation to the first, we have

$$x = d + \frac{d}{q-1} = \frac{qd}{q-1}.$$

8. Given $\left\{\begin{array}{l} x - y = d \\ x^2 + y^2 = a \end{array}\right\}$ to find the rest.

$x^2 - 2xy + y^2 = d^2$ by squaring,
$2x^2 \qquad + 2y^2 = 2a$ by mult. by 2,
$x^2 + 2xy + y = 2a - d^2$ by subt.
$x + y = \sqrt{2a - d^2}$ by extracting,
$x - y = d$ 1st equation,
$2x = d + \sqrt{2a - d^2}$ by add.

$$x = \frac{d + \sqrt{2a - d^2}}{2},$$

$2y = \sqrt{2a - d^2} - d$ by subt.

$$y = \frac{\sqrt{2a - d^2} - d}{2}.$$

9. Given $\left\{\begin{array}{l} x - y = d \\ x^2 - y^2 = b \end{array}\right\}$ to find the rest.

$x + y = \dfrac{b}{d}$ by dividing 2d by 1st.

$2x = d + \dfrac{b}{d} = \dfrac{d^2 + b}{d}$ by addition.

Therefore $x = \dfrac{d^2 + b}{2d}$,

$x + y = \dfrac{b}{d}$ 3d equation.

Hence $y = \dfrac{b - d^2}{2d}$ by subt.

10. Given $\left\{\begin{array}{l} xy = p \\ \dfrac{x}{y} = q \end{array}\right\}$ to find the rest.

$x^2 = qp$ by mult. 1st by 2d.

For $\dfrac{xy}{1} \times \dfrac{x}{y} = \dfrac{x^2 y}{y} = x^2$,

$x = \sqrt{qp}$ by extracting,

$y^2 = \frac{p}{q}$ dividing 1st by 2d.

For $\left.\frac{x}{y}\right) \frac{xy}{1} \left(\frac{xy^2}{x} = y^2.\right.$

Hence $y = \sqrt{\frac{p}{q}}$ by extracting.

11. Given $\left\{\begin{array}{l} xy = p \\ x^2 + y^2 = a \end{array}\right\}$ to find the rest.

$2xy = 2p$ by mult. by 2,

$x^2 + 2xy + y^2 = a + 2p$ by addition,

$x + y = \sqrt{a + 2p}$,

$x^2 - 2xy + y^2 = a - 2p$ by subt.

$x - y = \sqrt{a - 2p}$ by extracting,

$2x = \sqrt{a + 2p} + \sqrt{a - 2p}$ by addition,

$$x = \frac{\sqrt{a + 2p} + \sqrt{a - 2p}}{2}$$

$2y = \sqrt{a + 2p} - \sqrt{a - 2p}$ by subt.

$$y = \frac{\sqrt{a + 2p} - \sqrt{a - 2p}}{2}.$$

12. Given $\left\{\begin{array}{l} xy = p \\ x^2 - y^2 = d \end{array}\right\}$ to find the rest.

$x^2y^2 = p^2$ by squaring 1st.

$x^4 - 2x^2y^2 + y^4 = d^2$ square of 2d.

$4x^2y^2 = 4p^2$ by mult. by 4.

$x^4 + 2x^2y^2 + y^4 = d^2 + 4p^2$ by add.

$x^2 + y^2 = \sqrt{d^2 + 4p^2}$ extracting,

$x^2 - y^2 = d$ 2d equation,

$2x^2 = d + \sqrt{d^2 + 4p^2}$ by add.

$x^2 = \frac{d + \sqrt{d^2 + 4p^2}}{2}$ divide by 2,

$x = \sqrt{\frac{d + \sqrt{d^2 + 4p^2}}{2}}$ by extracting, And $2y^2 = \sqrt{d^2+4p^2}-d$ by subt.

$$y^2 = \frac{\sqrt{d^2+4p^2}-d}{2},$$

$$y = \sqrt{\frac{\sqrt{d^2+4p^2}-d}{2}}.$$

13. Given $\left\{\begin{array}{l}\frac{x}{y}=q\\ x^2+y^2=a\end{array}\right\}$ to find the rest.

From 1st $x=qy$ by mult. by y,

$x^2=q^2y^2$ by squaring,

$y^2=a-q^2y^2$ by subt.

$q^2y^2+y^2=a$ by transp.

$y^2=\frac{a}{q^2+1}$; for $\overline{q^2+1}\times y^2=q^2y^2+y^2$,

$x^2=a-\frac{a}{q^2+1}=\frac{q^2a}{q^2+1}$ by subt.

$$x=\sqrt{\frac{q^2a}{q^2+1}},$$

And $y=\sqrt{\frac{a}{q^2+1}}$.

14. Given $\left\{\begin{array}{l}\frac{x}{y}=q\\ x^2-y^2=b\end{array}\right\}$ to find the rest.

$x=qy$ mult. by y,

$x^2=q^2y^2$ squaring,

$x^2=b+y^2$ transp. 2d.

$q^2y^2=b+y^2$ being both $=x^2$,

$y^2=\frac{b}{q^2-1}$ dividing by q^2-1,

$x^2=b+\frac{b}{q^2-1}=\frac{q^2b}{q^2-1}$ by add.

$x=\sqrt{\frac{q^2b}{q^2-1}}$ by evolution,

And $y^2=\frac{b}{q^2-1}$.

Therefore $y = \sqrt{\dfrac{b}{q^2 - 1}}$ by evol.

15. Given $\left\{\begin{matrix} x^2 + y^2 = a \\ x^2 - y^2 = b \end{matrix}\right\}$ to find the rest.

$2x^2 = a + b$ by addition,

$x^2 = \dfrac{a+b}{2}$ dividing by 2,

$2y^2 = a - b$ by subt.

$y^2 = \dfrac{a-b}{2}$ dividing by 2.

Therefore $x = \sqrt{\dfrac{a+b}{2}}$ by evol.

And $y = \sqrt{\dfrac{a-b}{2}}$ by evol.

NOTE.—These 15 problems contain a valuable variety of arithmetical inquiry; in fact their solutions compose a whole system of arithmetic.

The pupil may make numerical calculations in each question, which will prove an excellent exercise.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. One bought some oxen for L.80; now had he bought four more for the same money, he would have paid L.1 less for each. How many did he buy?
Ans. 16.

2. There are two numbers whose difference is 3, and the difference of their cubes is 117. What are those numbers? Ans. 2 and 5.

3. Bought two remnants of cloth, one of which was six yards longer than the other, for L.3. 8s. and each of them cost as many shillings a-yard as there was yards therein. What was the length of each?
Ans. 8 and 2.

4. What two numbers are those, whose difference is 2, and the product of their cubes 42875?
Ans. 5 and 7.

5. One having sold a piece of cloth, which cost him L.30, found that if the price he sold it for was multiplied by his gain, the product would be equal to the cube of his gain. What was it? Ans. L.6.

6. A grazier bought as many sheep as cost him L.60, out of which he reserved 15, and sold the remainder for L.54, gaining 2s. a-head by them. How many sheep did he buy? Ans. 75.

7. A person bought a certain number of sheep for L.57. Having lost 8 of them, and sold the remainder at 8 shillings a-head profit, he is no loser by the bargain. How many sheep did he buy? Ans. 38.

8. It is required to divide 37 into two such parts, that the product of the squares of those parts may be 117964. What are the parts? Ans. 19 and 18.

9. It is required to divide 48 into two such parts, that the sum of their alternate quotients may be $5\frac{1}{5}$.

Ans. 40 and 8.

10. It is required to find two such numbers, the first of which may be to the second as the second is to 20; and the sum of the squares of the numbers sought may be 125. Ans. 5 and 10.

11. There are two numbers whose difference is 15, and $\frac{1}{2}$ their product is the cube of the lesser. What are those numbers? Ans. 3 and 18.

12. There are two numbers whose difference is 7, also their sum multiplied by the greater produces 345. What are those numbers? Ans. 15 and 8.

13. It is required to divide the number 24 into two such parts, that their product may be thirty-five times their difference. Ans. 14 and 10.

14. A and B, (who were 120 miles distant) set out to meet each other. A travelled 5 miles a-day, and the number of days at the end of which they met was greater by 3 than the number of miles which B went in a day. How many miles did each go?

Ans. 50 and 70.

15. To find four numbers, the first of which may be to the second as the third to the fourth; that the first may be to the fourth as 1 to 5; that the second may be to the third as 5 to 9; and the sum of the second and fourth may be 20. Ans. 3, 5, 9, 15.

16. One buys cloth for L.33. 15s. which he sells again for 48 shillings per piece, and gained by the bargain as much as one piece cost him. Required the number of pieces? Ans. 15.

17. To divide the number 6 into two such parts, that the sum of their cubes shall be 72.

Ans. 4 and 2.

OF CUBIC AND HIGHER EQUATIONS.

A CUBIC equation is that in which the third power of the unknown quantity enters, as $x^3 + ax^2 + bx = c$.

A biquadratic equation is that in which the fourth power of the unknown quantity enters, as $ax^4 + bx^3 + cx^2 + dx = e$.

And generally an equation is said to be of the 5th, 6th, &c. degree, according as the highest power of the unknown quantity is of any of these dimensions.

SOLUTION OF EQUATIONS BY APPROXIMATION.

THERE are many particular and prolix rules usually given for the solution of some of the above-mentioned powers or equations. But they may be all readily solved by approximation, that is, by methods which are continually bringing us nearer to the true value, till at last the error may be considered as nothing. Different methods have been proposed for this purpose, among these, the following is that which is commonly used, and is applicable to numeral equations of all dimensions.

RULE.—*Find by trial a number nearly equal to the root required, which call* r. *Then assume some letter, as* z, *to denote the difference between* r *and the true root* x;

instead of the unknown quantity x *in the given equation, substitute its equal* r + *or* — z, *by which a new equation will arise, involving only* z *and the known quantities, wherein all the terms that contain two or more dimensions of* z *may be neglected as inconsiderable. This being done, the value of* z *will be found by a simple equation, which added to or subtracted from that of* r, *according as* r *was taken too little or too great, and it will give the root required nearly.*

If the root thus found be not sufficiently exact, repeat the operation by substituting it in the equation, exhibiting the value of z, *and it will give a second correction of* z, *which being added to or subtracted from* r, *will give a nearer value of the root; and by repeating the same process, a value of* x *may be found to any degree of exactness.*

EXAMPLE.

1. Given $x^3 + x^2 + x = 90$, to find the value of x by approximation.

By substituting the numbers 3, 4, 5, &c. instead of x we obtain 39, 84, 155, instead of 90. Hence we find that the true root is more than 4, for 90 is greater than 84; but that it is less than 5, for 90 is less than 155; therefore it is 4 *plus* a certain quantity z.

Let therefore $4 = r$ and $r + z = x$

$$\text{Then } x^3 = r^3 + 3r^2z + 3rz^2 + z^3$$
$$x^2 = r^2 + 2rz + z^2$$
$$x = r + z.$$

And by neglecting the terms $3rz^2$, z^3, and z^2, as small in comparison with z, we have

$$r^3 + r^2 + r + 3r^2z + 2rz + z = 90.$$

Therefore $z = \dfrac{90 - r^3 - r^2 - r}{3r^2 + 2r + 1} = \dfrac{90 - 64 - 16 - 4}{48 + 8 + 1}$

$= \dfrac{6}{57} = {\cdot}10$, and $x = r + z = 4 + {\cdot}1 = 4{\cdot}1$ nearly. And again, if 4·1 be substituted for r, in the last equation, we have

$z = \dfrac{90 - r^3 - r^2 - r}{3r^2 + 2r + 1} = \dfrac{90 - 68{\cdot}921 - 16{\cdot}81 - 4.1}{50{\cdot}43 + 8{\cdot}2 + 1}$

$= {\cdot}00283.$

Therefore $x = 4\cdot1 + \cdot00283 = 4\cdot10283$; and so on to any required degree of exactness.

In the same manner may the root of a *pure equation* be found, and this is an easy method of approximating to the roots of numbers which are not perfect powers. As an example, let the equation $x^3 = 32$ be proposed to find its root or the value of x. Here, by a few trials, we find that x is greater than 3, and less than 4.

Therefore assume $3 = r$, and $r + z = x$; we have

$$x^3 = r^3 + 3r^2z + 3rz^2 + z^3 = 32.$$

Then by neglecting $3rz^2$ and z^3, we have $r^3 + 3r^2z = 32$.

Whence $z = \dfrac{32 - r^3}{3r} = \dfrac{32 - 27}{27} = \dfrac{5}{27} = \cdot18$,

and $x = r + z = 3 + \cdot18 = 3\cdot18$ nearly.

And again, by substituting $3\cdot18$ in place of r in the last equation, gives

$$z = \frac{32 - 32\cdot157}{30\cdot38} = \cdot005, \text{ \&c.}$$

and $x = r + z = 3\cdot18 - \cdot005 = 3\cdot175$, which is the root nearly.

EXAMPLES FOR PRACTICE.

1. Given $x^2 = 27$, to find the value of x, by approximation. Ans. $x = 5\cdot1961$.

2. Given $x^2 = 18$, to find the value of x. Ans. $x = 4\cdot2426$.

3. Given $x^3 = 78$, to find the value of x. Ans. $x = 4\cdot2726$.

4. Given $x^3 = 200$, to find the value of x. Ans. $x = 5\cdot848035$.

5. Given $x^4 = 36$, to find the value of x. Ans. $x = 2\cdot4494$.

6. Given $x^4 = 100$, to find x. Ans. $x = 3\cdot162277$.

7. Given $x^3 - 2x = 50$, to find the value of x. Ans. $x = 3\cdot86488$.

8. Given $x^3 - 17x^2 + 54x = 350$, to find the value of x. Ans. $x = 14\cdot95407$.

9. Given $x^6 + 4x^3 = 12$, to find the value of x.
Ans. $x = 1{\cdot}259921$.

10. Given $x^3 + 10x^2 + 5x = 2600$, to find the value of x.
Ans. $x = 11{\cdot}0068027$.

The roots of equations, of all dimensions, can also be found, to any degree of exactness, by the following easy rule of *Double Position.*

RULE.—*Find by trial two numbers as near the true root as possible. Substitute these numbers for the unknown quantity in the given equation, and mark the errors which arise from each of them. Multiply the difference of the numbers found by trial into the least error, and divide the product by the difference of the errors when they are alike, but by their sum when they are unlike, and add the quotient to the number belonging to the least error, when that number is too little, but subtract it when too great; the result will be the root nearly.*

Take this root and the nearest to the former, (or any nearer number), and proceeding in like manner, a root will be obtained nearer than before; and by again repeating the operation, the root will be had still nearer, and so on to any degree of exactness.

Having found one root, depress the equation one degree lower, by taking for a dividend the given equation, with the known term transposed, with its sign changed, to the unknown side of the equation, and for a divisor, take x *minus the root just found. Divide the said dividend by the divisor, and the quotient will be an equation depressed a degree lower than the given one. Then find another root, depress this last equation, and find a third root, and so on till the equation is brought to a quadratic, whose roots may be found by the rule for quadratic equations.*

REMARK.—The rule is founded on this supposition, that the first error is to the second, as the difference between the true and first supposed number is to the difference between the true and second supposed number, which, though not accurately true in all cases, yet it will be very nearly so.

That the rule is true according to the supposition, may be thus demoustrated. Let a and b be the two suppositions; A and B their results produced by the like operations; it is required to find the number from which N is produeed by a similar operation.

Let $x =$ the number required, $N - A = r$, $N - B = s$, then by supposition $r : s : : x - a : x - b$, and by division $r - s : s : : b - a : x - b$, that is $\frac{(b - a)s}{r - s} = x - b$, which is the rule when the assumed quantities are both too little. Next, let A and B be both greater than N, then $N - A = - r$, $N - B = - s$, but $- r : - s : : + r : + s$, wherefore $r - s : s : : a - b : b - x$, or $\frac{(a - b)s}{r - s} = b - x$, which is the rule when both quantities are too great.

Again, let one result A be too little, and the other B too great, then will r be positive, and s negative, and in this case $r + s : (- s$, or which is the same) $s : : a - b : b - x$.

Whence $\frac{(a - b)s}{r + s} = b - x$, which is the rule when the errors are unlike.

Note.—Among all the approximating rules that have been given by several mathematicians, there is no method so easy and general as the above, which will resolve the most difficult forms of equations of all degrees, however embarrassed by surds, compound quantities, &c. without any previous reduction.

EXAMPLES.

1. Given $x^3 + x^2 + x = 20$ to find the roots or values of x.

Here it will appear that x is some number between 2 and 3.

Assume those numbers for x, and the operation will stand thus.

1st Supposition.		2d Supposition.
2	x	3
4	x^2	9
8	x^3	27
14	Sums	39
20	but should be	20
-6	errors	$+19$

The sum of the errors $(19 + 6) = 25$,

Difference of numbers found by trial $=1$.
Least error $=6$.
$\frac{6\times1}{25}=\frac{6}{25}=\cdot24$ is the correction, and $x=2\cdot24$ nearly.

Secondly, let $2\cdot2$ and $2\cdot3$ be each put for x, and repeat the operation thus,

3d Supposition.		4th Supposition.
2·2	x	2·3
4·84	x^2	5·29
10·648	x^3	12·167
17·688	Sums	19·757
20	but should be	20
— 2·312	Errors	·243.

Their difference $(2\cdot312-\cdot243)=2\cdot069$,
Difference of numbers is $=1$. Least error $\cdot243$.
Th. $\frac{1\times\cdot243}{2\cdot069}=\cdot01174$ is the correction.
And $x=(2\cdot3+\cdot01174)=2\cdot31174$ extremely near.

Now, in order to find the other two roots, depress the given equation. We have $x^3+x^2+x-20=0$ for a dividend, and $x-2\cdot31174=0$ for a divisor; but because the second and third decimal figures in the divisor are small, and the work only an approximation, $x-2\cdot3$ may be used and the rest omitted, thus:

$$
\begin{array}{l}
x-2\cdot3)\;x^3+x^2+x-20\,(x^2+3\cdot3x+8\cdot59\\
\qquad\qquad\underline{x^3-2\cdot3x^2}\\
\qquad\qquad\qquad 3\cdot3x^2+x\\
\qquad\qquad\qquad \underline{3\cdot3x^2-7\cdot59x}\\
\qquad\qquad\qquad\qquad\quad 8\cdot59x-20\\
\qquad\qquad\qquad\qquad\quad \underline{8\cdot59x-19\cdot757}\\
\qquad\qquad\qquad\qquad\qquad\qquad -243
\end{array}
$$

The two roots of this equation $(x^2+3\cdot3\,x+8\cdot59=0)$ being found by the rule for quadratics, are $-1\cdot65\pm\sqrt{-5\cdot8675}$.

Wherefore the three roots of the given equation are

$2{\cdot}31174$, $-1{\cdot}65 + \sqrt{-5{\cdot}8675}$, and $-1{\cdot}65 - \sqrt{-58675}$ nearly, the two last of which are impossible.

2. Given $x^4 + 2x^3 + 3x^2 + 4x = 70$ to find x.

Let 2 and 3 be the supposed numbers, then

1st Supposition.		2d Supposition.
8	$4x$	12
12	$3x^2$	27
16	$2x^3$	54
16	x^4	81
52	Sums	174
70		70
-18	Errors	104

$\frac{1 \times 18}{18 + 104} = \frac{18}{122} = {\cdot}147 =$ the correction, wherefore $x = 2{\cdot}147$ nearly.

Now for a nearer approximation, let $2{\cdot}1$ and $2{\cdot}2$ be the supposed number, and we shall have,

3d Supposition.		4th Supposition.
8·4	$4x$	8·8
13·23	$3x^2$	14·52
18·522	$2x^3$	21·296
19.4481	x^4	23.4256
59·6001	Sums	68·0416
70		70
$-10{\cdot}3999$	Errors	$-1{\cdot}9584$

Differ. of numbers $(= 2{\cdot}2 - 2{\cdot}1) = 1$,

Diff. errors $(= 10{\cdot}3999 - 1{\cdot}9584) = 8{\cdot}4415$.

Wherefore $\frac{{\cdot}1 \times 1{\cdot}9584}{8{\cdot}4415} = {\cdot}0232$ the correct.

Whence $x = (2{\cdot}2 + {\cdot}0232) = 2{\cdot}2232$, and by depressing the equation, &c. as in the preceding example, the other roots will be obtained.

