3rd Ed.
1836

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A

CONCISE SYSTEM

OF

MATHEMATICS,

IN THEORY AND PRACTICE,

FOR THE

Use of Schools, Private Students, and Practical Men;

COMPREHENDING

ALGEBRA, ELEMENTS OF PLANE GEOMETRY, INTERSECTION OF PLANES, PRACTICAL GEOMETRY, PLANE AND SPHERICAL TRIGONOMETRY, WITH THEIR PRACTICAL APPLICATIONS; MENSURATION OF SURFACES AND SOLIDS, CONIC SECTIONS AND THEIR SOLIDS, SURVEYING, GAUGING, SPECIFIC GRAVITY, PRACTICAL GUNNERY, MENSURATION OF ARTIFICERS' WORK, STRENGTH OF MATERIALS, &c.

WITH

AN APPENDIX,

CONTAINING THE MORE DIFFICULT DEMONSTRATIONS OF THE RULES IN THE BODY OF THE WORK.

BY ALEXANDER INGRAM,

Author of Principles of Arithmetic, Elements of Euclid, &c.

THE THIRD EDITION,

THOROUGHLY REVISED, WITH MANY IMPORTANT ADDITIONS AND IMPROVEMENTS; BESIDES AN ACCURATE SET OF STEREOTYPED TABLES, COMPRISING LOGARITHMS OF NUMBERS, LOGARITHMIC SINES AND TANGENTS, NATURAL SINES AND TANGENTS, AREAS OF CIRCULAR SEGMENTS; SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS; TABLE OF JOISTING, &c.

BY JAMES TROTTER,

Of the Scottish Naval and Military Academy, Author of Lessons in Arithmetic, A Key to Ingram's Mathematics, &c.

ILLUSTRATED BY THREE HUNDRED AND FORTY WOOD-CUTS.

PUBLISHED BY OLIVER & BOYD, EDINBURGH;
AND SIMPKIN, MARSHALL, & CO., LONDON.

MDCCCXXXVI.

[Price Seven Shillings and Sixpence bound.]

ENTERED IN STATIONERS' HALL.

Printed by Oliver & Boyd,
Tweeddale Court, High Street, Edinburgh.

ADVERTISEMENT TO THE THIRD EDITION.

In preparing the present edition of INGRAM'S MATHEMATICS for the press, the most anxious care has been taken to introduce such improvements as might not only sustain but increase its high reputation.

To enumerate all the alterations and additions which have been made would occupy more space than would be found suitable in an advertisement; suffice it to say, that what was formerly given in the shape of an Appendix is now incorporated into the body of the work, in such a manner, that the Practical portion of each section is preceded by those Geometrical Theorems upon which the demonstrations of the rules depend,—an arrangement which was considered better adapted than the original one to initiate the Student in the principles of the science, and to enable him to apply them to the ordinary calculations of business. The properties of Conic Sections and their Mensuration have been presented under a distinct head, which is decidedly preferable to having some of the problems under Mensuration of Surfaces, and others under that of Solids. Such are the most important changes in the distribution of the materials.

To the section on Algebra have been added the articles on Ratios and Proportion, Cubic and Higher Equations, Exponential Equations, and Indeterminate Problems; while those on Series and Logarithms have been entirely re-written, and greatly extended. The principal propositions on the Intersection of Planes have been introduced at the end of the Elements of Plane Geometry. The Elements of Plane Trigonometry have been very considerably enlarged;—the equations which express the value of the trigonometrical lines, in terms of each other, are deduced from the definitions;—various useful analytical formulæ are investigated;—the signs of the trigonometrical lines, and the construction of the Tables of Sines, Tangents, &c., with their use, are fully explained.