3. Given $\sqrt{1 + x^2} + \sqrt{2 + x^2} + \sqrt{3 + x^2} = 10$ to find x.

Let 3 and 4 be the supposed numbers.

1st Supposition.		2d Supposition.
3·16227......	$\sqrt{1+x^2}$	......4·12310
3·31662......	$\sqrt{2+x^2}$	......4·24264
3·46410......	$\sqrt{3+x^2}$	......4·35889
9·94299........	Sums	...... 12·72463
10.......................		 10
— 0·05701........	Errors...	+ 2·72463

Wherefore $\frac{1 \times \cdot 05701}{2 \cdot 78164} = \cdot 0205$ the correction.

And $x = 30205$ nearly, and by repeating the operation a nearer value of x will be obtained.

EXAMPLES FOR PRACTICE.

1. Given $x^2 + 20x = 20$, to find x.
   Ans. $x = 4 \cdot 142135$.
2. Given $x^3 - 2x = 5$, to find x.
   Ans. $x = 2 \cdot 094251$.
3. Given $x^3 + x^2 + x = 100$, to find x.
   Ans. 4·264429.
4. Given $x^4 - 16x^3 + 40x^2 - 30x = -1$, to find x.
   Ans. $x = 1 \cdot 284724$.
5. Given $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321$, to find x. Ans. $x = 8 \cdot 414455$.
6. Given $\frac{20x}{\sqrt{(16+5x)}+x^2} + \frac{x\sqrt{(5+x^2)}}{25} = 34$, to find x. Ans. $x = 20\ 17$.
7. Given $(1+x^2)^{\frac{1}{2}} + (1+x^2)^{\frac{1}{3}} + (1+x^3)^{\frac{1}{4}} = 6 \cdot 5$, to find x. Ans. $x = 3 \cdot 03$[illegible]

QUESTIONS PRODUCING EQUATIONS OF THE HIGHER ORDERS.

1. What two numbers are those, the sum of whose squares is 58, and their product multiplied by the greater gives 147? Ans. 3 and 7.

2. It is required to divide £.16 so among two per-

sons, that the cube of the one's share may exceed the cube of the other's by 386. Ans. 9 and 7.

3. What two numbers are those, whose product is 945, and if from the square of the greater the lesser be taken, the remainder will be 1198?

Ans. 35 and 27.

4. What two numbers are those, whose product multiplied by the lesser will produce 225, but if their difference be multiplied by the greater is 36?

Ans. 9 and 5.

5. What two numbers are those, the sum of whose square roots is 4, and the difference of their cube roots is 1? Ans. 8·5773504, and 1·1476592.

ARITHMETICAL PROGRESSION.

If a rank or series of quantities increase or decrease by the continued addition or subtraction of the same quantity, then those quantities are said to be in arithmetical progression.

Thus, the numbers 1, 2, 3, 4, 5, 6, &c. which increase by the addition of (1) to each successive term, and the numbers 10, 8, 6, 4, 2, which decrease by the subtraction of (2) from each successive term, in an arithmetical progression.

In general, if a denote the first term of any arithmetical progression, and d the common difference, then may the series itself be expressed by a, $a+d$, $a+2d$, $a+3d$, $a+4d$, &c. increasing; and a, $a-a$, $a-2d$, $a-3d$, $a-4d$, &c. decreasing, by the addition and subtraction of the common quantity d.

In the foregoing series it is evident, that if there be but three terms, the sum of the extremes will be double to the mean. Thus a, $a+d$, $a+2d$, &c. increasing. Then $a+a+2d$, that is $2a+2d$, the sum of the extremes, is double of $a+d$ the mean. Or if the series be decreasing, as a, $a-d$, $a-2d$; then will $a+a-2d$; that is, $2a-2d$, the sum of the extremes, is double of $a-d$ the mean, and so in any other three terms of the series. In like manner, if there are four terms, then the sum of the two extremes will be equal to that of the two means, as a, $a+d$, $a+2d$, $a+3d$; that is, $a+a+3d=a+d+a+2d$.

Consequently, whatever be the number of terms in any arithmetical series, the sum of the two extremes will always be equal to the sum of any two means that are equally distant from those extremes; so in the series $a, a+d, a+2d, a+3d, a+4d, a+5d$, &c. $a+a+5d=a+d+a+4d=a+2d+a+3d$, &c.; and if the number of terms be odd, the sum of the two extremes will be double that of the mean.

Hence it follows, that if the sum of the extremes be multiplied by the number of terms in any arithmetical progression, the product will be double the sum of the series.

The reason of this will appear still more evident, by setting the terms of the series in an inverted order, under the same series in a direct order, and adding the corresponding terms together in that order. Thus in the series,

$$\begin{array}{lllll} a, & a+d, & a+2d, & a+3d, & a+4d. \\ a+4d, & a+3d, & a+2d, & a+d, & a, \\ \hline 2a+4d, & 2a+4d, & 2a+4d, & 2a+4d, & 2a+4d. \end{array}$$

Since the sum of any two terms $2a+4d$, is equal to the sum of the first and last term, it appears that the sum of the series is equal to the sum of the first and last term (or of any two terms equally distant from the first and last terms) multiplied by half the number of terms.

In every arithmetical progression we may observe the five following particulars:

$a=$ the first term,
$z=$ the last term,
$d=$ the common difference,
$n=$ the number of terms,
and $s=$ the sum of the series.

Then from the foregoing observations $2s = \overline{a + z}.n$ or $s = \overline{a + z}.\frac{n}{2}$ the sum of the series.

Also in these series it is easy to perceive, that the common difference (d) is added to the last term of the series as often as there are terms in it except the first; that is, the first term (a) has no difference added to it, but the last term has as many times (d) added to it, as it is distant from the first term.

Consequently, the difference between the two extremes is only the common difference (d) multiplied into the number of terms less unity or 1.

That is $z - a = d.\overline{n - 1}$, the difference between the two extremes.

Hence it follows, that if the difference between the extremes be divided by the number of terms less 1, the quotient will be the common difference of the series.

Thus, $d = \frac{z - a}{n - 1}$.

Now, by the help of the foregoing remarks, if any three of the five parts (a, z, d, n, s,) be given, the other two may be found.

Thus, given a, z, and n, to find s.

Then $2s = \overline{a + z}.n$,

or $s = \overline{a + z}.\frac{n}{2}$ as before.

When the first term a, is 0, or nothing, then the sum of the series will be expressed by the following equation, $s = \frac{zn}{2}$.

2. Given a, z, and n, to find d.

Then, as before, $\overline{n - 1}.d = z - a$.

Therefore $d = \frac{z - a}{n - 1}$.

3. Given a, z, and d, to find n.

By the last $d = \frac{z - a}{n - 1}$,

Then $nd - d = z - a$ by mult.

$nd = z - a + d$ by trans.

$$n = \frac{z - a + d}{d} = \frac{z - a}{d} + 1.$$

4. Given a, n, and s, to find z.

By 1st. $s = \frac{\overline{a + z} \cdot n}{2}$,

$2s = \overline{a + z} \cdot n$ by mult. by 2,

or $2s = na + nz$.

$z = \frac{2s - na}{n}$ by transp. and division.

Therefore $z = \frac{2s}{n} - a$.

5. Given z, n, and s, to find a.

By quest. 1st. $s = \frac{\overline{a + z} \cdot n}{2}$,

or $2s = a + z \cdot n$ mult. by 2,

$a + z = \frac{2s}{n}$ by transp. and division.

Therefore $a = \frac{2s}{n} - z$.

6. Given a, z, and s, to find n.

By quest. 1st. $s = \frac{\overline{a + z} \cdot n}{2}$,

or $2s = \overline{a + z} \cdot n$.

Therefore $n = \frac{2s}{a + z}$.

7. Given a, z, and d, to find s.

Then by placing the value of n in quest. 3d, equal to the value of n in quest. 6th, we have

$\frac{z - a}{d} + 1 = \frac{2s}{z + a}$,

$\frac{z^2 - a^2}{d} + z + a = 2s$ mult. by $z + a$.

Therefore $\frac{z^2 - a^2}{2d} + \frac{z + a}{2} = s$.

8. Given a, z, and s, to find d.

Then by the last equation we have $\frac{z^2 - a^2}{2d} + \frac{z + a}{2}$ $= s$.

$z^2 - a^2 + dz + da = 2ds$,

$z^2 - a^2 = 2ds - \overline{z + a} \, . \, d$.

Therefore $d = \frac{z^2 - a^2}{2s - z + a}$.

9. Given a, d, and n, to find z.

By quest. 3d. $z - a = nd - d$.

Therefore $z = a + d \, . \, \overline{n - 1}$.

10. Given z, d, and n, to find a.

By the last question,

$z - a = nd - d$.

Therefore $a = z - d \, . \, \overline{n - 1}$ by transp.

11. Given z, d, and n, to find s.

By quest. 1st. $s = \frac{\overline{a + z} \, . \, n}{2}$,

and by 10th. $a = z - d \, . \, \overline{n - 1}$.

Therefore $a + z = 2z - d \, . \, \overline{n - 1}$,

and $s = \overline{2z - d \, . \, \overline{n - 1}} \, . \, \frac{n}{2}$.

12. Given a, d, and n, to find s.

By quest. 9th. $z = a + d \, . \, \overline{n - 1}$.

Therefore $s = \overline{a + a + d \, . \, \overline{n - 1}} \, . \, \frac{n}{2}$,

or $s = \overline{2a + d \, . \, \overline{n - 1}} \, . \, \frac{n}{2}$.

13. Given a, d, and s, to find n.

By the last $s = \overline{2a + d \, . \, \overline{n - 1}} \, . \, \frac{n}{2}$,

$s = \frac{2an + dn^2 - dn}{2}$ by multiplication,

$n^2 + \frac{2a - d}{d} n = \frac{2s}{d}$ which is an adfected quadratic equation, and being resolved by completing the square, gives $n = \frac{\sqrt{8ds + (2a - d)^2} - 2a + d}{2d}$.

In like manner we may proceed to find any of the five quantities (a, z, d, n, s,) otherwise, that is, by varying or comparing these equations one with another, and from these produce new equations with other data; which we shall here omit, and leave them for the *learner's practice*, and shall in this place illustrate some of the foregoing equations, by a few numerical examples.

Ex. 1. Find the sum of the series, whose first term is 1, last 21, and the number of terms 11.

Then by equation 1, we have

$$\left.\begin{array}{l} a = 1, \\ z = 21, \\ n = 11, \end{array}\right\} \text{Therefore } s = \overline{a + z} \cdot \frac{n}{2},$$

$$s = \overline{1 + 21} \cdot \frac{11}{2},$$

$$s = \frac{22 \cdot 11}{2} = 121.$$

That is, $s = 121$.

2. One had 20 children that differed alike in their ages, the youngest was 5 years old, the eldest 43. What is the difference of their ages?

From equation 2d we have

$$\left.\begin{array}{l} a = 5 \\ z = 43 \\ n = 20 \end{array}\right\} d = \frac{z - a}{n - 1},$$

$$d = \frac{43 - 5}{20 - 1},$$

$$d = \frac{38}{19} = 2.$$

Therefore the difference of their ages is 2 years.

3. A man being asked how many children he had, answered, that his youngest child was 5 years old, and the eldest 43, and that he increased one in his family every two years. How many children had he?

Equation 3d we have

$$\left.\begin{array}{l} a = 5, \\ z = 43, \\ d = 2, \end{array}\right\} n = \frac{z - a}{d} + 1,$$

$$n = \frac{43 - 5}{2} + 1,$$

$$n = \frac{38}{2} + 1 = 19 + 1,$$

$n = 20$ the number of children required.

4. A man in 19 days went from Edinburgh to a certain place in the country; every day's journey was greater than the preceding one by three miles; his last day's journey was 60 miles. What was the first?

By equation 10th we have

$$\left.\begin{array}{l} z = 60, \\ d = 3, \\ n = 19, \end{array}\right\} \begin{array}{l} a = z - d \,.\, \overline{n - 1}, \\ a = 60 - 3 \,.\, \overline{19 - 1}, \\ a = 60 - 3 \,.\, 18, \\ a = 60 - 54 = 6. \end{array}$$

5. The first term of an arithmetical series is 1, the number of terms 16, and common difference 2. What is the last term, and sum of the series?

From equation 9th we have

$$\left.\begin{array}{l} a = 1, \\ d = 2, \\ n = 16, \end{array}\right\} \begin{array}{l} z = a + d \,.\, \overline{n - 1}, \\ z = 1 + 2 \,.\, \overline{16 - 1}, \\ z = 1 + 2 \,.\, 15, \\ z = 4 + 30 = 31. \end{array}$$

To find the sum, we have by equa. 12th,

$$s = \overline{2a + d \,.\, \overline{n - 1}} \,.\, \frac{n}{2}.$$

That is, $s = \overline{2 + 2 \,.\, \overline{16 - 1}} \,.\, \frac{16}{2}$,

$$s = 2 + 2 \,.\, 15 \,.\, \frac{16}{2},$$

$s = 32 \,.\, 8 = 256$, the sum required.

6. A gentleman buys clothes for his 7 sons; the values of the suits increase in arithmetical progression; the youngest cost 20s. and the whole amounted to 245s. How much was charged for the eldest?

By equation 5th we have

$$\left.\begin{array}{l} a=20, \\ s=245, \\ n=7, \end{array}\right\} \quad a=\frac{2s}{n}-z, \quad \text{Or } z=\frac{2s}{n}-a.$$

$$\text{Th. } z=\frac{245\cdot 2}{7}-20,$$

$$\text{or } z=\frac{490}{7}=70-20=50= \text{the value of the}$$

eldest.

7. A person spent £.100 the first year he kept his family, and afterwards increased his expenses each year by £.14, till he spent in all £.1192. Required the number of years?

$$\left.\begin{array}{l} \text{Here } a=100, \\ d=14, \\ s=1192, \end{array}\right\} \text{By equation 13 we have}$$

$$n=\frac{\sqrt{8ds+(2a-d)^2}-2a+d}{2d}$$

$$n=\frac{\sqrt{133504+34596}-200+14}{28}=8 \text{ the}$$

number of years required.

8. Two men A and B set out at the same time; A travels 8 miles per day, and B travels the first day 1 mile, the second day 2 miles, the third day 3 miles, &c. In how many days will B overtake A?

Let x be the number of days required;
Then A will travel $8x$ miles.

Also the sum of an arithmetical progression whose first term is 1, common difference 1, and number of terms x is expressed by the following equation,

$$\frac{\overline{x+1}.x}{2}=\frac{x^2+x}{2}.$$

$$\text{Th. B will travel } \frac{x^2+x}{2}.$$

$$\text{But } \frac{x^2+x}{2}=8x \text{ per quest.}$$

$x^2 + x = 16x$ mult. by 2,
$x + 1 = 16$ divid. by x,
$x = 16 - 1 = 15$ by transp.
Th. $x = 15$ the number of days when they will meet.

9. To find three numbers in arithmetical progression, the sum of whose squares shall be 1232, and the square of the mean shall exceed the product of the extremes by 16.

Let $x =$ the mean, and $y =$ the common difference.
Then $x - y$, x, and $x + y$ are the numbers;
But $x^2 = (x - y) \cdot (x + y) + 16$ per quest.
That is, $x^2 = x^2 - y^2 + 16$.
Th. $y^2 = 16$,
and $y = \sqrt{16} = 4$.
Now $(x - 4)^2 = x^2 - 8x + 16$,
and $x^2 = x^2$,
also $(x + 4)^2 = x^2 + 8x + 16$;
But $x^2 - 8x + 16 + x^2 + x^2 + 8x + 16 = 3x^2 + 32 = 1232$ by question,
$3x^2 = 1232 - 32 = 1200$ by transp.
$x^2 = 400$ by dividing by 3,
$x = \sqrt{400} = 20$ by evolution.
Therefore the numbers are 16, 20, and 24.

10. There are four numbers in arithmetical progression, of which the product of the two extremes is 22, and that of the means 40; what are the numbers?

Let $x =$ the less extreme,
and $y =$ the common difference.
Then x, $x + y$, $x + 2y$, $x + 3y$, the four numbers.
1st condition, $x(x + 3y) = 22$,
or $x^2 + 3xy = 22$.
2d condition $x, + y \cdot x + 2y = 40$,
or $x^2 + 3xy + 2y^2 = 40$.
1st equation, $x^2 + 3xy = 22$.
By subtraction $2y^2 = 18$,
By dividing by 2, $y^2 = 9$,
By evolution $y = \sqrt{9} = 3$.
By substituting 3 for y in the first gives $x^2 + 9x = 22$,

and comp. sq. $x^2+9x+\left(\frac{9}{2}\right)^2=\frac{169}{4}$,

By evol. $x+\frac{9}{2}=\pm\sqrt{\frac{169}{4}}=\frac{13}{2}$,

By transp. $x=\frac{13}{2}-\frac{9}{2}=\frac{4}{2}=2$.

Therefore the four numbers required 2, 5, 8, and 11.

11. To find three numbers in arithmetical progression, whose sum is a, and the sum of their squares b.

Let $x-y$, x, and $x+y$ be the numbers,
Then $x-y+x+x+y=3x=a$,
and $(x-y)^2+x^2+(x+y)^2=3x^2+2y^2=b$.

From equation 1st, $x^2=\frac{a^2}{9}$,

$\frac{a^2}{9}=\frac{b-2y^2}{3}$, both being $=x^2$.

Whence $a^2=3b-6y^2$,

$y^2=\frac{3b-a^2}{6}$,

$y=\sqrt{\frac{3b-a^2}{6}}$,

and $x=\frac{a}{3}$.

EXAMPLES FOR PRACTICE.

1. How many strokes does the hammer of a clock strike in 12 hours? Ans. 78.

2. The length of a garden is 94 feet; now if eggs be laid along the pavement a foot asunder, and be fetched up singly to a basket removed one foot from the first; how much ground does he traverse that does it?
Ans. 8930 feet.

3. A butcher bought 100 sheep, and gave for the first sheep 1s. and for the last £.9.19s.; I demand what he gave for the 100 sheep? Ans. £.500.

4. A man had eight sons that differed alike in their ages, the youngest was 4 years old, and the eldest 32; what was the difference of their ages?

Ans. 4 years.

5. A man is to travel from London to a certain place in 12 days, and to go but 3 miles the first day, increasing every day by an equal excess, so that the last day's journey may be 58 miles; what is the daily increase, and how many miles distant is the place from London? Ans. 5 daily inc. 366 distance.

6. A person travelling into the country, went 3 miles the first day, and increased every day 5 miles, till at last he went 58 miles in one day; how many days did he travel? Ans. 12.

7. A man takes out of his pocket at eight several times so many different numbers of shillings, every one exceeding the former by 6, the last was 46, what was the first? Ans. 4.

8. What is the last term of an arithmetical progression, whose first term is 6, common difference 8, and number of terms 20? Ans. 158.

9. A person began with spending £.200, and afterwards increased his expenses by an equal sum yearly till he spent £.500 the last year, and £.4550 altogether: Required the number of years, and annual increase. Ans. 13 and 25.

10. Two post-boys, A and B, set out at the same time from two cities which are 360 miles asunder, in order to meet each other. A rides 40 miles the first day, 38 the second, 36 the third, and so on, decreasing 2 miles every day; but B goes 20 miles the first day, 22 the second, 24 the third, &c. increasing 2 miles every day: In what number of days will they meet?

Ans. 6.

11. One being asked what were the several ages of his five children, answered, that the age of the eldest exceeded that of the second by 2 years; and by the same excess the second exceeded the third, the third

the fourth, &c.; and if the age of the eldest child were multiplied by the age of the youngest, it would produce 128. The age of each child is required.

Ans. 8, 10, 12, 14, 16.

12. The continual product of 4 numbers in arithmetical progression is 945; and their common difference 2: What are those numbers?

Ans. 3, 5, 7, 9.

13. To find three numbers in arithmetical progression, such that the sum of their squares may be 56, and the sum arising by adding together three times the first, and two times the second, and three times the third, may amount to 28. Ans. 2, 4, 6.

GEOMETRICAL PROGRESSION.

If a rank or series of quantities increase or decrease by the continual multiplication or division of the same quantity, then those quantities are said to be in Geometrical Progression.

Thus, 2, 4, 8, 16, 32, which increase by the continual multiplication of 2; and the numbers 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. which decrease by the continual division of 2, or the multiplication of $\frac{1}{2}$, are in geometrical progression.