Several additional problems are also inserted at the end of the tract on Surveying; and the New Rules for finding the Tonnage of Ships and Steam-vessels, as established by a late Act of Parliament, are given under their proper heads. Practical Gunnery, containing the principal theorems relating to Projectiles on Horizontal and Inclined Planes, a subject of great interest and importance, has been introduced. The Mensuration of Artificers' Work has been enriched by several New Rules, contributed by Mr DUFF, surveyor, Edinburgh, a gentleman who has long been professionally acquainted with the subject. The Editor is likewise indebted to the same eminent mathematician for the Tables of Joisting, and the Lengths of the Sides of Inscribed and Circumscribed Polygons, as well as for several excellent practical questions.

The article on the Strength of Materials is very much extended and improved. Tables of the strength and elasticity of various substances are given from the works of the best authors; as also those problems which are of most general use, on the strength of cast-iron beams, teeth of wheels, and on solid and hollow shafts.

The demonstrations of those rules which are not contained in the theorems which precede the practical part of each section are generally given in foot-notes; but several have been reserved for an Appendix, in consequence of their requiring the application of Fluxions.

Tables of Squares and Cubes, Square Roots and Cube Roots,—of Joisting,—of the Lengths of the Sides of Inscribed and Circumscribed Polygons,—and of Useful Numbers,—have also been supplied.

It is only necessary farther to state, that the whole work has been so thoroughly and carefully revised as scarcely to leave the possibility of an error of any magnitude in the results; and when it is considered that upwards of one hundred pages of valuable matter have been added to this impression, without any advance of price, the Publishers feel assured that it cannot fail to meet with an increase of that approbation which was so warmly bestowed upon the preceding editions.

EDINBURGH, *May* 1836.

ADVERTISEMENT TO THE SECOND EDITION.

AMONG the various branches of general knowledge, Mathematics have of late attracted a more than ordinary degree of attention. Nor is this to be wondered at, since it is a department of learning which—whether considered as unfolding, in its more recondite demonstrations, the most sublime discoveries that can engage the faculties of man, or viewed in a less elevated sphere, as assisting, directing, and perfecting his control over the rude elements of the material world—deserves to hold a very high rank in the scale of national education.

Of the works intended to facilitate and extend an acquaintance with this science, that of which a new edition is now offered to the public has obtained a marked preference in the most respectable seminaries, not only of Great Britain and Ireland, but also of America and the East Indies,—a circumstance which led the Publishers, some time ago, to consider in what manner it might be most efficiently improved, so as to keep pace with the expanding information and claims of the age. With this view, they made very careful inquiries among those most distinguished by their mathematical attainments, and collected useful information from every quarter. The numerous and important improvements which were the result of these inquiries are embodied in the work, and their *extent alone* has imposed the necessity of a change of the title—the former one, "A Concise System of Mensuration," in consequence of the more comprehensive plan of the volume, having become inappropriate :—

1. In that part of the work which treats of ALGEBRA various improvements have been made; and the conditions of the Questions, given in the form of Equations in the first edition, are now omitted, with the exception of a few difficult cases, as it was found that their insertion had a tendency to encourage indolence rather than to excite exertion.
2. The treatise on LAND-SURVEYING,—a subject of increasing importance,—is much improved, and the most modern methods are explained and illustrated by practical Examples.
3. That portion of the work devoted to GAUGING has been entirely recomposed, greatly extended, and adapted to the present standards; several useful Tables are introduced, and the Rules and Directions given for performing all the Computations by the Sliding Rule will be found so copious and explicit, as to make the use of that valuable and scientific instrument perfectly familiar to the student.
4. The section on the MENSURATION of ARTIFICERS' WORKS has likewise been rewritten; and the most approved methods of taking the dimensions of all the different kinds of work, together with the usual allowances and deductions, are explained at great length; while an entirely new head is added, on the Flexibility, Strength, and Fracture of Timber, which renders this part of the work complete, and adapts it to the purposes of practical men.

5. The LIMITS of RATIOS, FLUXIONS, and FLUENTS, previously forming an Appendix to the Algebra, are now incorporated with the General Appendix, which is so arranged as to exhibit a comprehensive and satisfactory view of the whole theory. And as an introduction to the study of NAVIGATION and NAUTICAL ASTRONOMY, a section on SPHERICAL TRIGONOMETRY, with examples of its application, has been inserted. Hence it will appear that no part of the science really valuable has been omitted.