Note.—The common multiplier or divisor is called the ratio, which shows the proportion or relation which one magnitude bears to another magnitude of the same kind with respect to quantity.

Proportion is the equality of ratios.

Three quantities are said to be proportional, when the ratio of the first to the second is equal to the ratio of the second to the third. As of the three quantities 2, 4, 8, where $\frac{4}{2} = \frac{8}{4} = 2$, both the same ratio.

Four quantities are said to be proportional, when the ratio of the first to the second, is the same as the ratio of the third to the fourth. As of the four 2, 4, 5, 10, where $\frac{4}{2} = \frac{10}{5} = 2$, both the same ratio.

Quantities are said to be in Geometrical Progression continued, or continually proportional, when the ratio is the same between every two adjacent terms; that is, when the first is to the second as the second to the third, as the third to the fourth, as the fourth to the fifth, and so on, all in the same common ratio. As in the quantities 2, 4, 8, 16, 32, 64, &c. where the common ratio is equal to 2.

In every series of continued proportionals, that quantity which is compared to another, is called the Antecedent of the ratio, and that quantity to which it is compared is called the Consequent.

As in the quantities 2 : 4 : : 4 : 8. Here 2 is the antecedent, and 4 is the consequent; and 4 the middle term is an antecedent, and 8 is its consequent. Whence it follows, that in every series in continued proportion, all the middle terms between the first and last terms are both antecedents and consequents: As in these, 2, 4, 8, 16, 32, 64. Here 4, 8, 16, 32, are both antecedents and consequents.

For 2 : 4 : : 4 : 8 : : 8 : 16 : : 16 : 32 : : 32 : 64. So that all the terms except the last are antecedents, and all the terms except the first are consequents.

The antecedents are homologous terms; and so are the consequents.

In general if a denote the first term of such a series, and r the common multiple or ratio, then may the series itself be represented by a, ar, ar^2, ar^3, ar^4, &c. increasing.

and a, $\frac{a}{r}$, $\frac{a}{r^2}$, $\frac{a}{r^3}$, $\frac{a}{r^4}$, &c. decreasing.

In any of these series it is evident, that if three quantities are in geometrical progression, the rectangle, or product of two extreme terms, will be equal to the square of the mean term.

Thus, a, ar, ar^2. Then $a \times ar^2 = ar \times ar = a^2r^2$,

or a, $\frac{a}{r}$, $\frac{a}{r^2}$. Then $a \times \frac{a}{r^2} = \frac{a}{r} \times \frac{a}{r} = \frac{a^2}{r^2}$.

If four quantities are in geometrical progression, the rectangle of the extremes is equal to the rectangle of the means. As in these a, ar, ar^2, ar^3, where $a \times ar^3 = ar \times ar^2 = a^2r^3$,

or a, $\frac{a}{r^2}$, $\frac{a}{r^3}$. Then $a \times \frac{a}{r^3} = \frac{a}{r} \times \frac{a}{r^2} = \frac{a^2}{r^3}$.

Consequently, if any number of quantities be continued proportionals, the rectangle of the extremes is equal to the rectangle of any two means that are equally distant from those extremes. As in these, a, ar, ar^2, ar^3, ar^4, ar^5, &c. where $a \times ar^5 = ar \times ar^4 = a^2r^5$; or $a \times ar^5 = ar^2 \times ar^3 = a^2r^5$.

If any number of quantities are in continued geometrical progression, then any one of the antecedents will be to its consequent as the sum of all the antecedents is to the sum of all the consequents. As in the series a, ar, ar^2, ar^3, ar^4, ar^5. Then $a : ar :: a + ar^2 + ar^4 : ar + ar^3 + ar^5$.

For $a \times ar + ar^3 + ar^5 = ar \times a + ar^2 + ar^4 = a^2r + a^2r^3 + a^2r^5$. That is, the rectangle of the extremes is equal to the rectangle of the means.

NOTE.—The ratio of any series in continued geometrical progression increasing, is found by dividing any of the consequents by its antecedent.

Thus $\frac{ar}{a} = r$, or $\frac{ar3}{ar^2} = r$.

But if the series be decreasing, the ratio is found by dividing any of the antecedents by its consequent. Thus, in the series a, $\frac{a}{r}$, $\frac{a}{r^2}$, $\frac{a}{r^3}$.

Then $\left.\frac{a}{r}\right) a \left(r\right.$, or $\left.\frac{a}{r^3}\right)\frac{a}{r^2}\left(r\right.$.

These preliminaries being premised, such equations may be deduced from them, as will solve all questions which may be proposed about quantities in continued geometrical proportion. In order to that,

Let a = the first term,
z = the last term,
r = the common ratio,
n = the number of terms,
s = the sum of the series.

Then the progression will be represented as before. Thus a, ar, ar^2, ar^3, ar^4, &c.; where the index of r in any term is less by unity than the number which denotes the place of that term in the series.

Hence n being the number of terms in the series, the last term will be ar^{n-1}. That is, $z = ar^{n-1}$, from which the value of any term may be found, for the foregoing equation involving a, z, r, n, may be reduced by division as under.

$$a = \frac{z}{r^{n-1}},$$

$$r^{n-1} = \frac{z}{a}, \text{ or } r = \left(\frac{z}{a}\right)^{\frac{1}{n-1}},$$

$$n = \frac{\text{Log. } z - \text{Log. } a}{\text{Log. } r} + 1;$$

NOTE.—As n only enters into these expressions as an exponent of r, its value cannot be commodiously exhibited in any other way than by logarithms.

The sum of any series of quantities increasing in geometrical progression as a, $+ar$, ar^2, ar^3, ar^4, ar^5, &c. where r is supposed to be greater than an unit.

Let $s = a + ar + ar^2 + ar^3 + ar^4$.

And if each side of this equation be multiplied by r, we have,

$$rs = ar + ar^2 + ar^3 + ar^4 + ar^5.$$

Hence, if the first equation be subtracted from the second, we have $rs - s = -a + ar^5$,

or $\overline{r - 1} \cdot s = ar^5 - a$.

Therefore $s = \dfrac{ar^5 - a}{r - 1}$,

or $s = \dfrac{r^5 - 1}{r - 1} \times a$.

If six terms of such a series were taken, their sum would, in like manner, be found to be equal to $\dfrac{ar^6 - a}{r - 1}$; if seven terms, to $\dfrac{ar^7 - a}{r - 1}$. Hence, if the number of terms be generally represented by n, $s = \frac{ar^n - a}{r-1}$. And if we substitute the values of a and its powers in the last equation, we have

$$s = \frac{zr - a}{r-1},$$

when the quantities decrease in geometrical progression, or when r is less than an unit.

For the convenience of calculation, therefore, it is better to transpose the equation $s = \frac{ar^n - a}{r-1}$ into $s = \frac{a - ar^n}{1-r}$, by multiplying the numerator and denominator of the fraction by -1.

When r is less than an unit, and the number of terms, or the value of n, is unlimited, r^n may be neglected. For it is evident that r^n decreases as n increases. Assume n to be indefinitely great, the value of r^n will be indefinitely small; so that, in the equation $s = \frac{a - ar^n}{1-r}$, ar^n may be considered as vanishing with respect to a, being, in this case, less than any assignable quantity, can have no effect on the value of such a series. Therefore,

$$s = \frac{1}{1-r} \times a,$$

$$\text{or } s = \frac{a}{1-r}.$$

We shall now (for the convenience of the *learner*) exhibit the foregoing equations in one view, and then illustrate them by a few practical examples.

Equa. 1st, $z = ar^{n-1}$.

Equa. 2d, $a = \frac{z}{r^{n-1}}$.

Equa. 3d, $r^{n-1} = \frac{z}{a}$, or $r = \left(\frac{z}{a}\right)^{\frac{1}{n-1}}$.

Equa. 4th, $n = \frac{\text{Log. } z - \text{Log. } a}{\text{Log. } r} + 1$.

Equa. 5th, $s=\frac{ar^n-a}{r-1}$.

Equa. 6th, $s=\frac{zr-a}{r-1}$.

When the quantities decrease in geometrical progression, we have,

Equa. 7th, $s=\frac{a-ar^n}{1-r}$.

When the number of terms are unlimited, or continued *ad infinitum*, their sum is expressed by

Equa. 8th, $s=\frac{a}{1-r}$.

EXAMPLES.

1. The number of terms is 7, the common ratio 2, and the first term 3: What is the last term?

$$\left.\begin{array}{r}\text{Here } a=3,\\ r=2,\\ n=7.\end{array}\right\}\text{ By equation 1st we have,}$$

$$z=ar^{n-1}.$$
$$\text{That is } z=3\cdot2^{7-1},$$
$$z=3\cdot2^6,$$
$$z=3\times64=192.$$

Therefore the last term is 192.

2. The number of terms is 7, the common ratio 2, and the last term 192: What is the first term?

$$\left.\begin{array}{r}\text{Here } z=192,\\ r=2,\\ n=7.\end{array}\right\}\text{ By equation 2d } a=\frac{z}{r^{n-1}}.$$

$$\text{Th. } a=\frac{192}{2^{7-1}},$$

$$a=\frac{192}{2^6},$$

$$\text{or } a=\frac{192}{64}=3, \text{ the first term.}$$

3. The first term is 3, the last 192, and the number of terms 7: What is the common ratio?

$$\left.\begin{array}{l}\text{Here } z=192,\\ a=3,\\ n=7.\end{array}\right\}\text{ By equation 3d } r=\left(\frac{z}{a}\right)^{\frac{1}{n-1}}.$$

$$\text{Then } r=\left(\frac{192}{3}\right)^{\frac{1}{7-1}},$$

$$\text{or } r=\left(\frac{192}{3}\right)^{\frac{1}{6}}=64^{\frac{1}{6}},$$

$$r=\sqrt[6]{64}=2 \text{ the common ratio.}$$

Hence we have a method of finding several mean proportionals between two given numbers, viz. divide the greater by the lesser, esteem the quotient a power whose index is greater by unity than the number of means proposed, and the root of this power extracted is the ratio, by which multiply the least of the two given numbers continually, and the several products are the means required. Thus,

To find five mean proportionals between 3 and 192.

$\frac{192}{3}=64$, and $\sqrt[6]{64}=2$.

Then $3\times2=6$, $6\times2=12$,

$12\times2=24$, $24\times2=48$, $48\times2=96$. So 6, 12, 24, 48, 96, are the mean proportionals required.

4. The first term is three, the last 192, and the common ratio 2: What is the number of terms?

$$\left.\begin{array}{r}\text{Here } z=192,\\ a=3,\\ r=2.\end{array}\right\}\text{Then by equation 4th we have}$$

$$n=\frac{\text{Log. } z-\text{Log. } a}{\text{Log. } r}+1,$$

$$n=\frac{2{\cdot}283301-0{\cdot}477120}{0{\cdot}301030}+1,$$

$$n=\frac{1{\cdot}806180}{0{\cdot}301030}+1=6+1=7, \text{ the number of terms.}$$

5. The first term is 1, the common ratio 3, and the number of terms 12: What is the sum of the series?

$$\left.\begin{array}{r}\text{Here } a=1,\\ r=3,\\ n=12,\end{array}\right\}\text{and } s=\frac{ar^n-a}{r-1}\text{ by equat. 5th.}$$

$$s = \frac{1 \times 3^{12} - 1}{3 - 1},$$

$$s = \frac{81^3 - 1}{3 - 1},$$

$$s = \frac{531441 - 1}{2},$$

$$s = \frac{531440}{2} = 265720, \text{ the sum required.}$$

6. The first term is 3, the last 192, and the common ratio is 2: What is the sum of the series?

Here $\left.\begin{array}{l} z = 192, \\ a = \ \ 3, \\ r = \ \ 2, \end{array}\right\}$ and $s = \frac{zr - a}{r - 1}$ by equat. 6th.

$$s = \frac{192 \times 2 - 3}{2 - 1},$$

$$s = \frac{384 - 3}{2 - 1},$$

$$s = 381, \text{ the sum.}$$

The ratio minus unity being 1, there is no occasion to divide by it.

7. Required the sum of ten terms of the series $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27}$, &c.

Here $\left.\begin{array}{l} a = \ 1, \\ r = \ \frac{2}{3}, \\ n = 10. \end{array}\right\}$ Therefore $s = \frac{a - ar^n}{1 - r}$ by equa. 7th.

That is, $s = \frac{1 - (\frac{2}{3})^{10}}{1 - \frac{2}{3}} = \frac{\overline{1 - (\frac{2}{3})^{10}} \times 3}{3 - 2} = \overline{1 - (\frac{2}{3})^{10}} \times 3$.

And $(\frac{2}{3})^{10} = \frac{2^{10}}{3^{10}} = \frac{1024}{59049}$.

Therefore $1 - (\frac{2}{3})^{10} = 1 - \frac{1024}{59049} = \frac{58025}{59049}$,

And $s = \frac{3 \times 58025}{59049} = \frac{174075}{59049}$, the sum required.

8. Required the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, &c. without limit.

Here $\left.\begin{array}{l} a = 1, \\ r = \frac{1}{2}, \end{array}\right\}$
n is unlimited.

Therefore by equation 8th we have

$$s = \frac{a}{1 - r},$$

$$s = \frac{1}{1 - \frac{1}{2}} = \frac{2}{2 - 1} = 2.$$

9. Required the sum of the series $\frac{3}{4} + \frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \frac{4}{27}$, &c. *ad infinitum.*

Here $\left.\begin{array}{l} a = \frac{3}{4}, \\ r = \frac{2}{3}, \end{array}\right\}$
n is unlimited.

Then by equation 8th

$$s = \frac{a}{1 - r},$$

$$s = \frac{\frac{3}{4}}{1 - \frac{2}{3}} = \frac{3}{4 - \frac{8}{3}} = \frac{9}{12 - 8} = \frac{9}{4}$$ the sum required.

These are sums to which the series would approximate to within the smallest limit, if sufficiently continued, but would never exactly complete.

Repeating and *circulating decimals* are quantities in geometrical progression; decreasing where $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. is the common ratio, according to the number of factors contained in the *repeating decimal.*

EXAMPLES.

1. Required the value of the circulating decimal ·3333, &c.

This decimal is represented by the geometrical series $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000}$, &c. whose first term is $\frac{3}{10}$, and common ratio $\frac{1}{10}$.

Here $\left.\begin{array}{l} a = \frac{3}{10}, \\ r = \frac{1}{10}, \end{array}\right\}$
n unlimited.

Therefore $s = \frac{a}{1-3}$ by equation 8th.

That is, $s = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{3}{10-1} = \frac{3}{9} = \frac{1}{3}$ the value required.

2. Required the value of ·323232, &c.

Here $\left.\begin{array}{l} a = \frac{32}{100}, \\ r = \frac{1}{100}, \end{array}\right\}$
n unlimited.

Therefore $s = \frac{a}{1-r}$.

That is, $s = \frac{\frac{32}{100}}{1-\frac{1}{100}} = \frac{32}{100-1} = \frac{32}{99}$.

10. Three numbers in geometrical progression are required, so that the difference of the first and second may be 6, and of the second and third 15.

Let x, y, and z, be the numbers required.

First condition, $x - y = 6$,
Second condition, $y - z = 15$.
From 1st. $x = y - 6$,
From 2d. $z = y + 15$.
But $x : y :: y : z$.
That is, $y - 6 : y :: y : y + 15$.
Then $y^2 + 9y - 90 = y^2$,
or $9y = 90$,
$y = 10$,
$x = y - 6 = 4$, and $z = y + 15 = 25$.

Therefore the three numbers required, are 4, 10, and 25.

11. It is required to find three numbers in geometrical progression, the sum of the first and second of which may be 14, and of the second and third 35.

Let x, y, and z, be the numbers required.

First condition, $x + y = 14$,
Second condition, $y + z = 35$.
From 1st. $x = y - 14$.
From 2d. $z = y - 35$.
But $x : y :: y : z$.

That is, $y - 14 : y :: y : y - 35$.

Then $y^2 + 49y - 490 = y^2$,

$49y = 490$,

$y = 10$,

$x = 14 - y = 4$, and $z = 35 - y = 25$;

so that the numbers are 4, 10, and 25.

12. It is required to find three numbers in geometrical progression, such that their sum may be 7, and the sum of their squares 21.

Let x, y, and z, be the numbers required.

First condition, $x + y + z = 7$,

Second condition, $x^2 + y^2 + z^2 = 21$.

But $x : y :: y : z$; therefore $y^2 = xz$.

From 1st. $x + z = 7 - y$.

By squaring $x^2 + 2xz + z^2 = 49 - 14y + y^2$,

But $2xz = 2y^2$,

By subt. $x^2 + z^2 = 49 - 14y - y^2$,

From 2d. $x^2 + z^2 = 21 - y^2$.

Hence $49 - 14y - y^2 = 21 - y^2$,

or $49 - 14y = 21$.

Therefore $14y = 49 - 21 = 28$ by trans.

$y = 2$ dividing by 14.

Again, $x + z = 7 - y = 5$.

We have $x^2 + 2xz + z^2 = 25$ by squaring.

But $4xz = 16$, for $xz = y^2$.

Therefore by $x^2 - 2xz + z^2 = 9$ by subt.

and $x - z = 3$ by evol.

Hence $x + z = 5$,

and $x - z = 3$.

By $2x = 8$ addition,

$x = 4$ by divid. by 2,

$2z = 2$ by subt.

$z = 1$ dividing by 2.

Therefore the numbers are 1, 2, and 4.

EXAMPLES FOR PRACTICE.

1. A farmer buys 6 cows, whose prices were in geometrical progression, the common ratio being 2, the price of the last was 96 crowns: What was the price of the first? Ans. 3 crowns.

2. A gentleman had 9 sons to whom he left his estate, divided into portions in geometrieal progression, the eommon ratio being 3; the youngest son got the least portion, being only L.50: What did the eldest son get? Ans. L.328,050.

3. A grazier bought 14 sheep, at a farthing for the first, a halfpenny for the second, still doubling the priee for every subsequent one, and was to pay only the priee of the last sheep for the whole: What sum had he to pay? Ans. L.8. 10s. 8d.

4. A gentleman purehased some aeres of ground, whose priees were in geometrical progression, the eommon ratio being 3; the price of the first acre was 3d. and of the last 59049d.: How many aeres did he purchase? Ans. 10.

5. A gentleman who had a daughter married on New-year's day, gave the husband towards her portion 4 shillings, promising to triple that sum the first day of every month, for nine months after the marriage; the sum paid on the first day of the ninth month was 26244 shillings: What was the lady's portion?

Ans. L.1968. 4s.

6. A eorn-merehant buys 12 staeks of wheat, and was to pay 2d. for the first staek, 6d. for the next, tripling the priee for every following staek: What sum had he to pay? Ans. L.2214. 6s. 8d.

7. The sum of 4 numbers in geometrical progression is 30, and the last term divided by the sum of the mean terms $\frac{4}{3}$ or $1\frac{1}{3}$: What are the numbers?

Ans 2, 4, 8, 16.

8. There are 3 numbers in geometrieal progression, whose produet is 64, and the sum of their eubes 584: What are the numbers?

Ans. 2, 4, 8

9. There are 3 numbers in geometrieal progression, whose sum is 21, and the sum of their squares 189: What are the numbers? Ans. 3, 6, 12.

INTEREST.

INTEREST is the allowance made for the loan or forbearance of a sum of money, which is lent for, or becomes due, at a certain time; this allowance being generally estimated at so much per cent per annum; that is, so much for the use of £.100 for a year; and this rate is by law fixed not to exceed £.5, or in other words, the greatest rate of interest must not exceed 5 per cent per annum.

Interest is either simple or compound.

Simple Interest is that which is allowed upon the principal only, for the whole time of the loan or forbearance. The money lent, is called the principal; the sum paid for the use of it, the interest. The interest of £.100 for one year, is called the rate per cent; and the sum of any principal and its interest together, the amount.

As the interest of any sum, for any time, is directly proportional to the principal sum, and to the time; therefore the interest of £.1 for one year being multiplied by any given principal sum, and by the time of its forbearance, in years and parts, will give its interest for that time. That is, if there be put

$r =$ rate of interest of £.1 per annum,
$p =$ any principal or sum lent,
$t =$ time in years or parts of a year,
$i =$ the interest of the given principal for the time t,
$a =$ the amount of the given principal and its interest for the given time.

Then £.1 : r : : £.p : $pr =$ the interest of p pounds for a year; likewise 1 year : pr : : t years : $prt =$ the interest of p pounds for t years, at r per cent per annum; and if p be added to this, the sum will be the amount, which is expressed in the following equation:

$$\text{1st, } a = p + prt.$$

From whence, by transposition, we have the following:

Equa. 2d, $p = \frac{a}{1 + rt}$.

Equa. 3d, $r = \frac{a - p}{pt}$.

Equa. 4th, $t = \frac{a - p}{pr}$.

Equa. 5th, $i = prt$.

By means of which equations all circumstances relating to the simple interest of money are readily computed.

EXAMPLES.

1. What is the interest and amount of £.320 for 3 years, at 4 per cent per annum?

Here $\left.\begin{aligned} p &= 320, \\ r &= \cdot 04, \\ t &= 3, \end{aligned}\right\}$ by equation 5th we have $i = prt$.

That is, $i = 320 \times \cdot 04 \times 3 =$ £.38. 4s. $=$ £.38. 8s. the interest.

And the amount by equation 1st is, $a = p + prt$; that is, $320 +$ £.38. 8s. $=$ £.358. 8s.

2. What principal, being put to interest at 5 per cent, will amount to £.1000 in 12 years?

Here $\left.\begin{aligned} a &= 1000, \\ t &= 12, \\ r &= \cdot 05, \end{aligned}\right\}$ by equa. 2d we have $p = \frac{a}{1 + rt}$.

Then $p = \frac{1000}{1 + \cdot 05 \times 12} = \frac{1000}{1 \cdot 6} =$ £.625, the principal required.

3. At what rate per cent will £.210 amount to £.399 in 20 years?

Here $\left.\begin{aligned} a &= 399, \\ p &= 210, \\ t &= 20, \end{aligned}\right\}$ by equa. 3d we have $r = \frac{a - p}{pt}$.

Then $r = \frac{399 - 210}{210 \times 20}$,

$r = \frac{189}{4200} = \cdot 045$ or $4\frac{1}{2}$ per cent per annum.

4. In how many years will £.210 amount to £.399, at $4\frac{1}{2}$ per cent?

Here $\left.\begin{array}{r} p = 210, \\ a = 399, \\ r = \cdot 045, \end{array}\right\}$ then equa. 4th gives $t = \frac{a-p}{pr}$.

That is, $t = \frac{399 - 210}{210 \times \cdot 045}$,

or $t = \frac{189}{9 \cdot 45} = 20$ years, the time required.

EXAMPLES FOR PRACTICE.

1. What is the interest of £.250 for $2\frac{1}{2}$ years, at 5 per cent? Ans. £.31. 10s.

2. What is the amount of £.284. 10s. for 7 years, at $3\frac{1}{2}$ per cent? Ans. £.354. 4s. $0\frac{1}{2}$d.

3. What principal, being put out at 5 per cent for $2\frac{1}{2}$ years, will amount to £.281. 10s.?

Ans. £.250.

4. At what rate per cent will £.284. 10s. amount to £.354. 4s. $0\frac{1}{2}$d. in 7 years?

Ans. $3\frac{1}{2}$.

5. In what time will £.672. 5s. amount to £.847. 17s. 6d. at $4\frac{3}{4}$ per cent?

Ans. $5\frac{1}{2}$ years.

6. In what time will £.378. 18s. amount to £.500. 9s. $3\frac{1}{2}$d. at 5 per cent per annum?

Ans. 6 years 5 mo. nearly.

DISCOUNT.

Discount is an allowance made on a bill or any other debt not yet become due, in consideration of present payment.

Bankers, merchants, &c. allow for discount a sum equal to the interest of the bill for the time before it becomes due, which, however, is not just; for, as the true value of the discount is equal to the difference between the debt and its present worth, it is equal only to the interest of that present worth, instead of the interest on the whole debt. And therefore the rule for finding the true discount is this:

As the amount of £.1 for the given rate and time is to the given sum or debt, so is the interest of £.1 for the given rate and time to the discount of the debt.

So if p be the principal or debt, r the rate of interest, and t the time,

Then as $1 + rt : p :: rt : \frac{prt}{1 + rt}$, which is the true discount.

Hence also $1 + rt : p :: 1 : \frac{p}{1 + rt}$, which is the present worth or sum to be received.

EXAMPLES.

1. What is the discount of £.150, due 9 months hence, at 5 per cent per annum?

Here $p = 150$,
$r = \cdot05$,
$t = \cdot75 = 9$ months.

Then the discount $= \frac{prt}{1 + rt}$.

That is, $= \frac{150 \times \cdot05 \times \cdot75}{1 + \cdot05 \times \cdot75}$,

or $= \frac{5\cdot625}{1\cdot0375} = 5\cdot4216 =$ £.5. 8s. 5d. the discount.

2. What is the present worth of £.405. 10s. due 4 years hence, at 5 per cent?

$$\left.\begin{aligned} \text{Here } p &= 405\cdot5, \\ r &= \cdot05, \\ t &= 4. \end{aligned}\right\}$$

Then the present worth $= \frac{p}{1 + rt}$.

That is, $= \frac{405\cdot5}{1 + 4 \times \cdot05} = \frac{405\cdot5}{1\cdot2} = 337\cdot9166$, &c. or £.337. 18s. 4d. the present worth.

COMPOUND INTEREST.

COMPOUND INTEREST is that which arises from any sum or principal in a given time, by increasing the principal, each payment, by the amount of that payment, and hence obtaining interest upon both principal and interest.

Let $p =$ the principal,
$r =$ the simple interest of £.1 for a year,
$t =$ the number of years,
$a =$ the amount at the end of t years,
and $R = (1 + r) =$ the amount of £.1 at the end of a year.

Then the particular amounts for the several times may be thus found, viz.

As £.1 is to its amount for any time, so is any principal to its amount for the same time. Thus,

$1 : R :: p \quad : pR \;=$ amount of p at the end of the first year.
$1 : R :: pR \quad : pR^2 =$ amount at the end of 2d year.
$1 : R :: pR^2 \quad : pR^3 =$ amount at the end of 3d year.
$1 : R :: pR^{t-1} : pR^t =$ amount at the end of t years.

So that at the end of t years the amount is equal pR^t, that is, $a = pR^t$; from which equation are deduced the following:

Equa. 1st, $a = pR^t$.

Equa. 2d, $p = \dfrac{a}{R^t}$.

Equa. 3d, $R = \sqrt[t]{\dfrac{a}{p}}$.

Equa. 4th, $t = \dfrac{\text{Log. } a - \text{Log. } p}{\text{Log. } R}$.

From which equations, any one of the quantities may be found when the rest are given.

The interest is found by subtracting the principal p *from the amount* a.

EXAMPLES.

1. What is the amount of £.500 for 4 years, at 5 per cent per annum, compound interest?

$$\left.\begin{array}{r} \text{Here } p = 500, \\ t = \quad 4, \\ R = 1{\cdot}05. \end{array}\right\} \text{By equa. 1st we have } a = pR^t.$$

That is, $a = 500 \times (1{\cdot}05)^4 = 607{\cdot}753125 =$ £.607. 15s. $0\frac{3}{4}$d. the amount required.

2. What sum at 4 per cent per annum, compound interest, would amount to £.200 in three years?

$$\left.\begin{array}{r} \text{Here } a = 200, \\ R = 1{\cdot}04, \\ t = \quad 3. \end{array}\right\} \text{By equa. 2d we have } p = \frac{a}{R^t}.$$

That is, $p = \dfrac{200}{(1{\cdot}04)^3}$,

$$\text{or } p = \frac{200}{1{\cdot}124864} = 177{\cdot}799 = \text{£}.177.\ 15\text{s. } 11\tfrac{3}{4}\text{d.}$$

the principal required.

3. At what rate per cent would £.500 amount to £.578. 15s. 3d. in 3 years, compound interest?

$$\left.\begin{array}{r} \text{Here } p = 500, \\ a = 578{\cdot}8125, \\ t = \quad 3. \end{array}\right\} \text{By equa. 3d, } R = \sqrt[t]{\frac{a}{p}}.$$

$$\text{That is } R = \sqrt[3]{\frac{578{\cdot}8125}{500}},$$

$$R = \sqrt[3]{1{\cdot}157625} = 1{\cdot}05.$$

Therefore $r = {\cdot}05 = 5$ per cent.

4. In what time will £.225 amount to £.260. 9s. $3\frac{3}{4}$d. at 5 per cent per annum compound interest?

$$\left.\begin{array}{r} \text{Here } p = 225, \\ a = 260{\cdot}465625, \\ R = 1{\cdot}05. \end{array}\right\}$$

Then by equa. 4th, $t = \dfrac{\text{Log. } a - \text{Log. } p}{\text{Log. } R}$, or $t = \frac{2\cdot41570 - 2\cdot35218}{0\cdot02119} = \frac{\cdot06352}{\cdot02119} = 3$ years, the time required.

EXAMPLES FOR PRACTICE.

1. Required the amount of £.300 for 4 years, at 5 per cent per annum compound interest.
Ans. £.364. 13s.

2. What is the amount of £.365 for 7 years, at $4\frac{1}{2}$ per cent per annum compound interest?
Ans. £.496. 14s. $3\frac{1}{2}$d.

3. What principal being put to interest will amount to £.346. 17s. in 7 years, at 5 per cent per annum?
Ans. £.246. 10s.

4. What principal being put to interest for 30 years, at $4\frac{1}{2}$ per cent compound interest, will amount to £.1872. 13s. 2d.? Ans. £.500.

5. At what rate per cent will £.246. 10s. amount to £.346. 17s. in 7 years, compound interest?
Ans. 5 per cent.

6. At what rate per cent will £.500 amount to £.1872. 13s. 2d. compound interest, in 30 years?
Ans. $4\frac{1}{2}$ per cent.

7. In what time will £.246. 10s. amount to £.346 17s. at 5 per cent compound interest?
Ans. 7 years.

8. In what time will £.530 amount to £.1872. 13s. 2d. at $4\frac{1}{2}$ per cent compound interest?
Ans. 30 years.

9. In what time will £.510 amount to £.685. 0s. $7\frac{1}{2}$d. at 5 per cent compound interest?
Ans. 5 years and 194 days.

APPENDIX.

EXERCISES IN SIMPLE EQUATIONS.

Question 1. To find a number which being multiplied by 3, subtracting 5 from the product, and the remainder divided by 2, if the number sought be added to the quotient, that the sum may be 40.

Solution.

Assume $x =$ the number.

Then $\frac{3x - 5}{2} + x = 40$ per quest.

$3x - 5 + 2x = 80$ by mult. by 2,

$5x = 85$ by transp.

Th. $x = 17$.

2. To find a number, which being multiplied by 12, and 48 added to the product, as much may be produced as if the number sought were multiplied by 18.

Solution.

Assume $x =$ the number.

Then $12x + 48 = 18x$ per quest.

$18x - 12x = 48$ by transp.

or $6x = 48$,

Th. $x = 8$.

3. To find a number, to which if 11 be added, and 7 subtracted from the same number, (viz. the first), the sum of the addition may be double of the remainder.

Solution.

Assume $x =$ the number.

Then $x + 11 = 2x - 14$ per quest.

$x = 2x - 25$ by transp.

Th. $x = 25$.

4. To find a number to which, if its double, treble, quadruple, &c. be added, the square of the same number may be produceed.

Solution.

Assume $x =$ the number.

Then $x + 2x + 3x + 4x = x^2$ per quest.

or $10x = x^2$.

Th. $10 = x$.

5. To find a number, which, if added to itself, and the sum multiplied by the same, and the same number still subtracted from the product, and lastly the remainder divided by the same, that it may produce 13.

Solution.

Assume $x =$ the number.

Then $\frac{2x^2 - x}{x} = 13$ per quest.

$2x^2 - x = 13x$ by mult. by x,

$2x^2 = 14x$ by transp.

Th. $2x = 14$,

and $x = 7$.

6. To divide the number 16 into two parts, so that the square of the greater part may exceed the square of the less by 32.

Solution.

Assume $x =$ the greater,

and $y =$ the lesser.

$$\left.\begin{array}{l}\text{1st cond. } x + y = 16 \\ \text{2d cond. } x^2 - y^2 = 32\end{array}\right\} \text{per quest.}$$

From 1st. $x = 16 - y$,

From 2d. $x^2 = 32 + y^2$.

Now, $32 + y^2 = (16 - y)^2$ both being $= x^2$,

or $32 + y^2 = 256 - 32y + y^2$ by sq.

$32y = 224$ by transp. and extr.

Th. $y = 7$,

and $16 - y = 9$.

Otherwise.

Assume $x =$ the greater,

Then $16 - x =$ the lesser.

$$\left.\begin{array}{l}\text{1st cond. } x + (16 - x) = 16 \\ \text{2d cond. } x^2 - (16 - x)^2 = 32\end{array}\right\}$$

From 2d. $x^2 = (16 - x)^2 + 32$,
or $x^2 = 256 - 32x + x^2 + 32$ by sq.
$32x = 288$ by transp. and extr.
Th. $x = 9$,
and $16 - x = 7$.

7. To divide the number 36 into two parts, so that if 12 be added to the first, and 6 to the second, the former may be double of the latter.

Solution.

Assume $x =$ the greater,
And $y =$ the lesser.
$$\left.\begin{array}{l}\text{1st cond. } x + y = 36,\\ \text{2d cond. } x + 12 = 2y + 12.\end{array}\right\}$$
From 1st. $x = 36 - y$,
From 2d. $x = 2y$.
But $2y = 36 - y$ both being $= x$,
$3y = 36$ by transp.
and $y = 12$,
also $x = 2y = 24$.

Otherwise.

Assume $x =$ the 1st,
Then $36 - x =$ the 2d.
Now, $x + 12 = \overline{36 - x + 6} \times 2$ per quest.
or $x + 12 = 72 - 2x + 12$,
$3x = 72$ by transp. and extr.
Th. $x = 24$,
and $36 - x = 12$.

8. A certain captain sends out $\frac{1}{3}$ of his soldiers + 10, there remains $\frac{1}{2}$ + 15: How many soldiers had he?

Solution.

Assume $x =$ the number.
Then $\frac{x}{2} + 10 + \frac{x}{3} + 15 = x$ per quest.
or $\frac{x}{2} + \frac{x}{3} + 25 = x$,
$x + \frac{3x}{2} + 75 = 3x$ by mult. by 3.

$$5x + 150 = 6x \text{ by mult. by } 2,$$
$$6x + 5x = 150 \text{ by transp.}$$
$$\text{or } x = 150.$$

9. There is an army, to which if you add $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of itself, and take away 5000, the sum total will be 100,000: What was the number of the army?

Solution.

Assume $x =$ the number.

$$\text{Then } x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} - 5{,}000 = 100{,}000 \text{ per quest.}$$
$$3x + \frac{2x}{3} + \frac{2x}{4} - 10{,}000 = 200{,}000 \text{ by mult. by } 2,$$
$$11x + \frac{6x}{4} - 30{,}000 = 600{,}000 \text{ by mult. by } 3,$$
$$50x - 120{,}000 = 2400{,}000 \text{ by mult. by } 4,$$
$$50x = 2520{,}000 \text{ by transp.}$$
$$\text{Th. } x = 50400.$$

10. To find two numbers in the proportion of 2 to 3, whose product, if they be multiplied by one another, shall be 54.

Solution.

$$\left.\begin{array}{l}\text{1st cond. } x : y :: 2 : 3, \\ \text{2d cond. } \quad xy = 54.\end{array}\right\} \text{per quest.}$$

From 1st. $3x = 2y$,

and $x = \frac{2y}{3}$.

From 2d. $x = \frac{54}{y}$.

$$\text{But } \frac{2y}{3} = \frac{54}{y} \text{ both being } = x,$$
$$2y = \frac{162}{y} \text{ by mult. by } 3,$$
$$2y^2 = 162 \text{ by mult. by } y,$$
$$y^2 = 81 \text{ by div.}$$
$$\text{Th. } y = \sqrt{81} = 9,$$
$$\text{and } x = \frac{2y}{3} = 6.$$

Otherwise.

Assume $x =$ the lesser,

Then $\frac{3x}{2} =$ the greater.

Now, $\frac{3x}{2} \times x = 54$ per quest.

or $\frac{3x^2}{2} = 54$,

$3x^2 = 108$ by mult. by 2,

$x^2 = 36$ by div.

Th. $x = \sqrt{36} = 6$,

and $\frac{3x}{2} = 9$.

11. To find two numbers whose ratio is to one another as 4 to 5, and the sum of the squares of both is 369.

Solution.

Assume $x =$ the lesser,

and $y =$ the greater.

$$\left.\begin{array}{l}\text{1st cond. } x : y :: 4 : 5, \\ \text{2d cond. } x^2 + y^2 = 369,\end{array}\right\} \text{per quest.}$$

From 1st. $4y = 5x$,

and $y = \frac{5x}{4}$.

From 2d. $y^2 = 369 - x^2$.

But $\left(\frac{5x}{4}\right)^2 = \frac{25x^2}{16} = 369 - x^2$ both being $= y^2$,

$25x^2 = 5904 - 16x^2$ by mult. by 16,

$41x^2 = 5904$ by transp.

$x^2 = 144$ by div.

Th. $x = \sqrt{144} = 12$,

and $y = \frac{5x}{4} = 15$.

12. A certain man hires a labourer on this condition, that for every day he wrought he should receive 12 pence, but for every day he was idle he should be mulcted 8 pence: when 390 days were passed, neither

of them were indebted to one another: How many days did he work, and how many was he idle?

Solution.

Assume $x =$ the days he wrought,
and $y =$ the days he idled.

$$\left.\begin{array}{ll}\text{1st cond.} & x + y = 390, \\ \text{2d cond.} & 12x = 8y,\end{array}\right\} \text{per quest.}$$

From 2d. $x = \dfrac{8y}{12}$.

Therefore $y + \dfrac{8y}{12} = 390$ per quest.

$20y = 4680$ by mult. by 12.

Th. $y = 234$.

and $\dfrac{8y}{12} = 156$.

Otherwise.

Assume $x =$ the days he wrought,
Then $390 - x =$ the days he idled.
But $12x = (390 - x)\,8$ per quest.
or $12x = 3120 - 8x$,
$20x = 3120$ by transp.
Th. $x = 156$,
and $390 - x = 234$.

13. A certain gentleman hires a servant, and promises him 24 pounds yearly wages, together with a cloak: At 8 months' end the servant obtains leave to go away, and instead of his wages receives a cloak + 13 pounds: What did the cloak cost?

Solution.

Assume x pounds $=$ the value of the cloak.
Then $x + 24$ pounds $=$ his wages for a year,
and $x + 13$ pounds $=$ his wages for 8 months.

And since he was paid in proportion to the time he served,

$$\text{Th. } \overset{\text{m.}}{12} : \overset{\text{m.}}{8} :: x + 24 : x + 13,$$

$12x + 156 = 8x + 192$ by mult. ext. and means,
$12x - 8x = 192 - 156$ by transp.
or $4x = 36$.
Th. $x = £.9$.

Otherwise.

Assume x pounds $=$ the value of the cloak.

Now, since he served 8 months $= \frac{2}{3}$ of his time, being only $\frac{1}{3}$ to serve; therefore there was due $\frac{2}{3}$ of his wages, as well as $\frac{2}{3}$ of the value of the cloak.

Th. $\frac{x}{3} = \frac{24 \times 2}{3} - 13$ per quest.

$x = \frac{144}{3} - 39$ by mult. by 3,

$3x = 144 - 117$ by mult. by 3,

or $3x = 27$.

Th. $x = £.9$.

14. A person being asked how old he was, answered, If I quadruple $\frac{2}{3}$ of my years, and add $\frac{1}{2}$ of them $+$ 50 to the product, the sum will be so much above 100 as the number of my years is now below 100.

Solution.

Assume his age $= x$ years.

Then $\frac{8x}{3} + \frac{x}{2} - 100 = 100 - x$ per quest.

$8x + \frac{3x}{2} - 300 = 300 - 3x$ by mult. by 3,

$19x - 600 = 600 - 6x$ by mult. by 2,

$25x = 1200$ by transp.

Th. $x = 48$.

15. One being asked what hour of the day it was, answered, The day at this time is 16 hours long; if now $\frac{1}{2}$ of the hours past be added to $\frac{2}{3}$ of the remainder, you will have the hour desired, reckoning from sun-rising.

Solution.

Assume x hours $=$ the time past,

Then $16 - x$ hours $=$ the remainder.

Now, $\frac{x}{2} + \frac{\overline{16 - x} \times 2}{3} = x$ per quest.

or $\frac{x}{2} + \frac{32 - 2x}{3} = x$,

$$x + \frac{64 - 4x}{3} = 2x \text{ by mult. by } 2,$$

$$3x + 64 - 4x = 6x \text{ by mult. by } 3,$$

$$7x = 64 \text{ by transp.}$$

$$\text{Th. } x = 9\tfrac{1}{7}.$$

16. From Norimberg to Rome are 140 miles: A traveller sets out at the same time from each of the two cities, one goes 8 miles a-day, the other 6: In how many days from their first setting out will they meet one another, and how many miles did each of them go?

Solution.

Sup. x days $=$ the time when they will meet.
Then the first travels $8x$ miles,
and the other $6x$ miles.
Also their sum $\overline{14x} = 140$ per quest.
and $x = 10$.
Th. the first travels $8x = 80$ miles,
and the other $6x = 60$ miles.

17. A certain messenger goes 6 miles every day; eight days after, another follows him, and he goes 10 miles every day: In what number of days will he come up to the first?

Solution.

Sup. they meet in x days.
Then $6x + 48 =$ miles travelled by the one,
and $10x =$ miles travelled by the other.
But $6x + 48 = 10x$ per quest.
$10x - 6x = 48$ by transp.
or $4x = 48$.
Th. $x = 12$.

18. A messenger goes 6 miles a-day, and after he had gone 56 miles another follows him, who goes 8 miles a-day: In how many days will he come up to him?

Solution.

Sup. they meet in x days.
Then $6x + 56 =$ miles travelled by the one,
and $8x =$ miles travelled by the other.

But $6x + 56 = 8x$ per quest.

$8x - 6x = 56$ by transp.

or $2x = 56$.

Th. $x = 28$.

19. One bought three books, whose prices were in proportion as 12, 5, 1. If the price of the first be doubled, of the second trebled, and of the third quadrupled, the sum of these products will as much exceed 10 crowns, as the sum of the prices of the greatest and middle is below 5: How much did the said books cost?

Solution.

Assume x crowns $=$ the value of the 3d book,

Then will $12x$, $5x$, and x, be their respective values.

But $24x + 15x + 4x - 10 = 5 - 17x$ per quest.

$60x = 15$ by transp.

Th. $x =$ 1s. 3d.

and $5x =$ 6s. 3d.

also $2x = 3$ crowns.

20. Suppose the number 50 were to be divided into two parts, so that the greater part being divided by y, and the less multiplied by 3, the sum of this product and the former quotient may make the same number proposed, which was 50.

Solution.

Assume $x =$ the greater,

Then $50 - x =$ the lesser.

and $\frac{x}{7} + 150 - 3x = 50$ per quest.

$x + 1050 - 21x = 350$ by mult. by 7,

$21x - x = 1050 - 350$ by transp.

or $20x = 700$.

Th. $x = 35$,

and $50 - x = 15$.

21. Let the number 20 be divided into two parts, so that the square of the lesser part being taken out of the square of the greater, may leave the very number proposed, which was 20, (or may leave the double, treble, &c. of the number proposed).

Solution.

Assume $x =$ the less part,
Then $20 - x =$ the greater.
Now, $(20 - x)^2 - x^2 = 20, 40, 60$, &c.
or $400 - 40x + x^2 - x^2 = 20, 40, 60$, &c.
By transp. $40x = 380, 360, 340$, &c.
Th. $x = 9\frac{1}{2}, 9, 8\frac{1}{2}$, &c.
and $20 - x = 10\frac{1}{2}, 11, 11\frac{1}{2}$, &c.

22. If a man gains 20 crowns a-week, how much must he spend a-week to have 500 crowns, together with the expense of 4 weeks, remaining at the year's end?

Solution.

Assume x crowns $=$ his expenditure per week.
Then $52 \times 30 - 52x = 4x + 500$ per quest.
or $1560 - 52x = 4x + 500$,
$56x = 1060$ by transp.
Th. $x = 18\frac{3}{4}$.

23. A labourer, after 40 weeks, in which he had been at work, lays up 28 crowns — the pay of three weeks, and finds that he had expended 36 crowns + the pay of 11 weeks: What pay did he receive a-week?

Solution.

Assume x crowns $=$ his pay per week,
Then in 40 weeks he earned $40x$.
Now, $28 - 3x + 36 + 11x = 40x$ per quest.
By transp. $32x = 64$.
Th. $x = 2$.

24. Two companions had got a parcel of guineas: says A to B, if you give me one of your guineas I shall have as many as you have left. Nay, replied B, if you give me one of your guineas, I shall have twice as many as you have left: How many guineas had each of them?

Solution.

Assume $x =$ A's guineas,
Then $x + 2 =$ B's.
Now, $\overline{x - 1} \times 2 = x + 3$ per quest.

$$\text{or } 2x - 2 = x + 3,$$
$$2x - x = 3 + 2 \text{ by transp.}$$
$$\text{or } x = 5.$$

25. A person bought two horses with the trappings, which cost £.100; which trappings, if laid on the first horse, both the horses would be of equal value, but if the trappings be laid on the other horse, he will be double the value of the first: How much did the said horses cost?

Solution.

First, on the supposition that the £.100 is the value of the trappings,

Sup. A cost x pounds,
and B y pounds.

$$\left.\begin{array}{l} \text{1st cond. } x + 100 = y, \\ \text{2d cond. } y + 100 = 2x. \end{array}\right\}$$

From 1st. $y = x + 100$,
From 2d. $y = 2x - 100$.
But $x + 100 = 2x - 100$.
Th. $x = 200$ by transp.
and $x + 100 = 300$.

Otherwise.

On the supposition that the £.100 is the value of the two horses with the trappings,

Assume x pounds $=$ the value of A,
and y pounds $=$ the value of the other,
also z pounds $=$ the value of the trappings.

$$\left.\begin{array}{l} \text{1st cond. } x + y + z = 100, \\ \text{2d cond. } x + z = y, \\ \text{3d cond. } \underline{y + z = 2x.} \end{array}\right\}$$

From 1st. $2y = 100$ by subt.
and $y = £.50$.
From 1st. $3x = 100$ by subt.
and $x = £.33\frac{1}{3}$.
also $z = 100 - \overline{x + y} = £.16\frac{2}{3}$.

26. A vintner has two sorts of wine, viz. A and B, which, if mixed in equal parts, a flaggon of mixed will cost 15d.; but if they be mixed in sesquialteral proportion, as if you should take two flaggons of A, as often

as you take three of B, a flaggon will cost 14d.: Required the price of each wine singly?

Solution.

Assume x pence $=$ the price of the one,
and y pence $=$ the price of the other.

$$\left.\begin{array}{l}\text{1st cond. } \dfrac{x+y}{2}=15, \\ \text{2d cond. } \dfrac{2x+3y}{5}=14.\end{array}\right\}$$

$$\begin{array}{r}\text{From 1st. } x+\ \ y=30, \\ \text{From 2d. } 2x+3y=70, \\ \text{3d mult. by 2, } \underline{2x+2y=60,} \\ \text{5th sub. from 4th. } y=10. \\ \text{Therefore } 30-y=20.\end{array}$$

27. A son asked his father how old he was: His father answered him thus, If you take away 5 from my years, and divide the remainder by 8, the quotient will be $\frac{1}{3}$ of your age; but if you add 2 to your age, and multiply the whole by 3, and then subtract 7 from the product, you will have the number of the years of my age: What was the age of the father and son?

Solution.

Assume x years $=$ the father's age,
and y years $=$ the son's.

$$\left.\begin{array}{l}\text{1st cond. } \dfrac{x-5}{8}=\dfrac{y}{3}, \\ \text{2d cond. } \overline{3y+6}-7=x,\end{array}\right\}\text{ per quest.}$$

From 1st. $x=\dfrac{8y}{3}+5.$

From 2d. $x=3y+6-7.$

But $\dfrac{xy}{3}+5=3y+6-7,$

$8y+15=9y+18-21$ by mult. by 3.

Th. $y=18$ by transp.

and $x=\dfrac{8y}{3}+5=53.$

28. To find two numbers, to the sum whereof, if you add 6, the whole shall be double of the greater, and if you subtract 2 from their difference, the remainder will be half of the least.

Solution.

Assume $x =$ the greater,
and $y =$ the lesser.

$$\left.\begin{array}{l} \text{1st cond. } x + y + 6 = 2x, \\ \text{2d cond. } x - y - 2 = \frac{y}{2}. \end{array}\right\}$$

From 1st. $x = y + 6$.

From 2d. $\frac{y}{2} = 4$ by subt.

and $y = 8$.

Th. $x = y + 6 = 14$.

29. To find two numbers, the product whereof is 240, and the triple of the greater divided by the lesser is 5.

Solution.

Assume $x =$ the greater,
and $y =$ the less.

$$\left.\begin{array}{l} \text{1st cond. } xy = 240, \\ \text{2d cond. } \frac{3x}{y} = \quad 5. \end{array}\right\}$$

From 1st. $x = \frac{240}{y}$,

From 2d. $x = \frac{5y}{3}$,

$$\frac{5y}{3} = \frac{240}{y},$$

$$5y = \frac{720}{y},$$

$$5y^2 = 720,$$

$$y^2 = 144.$$

Th. $y = \sqrt{144} = 12$,

and $\frac{240}{y} = 20$.

30. Two men had a mind to purchase a house rated at 1200 pounds; says A to B, if you give me $\frac{2}{3}$ of your money, I can purchase the house alone; but, says B to A, if you will give me $\frac{3}{4}$ of yours, I shall be able to purchase the house: How much money had each of them?

Solution.

Sup. A had x pounds,
And B y pounds,

$$\left.\begin{array}{l}\text{1st cond. } x+\dfrac{2x}{3}=1200, \\ \text{2d cond. } y+\dfrac{3x}{4}=1200,\end{array}\right\}\text{ per quest.}$$

From 2d. mult. by 4, $3x+4y=4800$,
Sub. 1st. mult. by 3, $3x+2y=3600$,
and there remains $2y=1200$,
Therefore $y=£.600$,
and by subt. $3x=2400$.
Therefore $x=£.800$.

31. A certain number of young men and maids had a reckoning to pay for a treat, being 37 crowns; and this was their condition, that every young man should pay 3 crowns, and every maid 2. Now, if the number of young men had been equal to the number of maids, observing the same conditions, the reckoning would have come to 4 crowns less than it did: How many young men and maids were there?

Solution.

Assume $x=$ the men,
And $y=$ the maids.

$$\left.\begin{array}{l}\text{1st cond. } 3x+2y=37, \\ \text{2d cond. } 2x+3y=33.\end{array}\right\}$$

By sub. $x-y=4$,
By transp. $x=y+4$.
From 2d. $2y+8+3y=33$ by subt.
By transp. $5y=25$.
Th. $y=5$,
and $x=y+4=9$.

32. A general who had fought a battle, upon reviewing his army, whose foot was thrice the number of his horse, finds that before the battle $\frac{1}{12}-120$ of his foot had deserted, and of his horse $\frac{1}{20}+120$, besides $\frac{1}{4}$ of his whole army were sent into garrison, (reckoning the sick and wounded) and $\frac{3}{8}$ of his army remained; the rest, who were wanting, being either slain or taken prisoners: Now, if you add 3000 to the number of the slain, the sum total will be equal to half the foot he had at the beginning: what were the number of each?

Solution.

Assume $x=$ the number of horse,
Then $x=$ the number of foot,
And $4x=$ the whole army,

Now, $\frac{1}{12}$ of $3x=\frac{3x}{12}=\frac{x}{4}$, & $\frac{x}{4}-120=$ foot deserted,

and $\frac{1}{20}$ of $x=\frac{x}{20}$, & $\frac{x}{20}+120=$ horse deserted,

Also $\frac{1}{4}$ of $4x=\frac{4x}{4}=x$, sent into garrison,

and $\frac{3}{8}$ of $4x=\frac{12x}{4}=\frac{3x}{2}$, remained.

Since the slain $+3000=\frac{1}{2}$ the foot, per quest.

Therefore $\frac{3x}{2}-3000=$ the slain.

Now, $\frac{x}{4}-120+\frac{x}{20}+120+x+\frac{3x}{2}+\frac{3x}{2}-3000$ $=4x$ per quest.

$$\text{Or } \frac{3x}{10}+4x-3000=4x,$$
$$3x+40x-30000=40x,$$
$$43x-40x=30{,}000,$$
$$\text{or } 3x=30{,}000 \text{ foot},$$
$$x=10{,}000 \text{ horse},$$
$$\frac{3x}{2}-3000=12{,}000 \text{ desert, or slain},$$
$$\frac{x}{4}-120\ =2380 \text{ deserters of foot},$$
$$\frac{x}{20}+120\ =620 \text{ deserters of horse}.$$

33. To divide 100 twice into two parts, so that the major part of the first division may be treble the minor part of the second division; and the major part of the second division may be double the minor part of the first.

Solution.

Assume $x =$ the minor part of the 1st division,
and $y =$ the minor part of the 2d,
Then $3y =$ the major part of the 1st,
and $2x =$ the major part of the 2d,

$$\left.\begin{array}{l}\text{1st cond. } 3y + x = 100, \\ \text{2d cond. } 2x + y = 100.\end{array}\right\}$$

From 1st. mult. by 2, $2x + 6y = 200$,
Sub. 2d. $\underline{2x + y = 100}$,
and there remains $5y = 100$.
Therefore $y = 20$,
and $3y = 60$.
Also $x = 100 - 3y = 40$,
and $2x = 80$.

34. To divide 30 twice into two parts, so that the major part of the first division, with the minor part of the second, may be 33; and the sum of the minor parts subtracted from the sum of the major, may leave 14.

Assume $x =$ the major part of the 1st division,
and $y =$ the major part of the 2d.
Then $30 - x =$ the minor part of the 1st,
and $30 - y =$ the minor part of the 2d.

$$\left.\begin{array}{l}\text{1st cond. } x + 30 - y = 33, \\ \text{2d cond. } \underline{x + y - 60 - x - y = 14}.\end{array}\right\}$$

From 1st. $x = y + 3$,
From 2d. $2x + 2y = 74$.
Th. $4y = 68$ by subt. and transp.
and $y = 17$.
Also $x = y + 3 = 20$,
and $30 - x = 10$.
Also $30 - y = 13$.

35. A man, his wife, and his son's age, make up 96 years, so that the husband's and son's years together make the wife's $+ 15$; but the wife's and the son's make husband's $+ 2$: What was the age of each?

Solution.

Assume x, y, $z=$ their respective ages.

$$\left.\begin{array}{l}\text{1st cond. } x+y+z=96,\\ \text{2d cond. } x+z=y+15,\\ \text{3d cond. } y+z=x+\ 2.\end{array}\right\}$$

From 1st. $2y+15=96$ by subt.
Th. $2y=81$ by transp.
and $y=40\frac{1}{2}$.
From 1st. $2x+2=96$ by subt.
and $x=47$.
Lastly, $z=96-x+y=8\frac{1}{2}$.

36. Three merchants from three different fairs meet together at an inn, where they reckon up their gains, and find them the sum of 780 crowns. Moreover, if you add the gain of the first and second, and subtract the gain of the third from the sum, there remains the gain of the first $+$ 82 crowns; but if you add the gain of the second and third, and from the sum subtract the gain of the first, there remains the gain of the third $-$ 43 crowns: What was the gain of each?

Assume their respective gains x, y, z.

$$\left.\begin{array}{l}\text{1st cond. } x+y+z=780,\\ \text{2d cond. } x+y-z=x+82,\\ \text{3d cond. } y+z-x=z-43.\end{array}\right\}$$

By addition $x+3y+z=x+z+819$,
By extr. $3y=819$.
Th. $y=273$.
From 3d. $x=y+43=316$,
From 2d. $z=y-82=191$.

37. Three merchants, A, B, and C, discourse thus together concerning their age: says B to A, your age added to mine is 54 years; says C to B, my age added to yours makes 78; and says A to C, my age added to yours is 72 years: What is the age of each?

Solution.

Let their respective ages be denoted by a, b, c.

$$\left.\begin{array}{l}\text{1st cond. } a+b=54,\\ \text{2d cond. } c+b=78,\\ \text{3d cond. } a+c=72.\end{array}\right\}$$

$2a + 2b + 2c = 204$ by addition,
$a + b + c = 102$ by div.
By sub. 1st from 5th, $c = 102 - 54 = 48$,
By sub. 5th from 2d, $b = 78 - c = 30$,
By sub. 7th from 1st, $a = 54 - b = 24$.

38. To find three numbers, so that the first and half of the remainder, the 2d and $\frac{1}{3}$ of the remainder, and the third and $\frac{1}{4}$ of the remainder, always make 34.

Solution.

Sup. the numbers x, y, z.

$$\left.\begin{array}{l} \text{1st cond. } x + \dfrac{y+z}{2} = 34, \\ \text{2d cond. } y + \dfrac{x+z}{3} = 34, \\ \text{3d cond. } z + \dfrac{x+y}{4} = 34, \end{array}\right\} \text{per quest.}$$

From 1st, $2x + y + z = 68$,
From 2d, $3y + x + z = 102$,
From 3d, $4z + x + y = 136$.
1st mult. by 2, $4x + 2y + 2z = 136$.
Now, their difference, $3y + 4z = 170$,
$4x + 5y + 6z = 306$ by addition.
2d sub. from 3d, $3z - 2y = 34$ remains.

From 1st difference, $z = \dfrac{170 - 3y}{4}$,

From 2d difference, $z = \dfrac{34 + 2y}{3}$.

But $\dfrac{170 - 3y}{4} = \dfrac{34 + 2y}{3}$ both being $= z$,

$510 - 9y = 136 + 8y$ by mult.
$17y = 374$ by transp.
Th. $y = 22$,

And $z = \dfrac{34 + 2y}{3} = 26$.

Also $x = 10$.

39. There are two numbers, whose sum is 56, and the lesser hath such proportion to the greater as 2 to 5: I demand the number.

Solution.

Assume $x =$ the lesser,
And $y =$ the greater.

$$\left.\begin{array}{l}\text{1st cond. } x + y = 56, \\ \text{2d cond. } 2:5::x:y.\end{array}\right\}$$

From 1st, $x = 56 - y$,

From 2d, $x = \frac{2y}{5}$.

But $\frac{2y}{5} = 56 - y$, both being $= x$,

$2y = 280 - 5y$ by mult. by 5,

$7y = 280$ by transp.

Th. $y = 40$,

And $x = 56 - y = 16$.

Otherwise.

Sup. $x =$ lesser,

Then $56 - x =$ greater.

Now, $x : 56 - x :: 2 : 5$ per quest.

$5x = 112 - 2x$ by mult. ext. and means,

$7x = 112$ by transp.

Th. $x = 16$,

And $56 - x = 40$.

Otherwise.

Sup. $x =$ lesser,

Then $\frac{5x}{2} =$ greater,

And $x + \frac{5x}{2} = 56$ per quest.

$7x = 112$ by mult. by 2.

Th. $x = 16$,

And $\frac{5x}{2} = 40$.

EXERCISES IN QUADRATIC EQUATIONS.

Quest. 1. To find a number, which being multiplied by 6, and the product subtracted from the square of the number to be found, the remainder will be 280.

Solution.

Assume $x =$ the number,

Then $x^2 - 6x = 280$ per quest.
$x^2 - 6x + 3^2 = 289$ by comp. the sq.
$x - 3 = \sqrt{289} = 17$ by evol.
$x = 17 + 3 = 20$ by transp.

2. To find a number, which being multiplied by 8, and the product added to the square of the number to be found, the sum will be 660.

Solution.

Assume $x =$ the number,
Then $x^2 + 8x = 660$ per quest.
$x^2 + 8x + 4^2 = 676$ by comp. the sq.
$x + 4 = \sqrt{676} = 26$ by evol.
$x = 26 - 4 = 22$ by transp.

3. To divide 140 into two parts, so that the product of these parts may equal the square of 56, that is, 3136.

Solution.

Assume x and $y =$ the numbers.
$\left.\begin{array}{l}\text{1st cond. } x + y = 140, \\ \text{2d cond. } xy = 3136,\end{array}\right\}$ per quest.
From 1st, $x = 140 - y$,
From 2d, $x = \dfrac{3136}{y}$.
But $140 - y = \dfrac{3136}{y}$ both being $= x$,
$140y - y^2 = 3136$ by mult. by y,
$y^2 - 140y = -3136$ by transp.
$y^2 - 140y + (70)^2 = -3136 + 4900 = 1764$,
$y - 70 = \sqrt{1364} = \pm 42$,
$y = 70 \pm 42 = 112$ or 28,
$x = 28$ or 112.

4. A set of boon companions dining at an inn, the reckoning in all came to 175s. But before the bill was paid off, two of them slunk away, and then the club of those that remained came to 10s. a man more: How many were there in company?

Solution.

Assume $x =$ the number,
Then $\dfrac{175}{x} + 10 = \dfrac{175}{x - 2}$ per quest.

$$175 + 10x = \frac{175x}{x-2} \text{ by mult. by } x,$$

$$175x + 10x^2 - 350 - 20x = 175x \text{ by mult. by } x-2,$$

$$10x^2 - 20x = 350 \text{ by transp. and extr.}$$

$$x^2 - 2x = 35 \text{ by div.}$$

$$x^2 - 2x + 1 = 36 \text{ by comp. the sq.}$$

$$x - 1 = \sqrt{36} = 6 \text{ by evol.}$$

$$x = 6 + 1 = 7 \text{ by transp.}$$

5. A man buys some ells of cloth for 70 crowns, and finds, that if he had 4 ells morc, he had then bought every ell 2 crowns cheaper: How many ells did he buy?

Solution.

Sup. $x =$ ells he bought,

Then $\frac{70}{x+4} = \frac{70}{x} - 2$ per quest.

$$70 = \frac{70x + 280}{x} - 2x + 8 \text{ by mult. by } x + 4,$$

$$70x = 70x + 280 - 2x^2 + 8x \text{ by mult. by } x,$$

$$2x^2 + 8x = 280 \text{ by transp. and exterm.}$$

$$x^2 + 4x = 140 \text{ by div.}$$

$$x^2 + 4x + 2^2 = 144 \text{ by comp. sq.}$$

$$x + 2 = \sqrt{144} = 12,$$

$$x = 12 - 2 = 10 \text{ by transp.}$$

6. To divide the number 21 into two parts, so that if the greater be divided by the less, and again the less by the greater, and then the first quotient being multiplied by 4, and the latter by 25, the numbers produced may be equal.

Solution.

Assume $x =$ the greater,

And $y =$ the lesser.

$$\left.\begin{array}{l} \text{1st cond. } x + y = 21, \\ \text{2d cond. } \frac{4x}{y} = \frac{25y}{x}, \end{array}\right\} \text{per quest.}$$

From 1st, $x = 21 - y$,

From 2d, $x^2 = \frac{25y^2}{4}$,

$$\frac{25y^2}{4} = (21 - y)^2,$$

$$\frac{25y^2}{4} = 441 - 42y + y^2 \text{ by squaring,}$$
$$25y^2 = 1764 - 168y + 4y^2,$$
$$21y^2 + 168y = 1764 \text{ by transp.}$$
$$y^2 + 8y = 84 \text{ by div.}$$
$$y^2 + 8y + 4^2 = 100 \text{ by comp. the sq.}$$
$$y + 4 = \sqrt{100} = 10 \text{ by evol.}$$
$$y = 10 - 4 = 6 \text{ by transp.}$$
$$x = 21 - y = 15.$$

7. A man buys a horse, which he sells again for 56 crowns, and gains as many crowns in 100 as the horse cost him: How much did he give for the horse?

Solution.

Assume x crowns $=$ the prime cost,
And y crowns $=$ the gain.

$$\left.\begin{array}{l}\text{1st cond. } x + y = 56, \\ \text{2d cond. } x : y :: 100 : x.\end{array}\right\}$$

From 1st, $x = 56 - y$,
From 2d, $x^2 = 100y$.
But $(56 - y)^2 = 100y$ being $= x^2$,
Or $3136 - 112y + y^2 = 100y$ by squaring,

$$y^2 - 212y = -3136 \text{ by transp.}$$
$$y^2 - 212y + (106)^2 = -3136 + 11236 = 8100,$$
$$y - 106 = \sqrt{8100} = 90,$$
$$y = 106 - 90 = 16,$$
$$x = 56 - y = 40.$$

Otherwise.

Assume $x =$ the prime cost,
Then $56 - x =$ the gain.
Now, $x : 56 - x :: 100 : x$ per quest.

$$x^2 = 5600 - 100x \text{ by mult. ext. and means,}$$
$$x^2 + 100x = 5600 \text{ by transp.}$$
$$x^2 + 100x + (50)^2 = 8100 \text{ by comp. the sq.}$$
$$x + 50 = \sqrt{8100} = 90,$$
$$x = 90 - 50 = 40.$$

8. A certain linen-draper buys two sorts of linen for 30 crowns, one finer, the other coarser. An ell of the finest cost as many crowns as he had ells; and also 28 ells of the coarsest at such a price, that 8 ells cost

as many crowns as one ell of the finest: How many ells of the finest linen did he buy, and what price did he give for them both?

Solution.

Sup. an ell of the finer cost x crowns,

Then an ell of the coarser cost $\frac{x}{8}$ crowns.

Also the price of the finer x^2 crowns,

And of the coarsest $\frac{28x}{8}=\frac{7x}{2}$.

Now, $x^2+\frac{7x}{2}=30$ per quest.

$x^2+\frac{7x}{2}+(\frac{7}{4})^2=30+\frac{49}{16}=\frac{529}{16}$ by comp. the sq.

$x+\frac{7}{4}=\sqrt{\frac{529}{16}}=\frac{23}{4}$ by evol.

$x=\frac{23}{4}-\frac{7}{4}=\frac{16}{4}=4$ by transp.

$x^2=16,$

$\frac{x}{8}=$ 2s. 6d.

9. Of three proportional numbers, there is the middle term given $=12$, and the difference of the extremes $=10$: Required the extremes.

Solution.

Sup. $x=$ the first,

Then $x:12::12:x\pm10,$

$x^2\pm10x=144$ by mult. ext. and means,

$x^2\pm10x+5^2=169$ by comp. the sq.

$x\pm5=\sqrt{169}=\pm13$ by evol.

$x=5\pm13=8$ or 18.

So the numbers are 8, 12, 18,

Or 18, 12, 8.

10. Of three proportional numbers there is given the sum of the first and second $=10$, and the difference of the second and third $=24$: Required the several numbers.

Solution.

Sup. the 1st, x,

Then the 2d, $10-x$,

And the 3d, $34 - x$.
Now, $x : 10 - x :: 10 - x : 34 - x$,
$100 - 20x + x^2 = 34x - x^2$,
$2x^2 - 54x = -100$,
$x^2 - 27x = -50$,
$x^2 - 27x + (\frac{27}{2})^2 = -50 + \frac{729}{4} = \frac{529}{4}$,
$x - \frac{27}{2} = \sqrt{\frac{529}{4}} = \frac{23}{2}$,
$x = \frac{27}{2} - \frac{23}{2} = \frac{4}{2} = 2$,
$10 - x = 8$,
$34 - x = 32$.

11. Of four proportional numbers there is given the third $= 12$, also the sum of the first and second $= 8$; besides the second number being subtracted from its square, the remainder is to be the fourth. Required the said numbers.

Solution.

Sup. the 2d, $= x$,
Then the 1st will be $= 8 - x$,
and the 4th, $= x^2 - x$.
Now, $8 - x : x :: 12 : x^2 - x$,
$8x^2 - x^3 - 8x + x^2 = 12x$,
$9x^2 - x^3 - 8x = 12x$,
$9x - x^2 - 8 = 12$,
$x^2 - 9x = -20$,
$x^2 - 9x + (\frac{9}{2})^2 - 20 + \frac{81}{4} = \frac{1}{4}$,
$x - \frac{9}{2} = \sqrt{\frac{1}{4}} = \frac{1}{2}$,
$x = \frac{9}{2} \pm \frac{1}{2} = 5$ or 4,
So the numbers are 3, 5, 12, 20,
or 4, 4, 12, 12.

12. Of four proportional numbers in continued proportion, there is given the sum of the means $= 24$, and likewise the sum of the extremes $= 56$: Required the said numbers, (supposing that the first is the least of all).

Solution.

Assume $x =$ the first,
and $y =$ the second.
Then $x : y :: 24 - y : 56 - x$.
From the three first, $24x - xy = y^2$,

and from the three last, $56y - xy = (24 - y)^2$.

$$x = \frac{y^2}{24 - y} = \frac{104y - y^2 - 576}{y},$$

$$y^3 = 2496y - 128y^2 - 13824 + 576y + y^3,$$

$128y^2 - 3072y = -13824$ by transp. and exterm.

$y^2 - 24y = -108$ by div. by 128.

$$y^2 - 24y + (12)^2 = -108 + 144 = 36,$$

$y - 12 = \sqrt{36} = \pm 6$ by evol.

Th. $y = 12 \pm 6 = 18$ or 6,

and $x = \dfrac{y^3}{24 - y} = 54$ or 2.

So the numbers are 54, 18, 6, 2,
or 2, 6, 18, 54.

13. Two country-women, A and B, carry 100 eggs together to market, and in the sale of them one took as much money as the other; but A (who had the largest and consequently the best eggs) says to B, Had I carried as many eggs as you, I should have had 18 pence for them; B replies, If I had brought as many eggs as you, I should have had but 8 pence for them: How many eggs had each?

Solution.

Sup. A had x eggs at y pence,
Then B had $100 - x$ at y pence.

$$\left.\begin{array}{l}\text{1st cond. } x : y :: 100 - x : 18, \\ \text{2d cond. } 100 - x : y :: x : 8.\end{array}\right\} \text{per quest.}$$

From 1st, $18x = 100y - xy$,
From 2d, $800 - 8x = xy$.

$$800 + 10x = 100y,$$

$$\text{or } 80 + x = 10y,$$

$$\text{Th. } \frac{80 + x}{10} = y.$$

$$\text{From 3d, } \frac{800 - 8x}{x} = y.$$

$$\text{But } \frac{x + x}{10} = \frac{800 - 8x}{x},$$

$$80 + x = \frac{8000 - 80x}{x},$$

$$80x + x^2 = 8000 - 80x,$$
$$x^2 + 160x = 8000 \text{ by transp.}$$
$$x^2 + 160x + (80)^2 = 8000 + 6400 = 14400,$$
$$x + 80 = \sqrt{14400} = 120,$$
$$x = 120 - 80 = 40, \text{ the eggs A had,}$$
$$y = \frac{80 + 40}{10} = 12 \text{ pence each received,}$$
$$\text{and } 100 - x = 60, \text{ the eggs B had.}$$

14. Two country-men, A and B, sell their corn at different prices: A sells 20 bushels; and B received for one bushel as many crowns as he sold bushels: A perceives that if he had sold as many bushels as B received crowns, he should then have received 252 crowns; but both together received 176 crowns: How many bushels did B sell, and what price had A?

Solution.

Sup. A sold at x shillings per bushel,
and B sold y bushels.
Then A received $20x$ shillings,
and B y^2 shillings.

$$\left.\begin{array}{l}\text{1st cond. } 20x + y^2 = 176, \\ \text{2d cond. } xy^2 = 252.\end{array}\right\}$$

$$\text{From 1st, } x = \frac{176 - y^2}{20},$$

$$\text{From 2d, } x = \frac{252}{y^2}.$$

$$\text{But } \frac{176 - y^2}{20} = \frac{252}{y^2} \text{ both being } = x,$$

$$176 - y^2 = \frac{5040}{y^2} \text{ by mult. by 20,}$$

$$176y^2 - y^4 = 5040 \text{ by mult. by } y^2,$$
$$y^4 - 176y^2 = -5040 \text{ by transp.}$$
$$\text{Comp. sq. } y^4 - 176y^2 + (88)^2 = -5040 + 7744 = 2704,$$
$$y^2 - 88 = \sqrt{2704} = 52 \text{ by evolution.}$$
$$y^2 = 88 - 52 = 36 \text{ by transp.}$$
$$y = \sqrt{36} = 6 \text{ by evolution,}$$
$$\text{Th. } x = \frac{252}{y^2} = \frac{252}{36} = 7.$$

14. Two merchants sell 21 ells of cloth: The first sells 1 ell for as many crowns as is $\frac{1}{5}$ of the number of ells that the second had; and the second sells 1 ell for as many crowns as is $\frac{1}{3}$ of the number of the ells that the first had. The sale being over, they had taken 48 crowns in all. How many ells did each sell, and at what price?

Solution.

Sup. the 1st sold x ells for y crowns,
Then the 2d sold $21-x$ ells for $48-y$ crowns;
and 1 ell of the 1st cost $\frac{x}{y}$ crowns,
also 1 ell of the 2d cost $\frac{48-y}{21-x}$ crowns.

$$\left.\begin{array}{l}\text{1st cond. } \dfrac{y}{x}=\dfrac{21-x}{5}, \\ \text{2d cond. } \dfrac{48-y}{21-x}=\dfrac{x}{3},\end{array}\right\}\text{per quest.}$$

From 1st, $y=\dfrac{21x-x^2}{5}$,

From 2d, $y=18$.

But $\dfrac{21x-x^2}{5}=18$.

$21x-x^2=90$ by mult. by 5,
$x^2-21x=-90$ by transp.
$x^2-21x+(\frac{21}{2})^2=-90+\frac{441}{4}=\frac{81}{4}$,

$$x-\tfrac{21}{2}=\sqrt{\tfrac{81}{4}}=\tfrac{9}{2},$$
$$x=\tfrac{21}{2}-\tfrac{9}{2}=\tfrac{12}{2}=6 \text{ ells},$$
$$\frac{y}{x}=\tfrac{18}{6}=3 \text{ crowns},$$
$$21-x=15 \text{ ells},$$
$$\frac{48-y}{21-x}=\frac{48-18}{21-6}=\tfrac{30}{15}=2 \text{ crowns}.$$

16. Two merchants have a parcel of silk, the first 40 ells, the second 90: the first sells for a crown $\frac{1}{3}$ of an ell more than the second. When the sale was over, they had taken between them 42 crowns. How many ells did each of them sell for a crown?

Solution.

Sup. B sold x ells for a crown,
Then A sold $x+\frac{1}{3}$ ells for a crown.

Now, $\dfrac{40}{x+\frac{1}{3}}+\dfrac{90}{x}=42$ per quest.

$$40x+90x+30=42x^2+14x,$$
$$42x^2-116x=30 \text{ by transp.}$$
$$21x^2-58x=15,$$
$$x^2-\frac{58x}{21}=\tfrac{15}{21},$$
$$x^2-\frac{58x}{21}+(\tfrac{29}{21})^2=\tfrac{15}{21}+\tfrac{841}{441},$$
$$x-\tfrac{29}{21}=\sqrt{\tfrac{1156}{441}}=\tfrac{34}{21},$$
$$x=\frac{29+34}{21}=\tfrac{63}{21}=3 \text{ ells.}$$

Then A sold $x+\frac{1}{3}=3\frac{1}{3}$ ells.

17. To find a number, to the quadruple of which if you add 91, the whole shall be to the square of the number sought as 3 to 4.

Solution.

Assume $x=$ the number.
Then $4x+91:x^2::3:4$ per quest.

$$16x+364=3x^2,$$
$$3x^2-16x=364,$$
$$x^2-\frac{16x}{3}=\tfrac{364}{3},$$
$$x^2-\frac{16x}{3}+(\tfrac{8}{3})^2=\tfrac{64}{9}+\tfrac{1092}{9}=\tfrac{1156}{9},$$
$$x-\tfrac{8}{3}=\sqrt{\tfrac{1156}{9}}=\tfrac{34}{3},$$
$$x=\tfrac{34}{3}+\tfrac{8}{3}=\tfrac{42}{3}=14.$$

18. To find a number, from the double of which if you subtract 12, the square of the remainder, less 1, will be 9 times the number sought.

Solution.

Assume $x=$ the number.
Then $(2x-12)^2-1=9x$ per quest.

$$4x^2-48x+144-1=9x,$$

$$4x^2 - 57x = -143,$$
$$x^2 - \frac{57x}{4} = -\tfrac{143}{4},$$
$$x^2 - \frac{57x}{4} + (\tfrac{57}{8})^2 = -\tfrac{143}{4} + \tfrac{3249}{64} = \tfrac{961}{64},$$
$$x - \tfrac{57}{8} = \sqrt{\tfrac{961}{64}} = \pm \tfrac{31}{8},$$
$$x = \tfrac{57}{8} \pm \tfrac{31}{8} = 11 \text{ or } 3\tfrac{1}{4}.$$

19. To divide the number 19 into two parts, so that the sum of the square of the parts will be 193.

Solution.

Sup. The parts x and y.

$$\left.\begin{array}{l}\text{1st cond. } x + y = 19, \\ \text{2d cond. } x^2 + y^2 = 193.\end{array}\right\}$$

Now, $(19 - y)^2 + y^2 = 193$ by subt.

or $361 - 38y + y^2 + y^2 = 193$ by sq.

$2y^2 - 38y = -168$ by transp.

$y^2 - 19y = -84$ by div. by 2,

$$y^2 - 19y + (\tfrac{19}{2})^2 = -84 + \tfrac{361}{4} = \tfrac{25}{4},$$
$$y - \tfrac{19}{2} = \sqrt{\tfrac{25}{4}} = \pm \tfrac{5}{2},$$
$$y = \tfrac{19}{2} \pm \tfrac{5}{2} = 12 \text{ or } 7,$$
$$x = 19 - y = 7 \text{ or } 12.$$

Otherwise.

Assume $x =$ the one part,

Then $19 - x =$ the other.

Now, $x^2 + (19 - x)^2 = 193$ per quest.

Or $x^2 + 361 - 38x + x^2 = 193$,

$2x^2 - 38x = -168$ by transp.

$x^2 - 19x = -84$ by div. by 2,

$$x^2 - 19x + (\tfrac{19}{2})^2 = -84 + \tfrac{361}{4} = \tfrac{25}{4},$$
$$x - \tfrac{19}{2} = \sqrt{\tfrac{25}{4}} = \pm \tfrac{5}{2},$$
$$x = \tfrac{19}{2} \pm \tfrac{5}{2} = 12 \text{ or } 7,$$
$$19 - x = 7 \text{ or } 12.$$

20. To divide 7 into two parts, so that the difference of the squares, which are made from the triple of the lesser part, and the double of the greater, may be 17?

Solution.

Sup. the lesser x, and greater y.

1st cond. $x+y=7,$
2d cond. $9x^2-4y^2=17.$ $\Big\}$

From 1st, $x=7-y$.

Now, $(7-y)^2\times 9-4y^2=17$ by subt.

Or $441-126y+9y^2-4y^2=17$,

$$5y^2-126y=-424 \text{ by transp.}$$

$$y^2-\frac{126y}{5}+\left(\frac{63}{5}\right)^2=-\frac{424}{5}+\frac{3969}{25},$$

$$y-\tfrac{63}{5}=\sqrt{\tfrac{1849}{25}}=\tfrac{43}{5},$$

$$y=\tfrac{63}{5}-\tfrac{43}{5}=\tfrac{20}{5}=4,$$

$$x=7-y=3.$$

Otherwise.

Assume $x=$ the lesser,

Then $7-x=$ the greater,

And $9x^2-(7-x)^2\times 4=17$.

Or $9x^2-196+56x-4x^2=17$ by sq.

$5x^2+56x=213$ by transp.

$$x^2+\frac{56x}{5}=\frac{213}{5},$$

$$x^2+\frac{56x}{5}+\left(\frac{28}{5}\right)^2=\frac{784}{25}+\frac{213}{5}=\frac{1849}{25},$$

$$x+\frac{28}{5}=\sqrt{\frac{1849}{25}}=\frac{43}{5},$$

$$x=\frac{43-28}{5}=\frac{15}{5}=3,$$

$$7-x=7-3=4.$$

21. A man buys a piece of linen, and by selling it again, he gains 12 crowns $-\frac{1}{10}$ of what he bought it for; and finds by this means that he had gained as much for 100 crowns as the linen cost him: What price was the linen bought and sold at?

Solution.

Sup. x crowns $=$ prime cost,

Then $x+12-\dfrac{x}{10}=$ sum sold for.

Now, $x:12-\dfrac{x}{10}::100:x$,

$$x^2 = 1200 - 10x \text{ by mult. ext. and means,}$$
$$x^2 + 10x = 1200 \text{ by transp.}$$
$$x^2 + 10x + 5^2 = 1225 \text{ by comp. the sq.}$$
$$x + 5 = \sqrt{1225} = 35 \text{ by evol.}$$
$$x = 35 - 5 = 30 \text{ by transp.}$$

Th. $x + 12 - \frac{x}{10} = 39.$

22. A man buys 18 ells of cloth of different sorts and colours, suppose red and black; what he bought of each cost 40 crowns; and he pays for every ell of red cloth 1 crown more than for the black: How many ells of each did he buy?

Solution.

Sup. he bought x ells of red,
Then $18 - x =$ ells of black,

And $\frac{40}{x}$ crowns $=$ an ell of red.

Also $\frac{40}{18 - x}$ crowns $=$ an ell of black.

$$\text{Now, } \frac{40}{x} = \frac{40}{18 - x} + 1 \text{ per quest.}$$
$$40 = \frac{40x}{18 - x} + x,$$
$$720 - 40x = 40x + 18x - x^2,$$
$$x^2 - 98x = -720 \text{ by transp.}$$
$$x^2 - 98x + (49)^2 = -720 + 2401 = 1681,$$
$$x - 49 = \sqrt{1681} = 41,$$
$$x = 49 - 41 = 8,$$
$$18 - x = 18 - 8 = 10.$$

23. A man buys 120 pounds of pepper, and as many of ginger; and received for a crown one pound of ginger more than of pepper: So that the whole price of the pepper came to 6 crowns more than the price of ginger: How many pounds of each did he buy for a crown?

Solution.

Assume x crowns $=$ the price of the ginger,
Then $x + 6$ crowns $=$ the price of the pepper.

$$\overset{\text{cr.}}{x} : \overset{\text{lib.}}{120} :: \overset{\text{cr.}}{1} : \frac{120}{x} = \text{libs. of ginger for 1 crown,}$$

$$x+6 : 120 :: 1 : \frac{120}{x+6} = \text{libs. of pepper for 1 crown.}$$

Now, $\frac{120}{x} = \frac{120}{x+6} + 1$ per quest.

$$120x + 720 = 120x + x^2 + 6x,$$

$$x^2 + 6x = 720 \text{ by exter.}$$

$$x^2 + 6x + 3^2 = 729 \text{ by comp. the sq.}$$

$$x + 3 = \sqrt{729} = 27,$$

$$x = 27 - 3 = 24,$$

$$x + 6 = 30,$$

$$\frac{120}{24} = 5 \text{ libs. of ginger for 1 crown,}$$

$$\frac{120}{30} = 4 \text{ libs. of pepper for 1 crown,}$$

24. A man buys 80 pounds of pepper, and 36 pounds of saffron, so that for 8 crowns he had 14 pounds of pepper more than he had of saffron for 26 crowns, and what he laid out amounted to 188 crowns: How many pounds of pepper had he for 8 crowns, and how many of saffron for 26?

Solution.

Sup. x pounds of pepper for 8 crowns,
And y pounds of saffron for 26 crowns.

Then $\overset{\text{lib.}}{x} : \overset{\text{crs.}}{8} :: \overset{\text{lib.}}{80} : \overset{\text{crs.}}{\frac{640}{x}}$ = cost of pepper,

And $y : 26 :: 36 : \frac{936}{y}$ = cost of saffron.

But $\frac{936}{y} = \frac{640}{x} = 188$ per quest.

$$936x + 640y = 188xy \text{ by mult.}$$

$$234x + 160y = 47xy \text{ by div. by 4,}$$

$$47xy - 234x = 160y \text{ by transp.}$$

$$\text{Th. } x = \frac{160y}{47y - 234}.$$

Also $x=y+14=\frac{160y}{47y-234}$,

$47y^2+424y-3276=160y$,

$47y^2+264y=3276$ by transp.

$$y^2+\frac{264y}{47}+\left(\frac{132}{47}\right)^2=\frac{3276}{47}+\frac{17424}{2209}=\frac{171396}{2209},$$

$$y+\frac{132}{47}=\sqrt{\frac{171396}{2209}}=\frac{414}{47},$$

$$y=\frac{132+414}{47}=\frac{282}{47}=6,$$

$$x=y+14=6+14=20.$$

25. A and B between them owe 174 pounds; A pays 8 pounds a-day, and B pays the first day 1 pound, the second 2, the third 3, and so on: In how many days will they clear the debt, and how much did each of them owe?

Solution.

Sup. in x days; and since A's payments in x days are a series of pounds in arithmetical progression, whereof the number of terms is x, the greatest term x, and the least term 1.

Now, $\overline{x+1}\times\frac{x}{2}=\frac{x^2+x}{2}=$ A's payment in x days,

Also $8x=$ the pounds B paid.

But $\frac{x^2+x}{2}+8x=174$ per quest.

$x^2+x+16x=348$ by mult. by 2,

Or $x^2+17x=348$,

$$x^2+17x+\left(\frac{17}{2}\right)^2=348+\frac{289}{4}=\frac{1681}{4}.$$

$$x+\frac{17}{2}=\sqrt{\frac{1681}{4}}=\frac{41}{2},$$

$$x=\frac{41-17}{2}=\frac{24}{2}=12,$$

$$\frac{x^2+x}{2}=\frac{(12)^2+12}{2}=\frac{144+12}{2}=£.78.$$

$$8x=12\times8=£.96.$$

26. A certain man intends to travel as many days as he has crowns: It happens that every following day of his journey he had as many crowns as he had the day before, besides 2 crowns over and above; and when he came to his journey's end he finds he had in all 45 crowns: How many crowns had he at first?

Solution.

Sup. $x =$ crowns he had.

Then the sum of an arithmetical progression, whose first term is x, common difference 2, and number of terms x is $2x^2 - x$.

Now, $2x^2 - x = 45$ per quest.

$$x^2 - \tfrac{1}{2}x = \frac{45}{2}.$$

$$x^2 - \tfrac{1}{2}x + \frac{1}{16} = \frac{45}{2} + \frac{1}{16} = \frac{361}{16},$$

$$x - \tfrac{1}{4} = \sqrt{\frac{361}{16}} = \frac{19}{4},$$

$$x = \frac{19+1}{4} = 5.$$

27. A certain traveller goes 9 miles a-day; three days after, another follows him, who the first day travels 4 miles, the second 5, and the third 6, and so on, gaining a mile every day: In what time will he overtake the former?

Solution.

Sup. in x days.

Then the first travels $\overline{x+3} \times 9 = 9x + 27$ miles, and the second travels the sum of an arithmetical progression increasing, whose first term is 4, common difference 1, and number of terms $x = 4x + \frac{x^2 - x}{2} \times 1$ $= \frac{8x + x^2 - x}{2}.$

But $\frac{8x + x^2 - x}{2} = 9x + 27,$

$$8x + x^2 - x = 18x + 54 \text{ by mult. by } 2,$$

$$x^2 - 11x = 54 \text{ by transp.}$$

$$x^2 - 11x + \left(\frac{11}{2}\right)^2 = 54 + \frac{121}{4} = \frac{216 + 121}{4} = \frac{337}{4},$$

$$x - \frac{11}{2} = \sqrt{\frac{337}{4}} = \frac{18\cdot3578}{2},$$

$$x = \frac{18\cdot3578}{2} + \frac{11}{2} = 14\cdot6789 \text{ days.}$$

28. Two travellers set out at the same time from two cities, the one from A, and the other from B, which are 170 miles distant from one another: one of them goes 6 miles every day; and the other 2 miles the first day, $2\frac{1}{2}$ the second, 3 the third, and so on, adding $\frac{1}{2}$ a mile to every day's journey: In what time will they meet with one another?

Solution.

Sup. in x days.

Then the 1st goes $6x$ miles,

And the 2d, $2x + \frac{x^2 - x}{2} \times \frac{1}{2} = \frac{8x + x^2 - x}{4}$,

But $\frac{8x + x^2 - x}{4} + 6x = 70$ per quest.

$8x + x^2 - x + 24x = 280$ by mult. by 4,

$x^2 + 31x = 280$ by transp.

$$x^2 + 31x + (\tfrac{31}{2})^2 = 280 + \frac{961}{4} = \frac{961 + 1120}{4} = \frac{2081}{4},$$

$$x + \frac{31}{2} = \sqrt{\frac{2081}{4}} = \frac{45\cdot618}{2},$$

$$x = \frac{45\cdot618}{2} - \frac{31}{2} = 7\cdot309 \text{ days.}$$

29. Again, Two travellers set out at the same time from two cities, the one from A, and the other from B, which are 120 miles distant from one another; the first goes 5 miles a-day, and the other 3 miles less than the number of days in which they meet: When will they meet?

Solution.

Sup. x days = the time of meeting.

Then the one travels $5x$ miles,

and the other $\overline{x - 3} \times x = x^2 - 3x$ miles.

But $x^2 - 3x + 5x = 120$ per quest.

or $x^2 + 2x = 120$,

$x^2 + 2x + 1 = 121$ by comp. the sq.

$x + 1 = \sqrt{121} = 11$,

$x = 11 - 1 = 10$ by transp.

30. A post sets out from A towards B, who travels 8 miles a-day: after he had gone 27 miles, another sets out from B to meet him, who goes every day $\frac{1}{20}$ of the whole journey or distance of the places A and B, and meets the first post after so many days as is $\frac{1}{20}$ of the said distance. Required the distance of A and B?

Solution.

Sup. the distance x miles.

Then will the time they meet be $\frac{x}{20}$,

and the first will have travelled $27 + \frac{2x}{5}$ miles,

and the other $\frac{x}{20} \times \frac{x}{20} = \frac{x^2}{400}$ miles.

But $27 + \frac{2x}{5} + \frac{x^2}{400} = x$ per quest.

$10800 + 160x + x^2 = 400x$,

$x^2 - 240x = -10800$,

$x^2 - 240x + (120)^2 = -10800 + 14400 = 3600$,

$x - 120 = \sqrt{3600} = 60$,

$x = 60 + 120 = 180$.

31. Two merchants, A and B, go partners; B brings 420 crowns, and A receives out of the gain 52 crowns, and the sum of both their shares is 854 crowns. How much did A bring, and how much did B receive out of the gains?

Solution.

Sup. A's stock $= x$ crowns.

Then A's stock : A's gain : : B's stock : B's gain.

That is, $x : 52 :: 420 : \frac{21840}{x}$.

But $x + 52 + 420 + \frac{21840}{x} = 854$ per quest.

$$x^2 + 52x + 420x + 21840 = 854x,$$
$$x^2 - 382x = -21840,$$
$$x^2 - 382x + (191)^2 = -21840 + 36481 = 14641,$$
$$x - 191 = \sqrt{14641} = 121,$$
$$x = 121 + 191 = 312, \text{ A's stock,}$$
$$\frac{21840}{x} = 70, \text{ B's gain.}$$

32. A son asked his father how old he was? his father replied thus, If you take 4 from my age, the remainder will be thrice the number of your years; but if you take 1 from your age, half the remainder will be the square root of my age. Required the age of the father and son?

Solution.

Assume x years $=$ the father's age,
and y years $=$ the son's.

$$\left.\begin{array}{l} \text{1st cond. } x - 4 = 3y, \\ \text{2d cond. } \dfrac{y-1}{2} = \sqrt{x}, \end{array}\right\} \text{per quest.}$$

From 1st, $x = 3y + 4$,

From 2d, $\left(\dfrac{y-1}{2}\right)^2 = x$;

or $\dfrac{y^2 - 2y + 1}{4} = x$, or $3y + 4$,

$y^2 - 2y + 1 = 12y + 16$ by mult. by 4,

$y^2 - 14y = 15$ by transp.

$y^2 - 14y + 7^2 = 64$ by comp. the sq.

$y - 7 = \sqrt{64} = 8,$

$y = 8 + 7 = 15$ by transp.

$x = 3y + 4 = 49.$

33. To find two numbers, the sum of whose squares may be 317, and the product, if they be multiplied by one another, may be 154.

Solution.

Sup. the two numbers x and y.

$$\left.\begin{array}{l} \text{1st cond. } x^2 + y^2 = 317, \\ \text{2d cond. } xy = 154, \end{array}\right\} \text{per quest.}$$

From 2d, $x=\frac{154}{y}$.

Now, $\left(\frac{154}{y}\right)^2+y^2=317$ by subt.

$$\text{or } \frac{23716}{y^2}+y^2=317 \text{ by sq.}$$

$$23716+y^4=317y^2 \text{ by mult. by } y^2,$$

$$y^4-317y^2=-23716 \text{ by transp.}$$

$$y^4-317y^2+\left(\frac{317}{2}\right)^2-23716+\frac{100489}{4},$$

$$y^2-\frac{317}{2}=\sqrt{\frac{5625}{4}}=\frac{75}{2},$$

$$y^2=\frac{317-75}{2}=\frac{242}{2}=121,$$

$$y=\sqrt{121}=11,$$

$$x=\frac{154}{y}=\frac{154}{11}=14.$$

34. To find two numbers, whose product may be 108, and the difference of their squares 63.

Solution.

Sup. the two numbers x and y.

$$\left.\begin{array}{r}\text{1st cond. } xy=108,\\ \text{2d cond. } x^2-y^2=63,\end{array}\right\}\text{ per quest.}$$

$$\text{From 1st, } x=\frac{108}{y}.$$

Now, $\left(\frac{108}{y}\right)^2-y^2=63$,

$$\text{or } \frac{11664}{y^2}-y^2=63 \text{ by sq.}$$

$$11664-y^4=63y^2 \text{ by mult. by } y^2,$$

$$y^4+63y^2=11664 \text{ by transp.}$$

$$y^4+63y^2+\left(\frac{63}{2}\right)^2=11664+\frac{3969}{4}=\frac{50625}{4},$$

$$y^2+\frac{63}{2}=\sqrt{\frac{50625}{4}}=\frac{225}{2},$$

$$y^2 = \frac{225 - 63}{2} = \frac{162}{2} = 81,$$
$$y = \sqrt{81} = 9,$$
$$x = \frac{108}{y} = \frac{108}{9} = 12.$$

35. Two farmers sell two sorts of corn: A sells 6 bushels; B receives in all for his 20 crowns. Now, says B to A, If we add the number of my bushels to the number of your crowns, the sum will be 28. Says A to B, And if I add the square of my crowns to the square of your bushels, the sum will be 424. How many bushels did B sell, and how many crowns did A receive?

Solution.

Sup. A received x crowns,
and B sold y bushels.

$$\left.\begin{array}{r} \text{1st cond. } x + y = 28, \\ \text{2d cond. } x^2 + y^2 = 424, \end{array}\right\} \text{per quest.}$$

From 1st, $x = 28 - y$.

Now, $(28 - y)^2 + y^2 = 424$ by subt.

$784 - 56y + 2y^2 = 424$ by sq.

$2y^2 - 56y = -360$ by transp.

$y^2 - 28y = -180$ by div. by 2,

$y^2 - 28y + (14)^2 = -180 + 196 = 16,$

$y - 14 = \sqrt{16} = 4,$

$y = 4 + 14 = 18$ crowns,

$x = 28 - y = 10$ bushels.

36. To find two numbers, the first of which $+ 2$, multiplied into the second $- 3$, may produce 110; and, on the contrary, the first $- 3$, multiplied by the second $+ 2$, may produce 80.

Solution.

Sup. $x =$ the first,
and $y =$ the second.

$$\left.\begin{array}{r} \text{1st cond. } (x + 2)(y - 3) = 110, \\ \text{2d cond. } (x - 3)(y + 2) = 80. \end{array}\right\} \text{per quest.}$$

From 1st, $xy + 2y - 3x - 6 = 110,$

From 2d, $xy - 3y + 2x - 6 = 80,$

$$5y - 5x = 30,$$
$$y - x = 6,$$
$$y = x + 6.$$

Now, $(x + 2)(x + 6 - 3)$, or $(x + 2)(x + 3) = 110$ by substitution,

or $x^2 + 5x + 6 = 110$,

$x^2 + 5x = 104$ by transp.

$$x^2 + 5x + \left(\frac{5}{2}\right)^2 = 104 + \frac{25}{4} = \frac{416 + 25}{4} = \frac{441}{4},$$

$$x + \frac{5}{2} = \sqrt{\frac{441}{4}} = \frac{21}{2},$$

$$x = \frac{21 - 5}{2} = \frac{16}{2} = 8.$$

$$y = x + 6 = 14.$$

37. A general disposing his army into a square battalion, finds he has 284 soldiers over and above; but increasing each side with one soldier, he wants 25 soldiers to fill up the square: How many soldiers had he?

Solution.

Suppose $x =$ the side of the square.

Then $x^2 + 284 = (x + 1)^2 - 25$ per quest.

or $x^2 + 284 = x^2 + 2x + 1 - 25$,

$284 = 2x + 1 - 25$ by exter. x^2,

$2x = 308$ by transp.

$x = 154$ dividing by 2.

Therefore $154 \times 154 + 284 = 24000$, the number of soldiers.

38. Let 969 soldiers be drawn up into an oblong battalion, so that the difference of the greater and lesser sides is 40. Required the number of the soldiers in each rank, in length and breadth?

Solution.

Sup. $x =$ the number in length,

$y =$ the number in breadth.

Then $\left.\begin{aligned} xy &= 969, \\ x - y &= 40, \end{aligned}\right\}$ per question,

$x^2 - 2xy + y^2 = 1600$ the 2d squared,

$4xy = 3876$ the 1st mult. by 4,

$$x^2 + 2xy + y^2 = 5476 \text{ by addition,}$$
$$x + y = \sqrt{5476} = 74 \text{ by evol.}$$
$$x - y = 40 \text{ the 2d equa.}$$
$$2x = 114 \text{ by addition,}$$
$$x = 57 \text{ divid. by 2,}$$
$$\text{and } 2y = 34 \text{ by subt.}$$
$$y = 17 \text{ by division.}$$

So that the number in length is 57, and in breadth 17.

39. Again, let 480 soldiers be drawn up into an oblong battalion, so that the sum of the greater and lesser sides is 52. Required the number of soldiers in each rank, in length and breadth?

Solution.

Sup. $x =$ number in length,
and $y =$ number in breadth.

$$\left.\begin{array}{l}\text{Then } xy = 480, \\ x + y = 52,\end{array}\right\} \text{per question,}$$
$$x^2 + 2xy + y^2 = 2704 \text{ the 2d squad.}$$
$$4xy = 1920 \text{ the 1st} \times 4,$$
$$x^2 - 2xy + y^2 = 784 \text{ by subt.}$$
$$x - y = \sqrt{784} = 28 \text{ by evol.}$$
$$x + y = 52 \text{ the 2d equa.}$$
$$2x = 80 \text{ by add.}$$
$$x = 40 \text{ by} \div 2,$$
$$2y = 24 \text{ by subt.}$$
$$y = 12 \text{ by} \div 2.$$

That is, $x = 40$, and $y = 12$.

40. Let 600 soldiers be disposed into an oblong battalion, which the colonel, willing to make broader, finds, that if he takes away 10 ranks from the length, he shall augment the breadth with two ranks: What was the number of his soldiers through every rank, in length and breadth?

Solution.

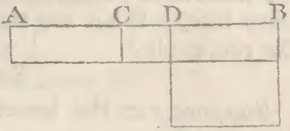
Let the length be $= x$,
and the breadth $= y$.

Then $(x - 10)(y + 2) = 600$ per quest.
or $xy - 10y + 2x - 20 = 600 = xy$,

$2x - 10y = 20$ by transposition,
$x - 5y = 10$ by dividing by 2,
and $x = \frac{600}{y}$.
But $5y + 10 = \frac{600}{y}$ both $= x$,
$5y^2 + 10y = 600$ by $\times\, y$,
$y^2 + 2y = 120$ by $\div\, 5$,
$y^2 + 2y + 1 = 121$ by comp. sq.
$y + 1 = \sqrt{121} = 11$ by evol.
$y = 11 - 1 = 10$ by transp.
and $x = \frac{600}{y} = \frac{600}{10} = 60$.

GEOMETRICAL PROBLEMS.

1. LET the line AB (of 90 parts), be divided any how in C, so that AC may be 42, BC 48 ; it is required to divide the same line again in another point, for example, in D, so that the rectangle ADC may be equal to the square DB ; let the segment CD be inquired, which being obtained, AD and DB will be known.

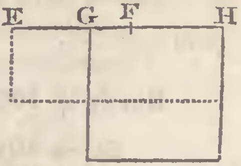


Solution.

Suppose $x = DC$, then will $48 - x = DB$, and $42 + x = AD$,
and $(42 + x)\,x = (48 - x)^2$ per quest.
or $42x + x^2 = 2304 - 96x + x^2$,
$42x = 2304 - 96x$ taking x^2 from both,
$138x = 2304$ by transp.
$x = 16{\cdot}695$ dividing by 138.

Hence CD $= 16{\cdot}695$, DB $= 31{\cdot}305$, and AD $= 58{\cdot}695$.

2. Let the line EF be divided any how in G, so that EG may be 6, FG 4. It is required to produce the right line EF to H, so that the rectangle EHF, may be equal to the square GH. The length HF is required.



Solution.

Assume $x = \text{FH}$. Then $\text{EH} = 10 + x$, and $\text{GH} = 4 + x$,

and $(10 + x)\,x = (4 + x)^2$ per quest.

or $10x + x^2 = 16 + 8x + x^2$,

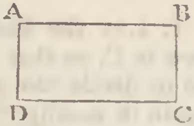
$10x = 16 + 8x$ taking x^2 from both sides,

$2x = 16$ by transp.

$x = 8$ dividing by 2.

So $\text{HF} = 8$.

3. In the rectangle ABCD, the difference of the greater side AB, and of the lesser BC is 12; but the difference of the squares of the sides 1680: What are the sides of the rectangle?



Solution.

Suppose $x =$ the lesser side, then the greater will be $x + 12$,

and $(x + 12)^2 - x^2 = 1680$ per quest.

or $x^2 + 24x + 144 - x^2 = 1680$,

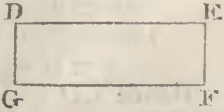
$24x + 144 = 1680$,

$24x = 1536$ by transp.

$x = 64$ by $\div\ 24$,

and $64 + 12 = 76$. Therefore $\text{AB} = 76$, and $\text{BC} = 64$.

4. The length DE of the rectangle DEFG, is twice the breadth EF, and the sum of the squares of the length and breadth is 10 times the sum of the two sides DE and EF: What are the sides of the rectangle DEFG?



Solution.

Suppose the breadth x, the length $2x$, and their sum $3x$.

$x^2 + 4x^2 = 5x^2 =$ the sum of the squares of the length and breadth.

Now, $5x^2 = 3x \times 10 = 30x$ per quest.

$5x = 30$ by dividing by x,

$x = 6$ by dividing by 5.

So $6 =$ the breadth EF,

and $6 \times 2 = 12 =$ the length DE.

5. To find the side of a square, whose area is to the sum of the sides in a given ratio as 45 to 12.

Solution.

Suppose $x =$ the side of the square, then the area is x^2, and the sum of the sides is $4x$,

and $x^2 : 4x :: 45 : 12$ per quest.

or $x^2 : 4x :: 15 : 4$,

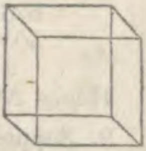
$4x^2 = 60x$ mult. ext. and means,

$4x = 60$ by dividing by x,

$x = 15$ by dividing by 4.

Therefore the side of the square is 15.

6. To find the side of a cube, whose superfices is to its solidity in a given ratio as 6 to 11.



Solution.

Suppose $x =$ the side of the cube, $6x^2$ its superfices, and x^3 its solidity.

Then $6x^2 : x^3 :: 6 : 11$ per quest.

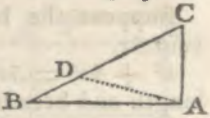
$66x^2 = 6x^3$ mult. ext. and means,

$66 = 6x$ by dividing by x^2,

$11 = x$ by dividing by 6.

So the side of the cube is 11.

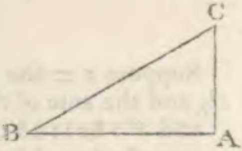
7. In the right angled triangle ABC, is given the base $AB = 9$, and the difference of the other sides, that is, the segment $BD = 3$: Required the sides AB and BC.



Solution.

Sup. $x = AC$ or CD, then $x + 3 = BC$.
But $BC^2 = AC^2 + AB^2$ by 47, *Euc.* 1.
That is, $(x+3)^2 = x^2 + 81$,
or $x^2 + 6x + 9 = x^2 + 81$,
$6x + 9 = 81$ by taking x^2 from both sides,
$6x = 72$ by transp.
$x = 12$ by dividing by 2.
Hence $AC = 12$, and $BC = 12 + 3 = 15$.

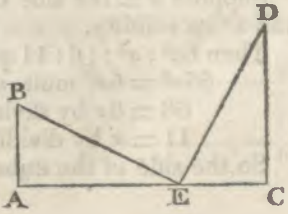
8. In the right angled triangle ABC, is given the base $AB = 5$, and the sum of the other sides $AB + BC = 25$: Required the sides AC and BC severally.



Solution.

Suppose $x = AC$, then $25 - x = BC$.
But $BC^2 = AC^2 + AB^2$ by 47. *Euc.* 1.
That is, $(25 - x)^2 = x^2 + 5^2$,
or $625 - 50x + x^2 = x^2 + 25$,
$625 - 50x = 25$ taking x^2 from both sides,
$50x = 600$ by transp.
$x = 12$ dividing by 50.
Hence $AC = 12$, and $BC = 25 - 12 = 13$.

9. Suppose two towers, AB 180 feet high, and CD 240; at the distance AC, 360 feet, a ladder is to be set up on the line AC, at some point, suppose in E, of such length, as from thence it may reach the top of both the towers: Required the point E in the line of distance, as also the length of the ladder EB or ED.

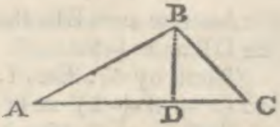


Solution.

Sup. $x = \text{AE}$, then $360 - x = \text{EC}$,
And $180^2 + x^2 = 240^2 + (360 - x)^2$ p. quest.
Or $32400 + x^2 = 57600 + 129600 - 720x + x^2$,
$32400 = 57600 + 129600 - 720x$,
$720x = 187200 - 32400$ by transp.
$720x = 154800$,
$x = 215$ by dividing by 720.

Hence $\text{AE} = 215$, and $\text{EC} = 360 - 215 = 145$, $\text{BE} = \text{DE} = \sqrt{\text{AE}^2 + \text{AB}^2} = \sqrt{215^2 + 180^2} = 280{\cdot}44$, the length of the ladder.

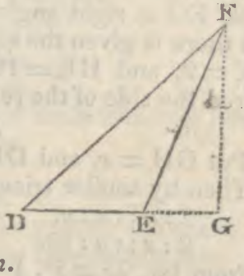
10. In the triangle ABC, the several sides $\text{AB} = 13$, $\text{AC} = 14$, $\text{BC} = 15$ are given, and the perpendicular BD being drawn: Required the segment of the base AD and DC.



Solution.

Sup. $\text{AD} = x$, then $\text{DC} = 14 - x$,
But $13^2 - x^2 = \text{DB}^2 = 15^2 - (14 - x)^2$ by 47. *Euc.* 1.
That is, $169 - x^2 = 225 - 196 - 28x - x^2$,
$28x = 365 - 225 = 140$ by transp.
or $28x = 140$,
$$\left.\begin{array}{r} x = 5 = \text{AD}, \\ \text{and } 14 - 5 = 9 = \text{DC}, \end{array}\right\} \text{segments.}$$

11. In the obtuse angled triangle DEF, the several sides are, viz. $\text{DE} = 11$, $\text{EF} = 13$, $\text{DF} = 20$, and the perpendicular FG being let full upon the base produced: Required the prolongation of the base EG.



Solution.

Sup. $\text{EG} = x$, then $\text{DG} = 11 + x$,
But $20^2 - (11 + x)^2 = \text{GF}^2 = 13^2 - x^2$ by 47, *Euc.* 1.

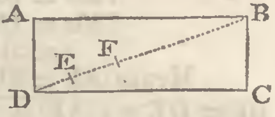
That is, $400 - 121 - 22x - x^2\ 169 - x^2$,

$22x = 110$ by transp.

$x = 5$ dividing by 22.

Therefore $EG = 5$.

12. In the rectangle ABCD, is given the difference between the length AB and the diagonal BD, that is, $DE = 2$; and likewise the difference between the breadth AB and the diagonal BD, that is, $FB = 9$: Required the sides of the rectangle AB and AD.



Solution.

Assume $x = EF$, then $AB = BE = x + 9$, and $AD = DF = x + 2$.

Then by 47. *Euc.* 1. we have $(x + 9)^2 + (x + 2)^2 = (x + 9 + 2)^2 = (x + 11)^2$.

That is, $x^2 + 18x + 81 + x^2 + 4x + 4 = x^2 + 22x + 121$,

or $2x^2 + 22x + 85 = x^2 + 22x + 121$ by add.

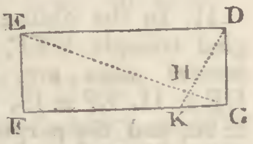
$x^2 = 121 - 85 = 36$ by transp.

$x = \sqrt{36} = 6$ by evolution.

Therefore $6 + 9 = 15 = AB$,

and $6 + 2 = \ \ 8 = AD$.

13. In the rectangle DEFG, the right line DK is drawn from the angle D to the opposite side, cutting the diagonal EG at right angles in H, and there is given the segments $HK = 2$, and $HE = 16$: Required the side of the rectangle.



Solution.

Put $GH = x$, and $DH = y$.

Then by similar triangles we have

$2 : x :: x : y$,

$2 : x :: y : 16$.

$$\left.\begin{array}{l}\text{From 1st, } 2y = x^2, \\ \text{From 2d, } xy = 32,\end{array}\right\}\text{ by mult. ext. and means,}$$

or $y = \dfrac{x^2}{2}$ dividing by 2,

and $y = \frac{32}{x}$ dividing by x.

But $\frac{x^2}{2} = \frac{32}{x}$ both being $= y$,

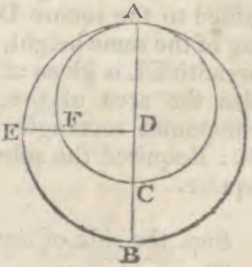
$x^3 = 64$ mult. by 2 and x,

$x = \sqrt[3]{64} = 4$ by evolution,

and $y = \frac{32}{4} = 8$.

Now, DE, the length of the rectangle, is equal to $\sqrt{8^2 + 16^2} = \sqrt{64 + 256} = 17{\cdot}888$. And DG, the breadth, is equal $\sqrt{x^2 + y^2} = \sqrt{16 + 64} = 8{\cdot}944$.

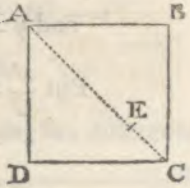
14. Let there be a circle whose diameter is AB, with another less circle whose diameter, which is AC, touches within in A, and from the centre of the greater circle D draw the radius DE at right angles to AB, cutting the periphery of the lesser circle in F. Now, there is given BC, the difference of the diameters, $= 9$, with the segment $\text{EF} = 5$: Required the diameters AB and AC of the said circles.



Solution.

Put $2x = \text{AB}$, then $x = \text{AD}$ or DB,
$x - 5 = \text{DF}$, and $x - 9 = \text{DC}$.
Then by 13. *Euc.* 6. we have
$x : x - 5 :: x - 5 : x - 9$,
$x^2 - 9x = x^2 - 10x + 25$ mult. ext. and means,
$10x - 9x = 25$ by transp.
or $x = 25$ by subt.
Hence $25 = \text{AD}$ or DB,
$25 \times 2 = 50 = \text{AB}$ the greater diameter,
and $50 - 9 = 41 = \text{AC}$ the lesser diameter.

15. In the square ABCD, is given the difference of the diagonal and the side, that is, $EC = 6$: Required the side of the square.



Solution.

Call the side of the square x.

Then by 47. *Euc.* 1. we have $x^2 + x^2 = (x + 6)^2$.

That is, $2x^2 = x^2 + 12x + 36$,

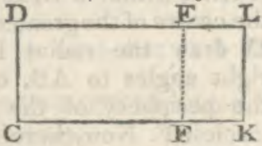
$x^2 - 12x = 36$ by transp.

$x^2 - 12x + 6^2 = 72$ by comp. sq.

$x - 6 = \sqrt{72} = 8{\cdot}4852814$ by evol.

$x = 8{\cdot}4852814 + 6 = 14{\cdot}4852814 =$ the side of the square.

16. The rectangle EK is added to the square DF, being of the same height, whose breadth EL is given $= 2$, and also the area of the whole compound rectangle $DK = 60$: Required the side of the square.



Solution.

Sup. the side of the square $= x$.

Then $\overline{x + 2} \times x = 60$ per quest.

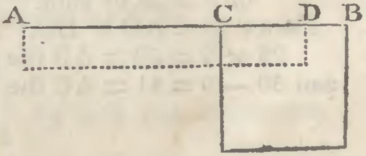
or $x^2 + 2x = 60$,

$x^2 + 2x + 1 = 61$ by comp. sq.

$x + 1 = \sqrt{61} = 7{\cdot}81$ by evol.

$x = 7{\cdot}81 - 1 = 6{\cdot}81$ the side of the square.

17. Let the line AB be divided in C, so that AC may be 8 and CB 6. We are to divide the same line AB in D, so that the rectangle under AD and DC may be equal to the rectangle under AC and CB, or to the product from 8 and 6, which is 48: Required the segment CD.



Solution.

Sup. $CD = x$, then $AD = 8 + x$,

and $\overline{8 + x} \times x = 48$ by the quest.

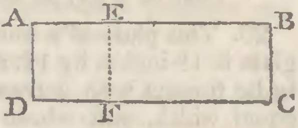
or $x^2 + 8x = 48$,

$x^2 + 8x + 4^2 = 64$ by comp. sq.

$x + 4 = \sqrt{64} = 8$ by evolution,

$x = 8 - 4 = 4 = CD$.

18. Let there be a rectangular garden ABCD, the length of which AB is thrice the breadth AD, and reckoning 18 perches from B towards A, that is BE, and drawing EF parallel to AD, let the area of the remaining rectangle ED be given = 120 square perches: What is the length and breadth of the said garden?



Solution.

Put $x = AD$, $3x = AB$, and $3x - 18 = AE$.

Then $\overline{3x - 18} \times x = 120$ per quest.

or $3x^2 - 18x = 120$,

$x^2 - 6x = 40$ by $\div 3$,

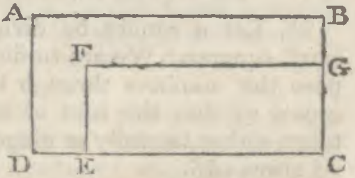
$x^2 - 6x + 3^2 = 49$ comp. sq.

$x - 3 = \sqrt{49} = 7$ by evol.

$x = 7 + 3 = 10 = AD$ breadth,

$3x = 30 = AB$ length.

19. In a certain rectangular garden, the length of which AB is 22 perches, and the breadth AD is 10, the walk DG is to be made in a situation parallel to the sides of the figure, so that the area of the said walk or gnomon DG may be equal to the remaining rectangle FC, or that the gnomon DG may be half of the whole figure ABCD proposed: Required the breadth of the said gnomon DE or BG.



Solution.

Sup. $x =$ the breadth, then $22 - x = FG$ or EC, and $10 - x =$ EF or GC. Then by the question we have

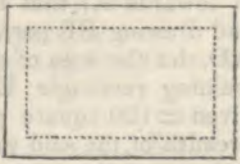
$$\overline{22 - x} \times \overline{10 - x} = \frac{22 \times 10}{2},$$

or $220 - 32x + x^2 = 110$,

$$x^2\ 32x = -110 \text{ by transp.}$$
$$x^2 - 32x + 16^2 = 256 - 110 = 146 \text{ by comp. sq.}$$
$$x - 16 = \sqrt{146} = 12{\cdot}083 \text{ by evol.}$$
$$x = 16 - 12{\cdot}083 = 3{\cdot}917 \text{ by transp.}$$

That is, $3{\cdot}917 =$ DE or BG the breadth required,

20. The plate of a looking-glass is 18 inches by 12, and is to be framed with a frame of equal width, and whose area is to be equal to that of the glass. The width of the frame is required.



Solution.

Sup. $x =$ the breadth of the frame, then $18 + 2x =$ length of the glass and frame, $12 + 2x =$ the breadth of both glass and frame. Hence, per quest.

$$\overline{2x + 18} \times \overline{2x + 12} = 18 \times 12 \times 2,$$

or $4x^2 + 60x + 216 = 432$ by mult.

$$x^2 + 15x + 54 = 108 \text{ by } \div 4,$$
$$x^2 + 15x = 180 - 54 = 54 \text{ by transp.}$$
$$x^2 + 15x + 56{\cdot}25 = 110{\cdot}25 \text{ comp. sq.}$$
$$x + 7{\cdot}5 = \sqrt{110{\cdot}25} = 10{\cdot}5 \text{ by evol.}$$
$$x = 10{\cdot}5 - 7{\cdot}5 = 3 \text{ the breadth.}$$

21. Let a square be divided into 9 small squares. We are to find and dispose the numbers through the several areas, so that the sum of every three taken either laterally or diagonally may be always 15.

a	b	c
d	e	f
g	h	m

Solution.

Suppose it done, and represented in its proper form, by the following symbols placed as above:

Then by the question we have the following equations:

$$\left.\begin{array}{l} a+e+m=15, \\ b+e+h=15, \\ c+e+g=15. \end{array}\right\} \text{Required } e \text{ the middle number.}$$

Their sum $= a+b+c+3e+m+h+g=45$.

Again, $\left.\begin{array}{l} a+b+c=15, \\ m+h+g=15. \end{array}\right\}$ Added together, we have $a+b+c+m+h+g=30$. This subtracted from 4th equation gives

$$3e=15,$$
$$e=\tfrac{15}{3}=5 \text{ the middle number.}$$

Or by numbers thus,

First, the sum of the progressional numbers are $1+2+3+4+5+6+7+8+9=45$.

Then $3=$ the number of rows.

Therefore $\frac{45}{3}=15=$ the sum of each side or rank, and $\frac{15}{3}=5$ the middle number, as before.

Again, to find the corner figures, and first to find the figures represented by a. Beginning with 1, I find the corner letter a, or any other corner letter, cannot be 1; for if a be $=1$, then m must be $=9$, $b+c=15-1=14$; as also, $d+g=15-1=14$. But there remains no two numbers after 5, 1, and 9, whose sum is 14, but 6 and 8.

Then, if any of these figures were b, the other would be c, and then no figures would remain for the value of either d or g; wherefore a is not $=$ to 1, nor any corner letter $=$ to 1 or 9.

Now, 3 cannot be $=$ to a; for if it were, then m should be $=7$: $b+c=15-3=12$; as also, $d+g=15-3=12$; but there remain no two numbers after 5, 3, and 7, whose sum is 12, but 8 and 4, which cannot answer to b and c, and d and g; wherefore a, or any other corner letter, is not $=$ to 3, neither is m nor any other corner letter $=$ to 7.

From what has been said, it is plain, (that if the question proposed is capable of being solved), the corner letters are all even numbers; wherefore if $a=2$, m will be $=8$, and e must either be $=4$ or 6. Let $e=4$, then $g=6$, $b=9$, $d=7$, $f=3$, and $h=1$; and so the square will be completed as was required.

2	9	4
7	5	3
6	1	8

But if $c = 6$, (a being 2), then $g = 4$, $b = 7$, $d = 9$, $f = 1$, and $h = 3$, then the square will stand thus,

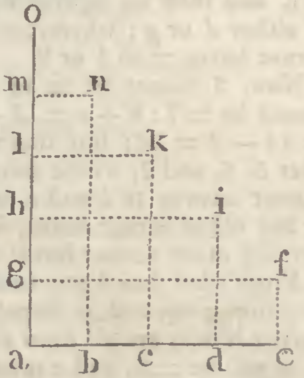
2	7	6
9	5	1
4	3	8

Theorem.

22. Let any number whatsoever be given, if you subtract every less number from that which is the next greatest: I say, that the sum of those differences is equal to the difference of the greatest and least number.

Demonstration.

It is evident from the annexed figure that $ao - nb = l = om$, $nb - kc = lm$, $kc - id = lh$, $id - fe = hg$; now it is also evident that the sum of these differences, that is, $om + ml + lh + hg = og = ao$.



THE END.

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