6. To adapt the work to *all the purposes of teaching*, due regard has also been paid to the variety of Exercises added to each Problem, which will be found more than double the number contained in the first impression,—an addition of the highest importance in a text-book.

7. Besides many new and useful Tables interspersed throughout the work, there are now added TABLES of the LOGARITHMS of NUMBERS from 1 to 10,000, of LOGARITHMIC SINES and TANGENTS to every Degree and Minute, and of NATURAL SINES and TANGENTS to every Five Minntes of the Quadrant. These have all been carefully stereotyped from new types, and, in order to obtain the utmost possible accuracy, rigidly collated with the Tables of BRIGGS, VLACQ, SHERWIN, GARDINER, CALLET, TAYLOR, HUTTON, BABBAGE, BORDA, and also with those of GALBRAITH, which are especially distinguished for accuracy. The Table of the AREAS of CIRCULAR SEGMENTS has likewise been collated with several other more extensive ones.

Such is a brief and cursory view of the leading features now introduced into this edition. But, exclusive altogether of the great amount of new matter, and independent of many minor improvements, the whole work has undergone a careful, rigorous, and minute revision;—what was obscure has been illustrated, what was defective has been supplied, and the errors which had formerly escaped notice have been corrected. With the view of securing perfect accuracy, the Author availed himself of the assistance of an eminent Mathematician in examining every calculation; and although it would be presumptuous to assert that the work is immaculate, yet the Publishers feel assured that no error of importance has been allowed to remain.

Finally, when the Publishers consider the success attending the work in a less perfect shape, they confidently hope that the variety and importance of the contents in the present edition, as well as the perspicuous and familiar manner in which these are treated, taken along with the extensive additions and improvements introduced throughout, will give it a still higher claim to public favour, and render it better calculated for facilitating the acquirement of mathematical knowledge, and disseminating a taste for that science among all classes of students. As an additional recommendation, they may venture to affirm, that while it is in many respects the *most complete*, it is unquestionably the *cheapest* work of the kind ever published.

EDINBURGH, *January* 22, 1830.

ORIGINAL PREFACE.

Several treatises on Mensuration have made their appearance within the last fifty years, and among these, Dr Hutton's large work has deservedly acquired the highest celebrity. It treats fully both of the theory and practice of the science, and may be consulted with advantage by persons employed in every kind of measurement. But the scientific part of that work can be read by such only as are well acquainted with the higher branches of Mathematics, and hence the student must have frequent recourse to other publications, to enable him to understand it; while the practical part involves such a multiplicity of rules for the same thing, without distinguishing sufficiently the various cases in which they may be applied, that he is liable to be perplexed with their variety; and nothing has been done by later writers to remove the difficulty.

A book on Mensuration is therefore still wanted, embracing the whole theory and practice in such a way, that both, though kept separate, may be rendered intelligible to every reader, without the necessity of having recourse to other publications, and arranged so as to comprise a complete system in a small compass. Such are the objects of the present publication.

The practical part of this work consists of plain rules for performing the various operations requisite in *Trigonometry, Mensuration, Surveying, Gauging, &c.* These rules are illustrated by proper examples, one or more of which is wrought for the assistance of the learner. A demonstration of the rule is sometimes annexed to it in the form of a note, when this can be done in an easy and concise manner; but the more difficult demonstrations are reserved for the Appendix.

By pursuing this method, the Author has endeavoured to render his book fit for the use of every person who wishes to study Mensuration with facility and success. The treatise on Practical Geometry, which precedes the Trigonometry, will enable the student to draw his figures; while the rules delivered in the following parts of the work will direct him how to find their contents, and the lengths of their lines; and a little reflection will qualify him to compare these lengths or contents with one another. Hence, this work will be found a most useful guide to practical measurers, and well adapted for the use of schools. The rules may be applied directly in all ordinary cases; but if any shall occur which requires investigation, the method of conducting the process may be learned from the treatise on Algebra prefixed to the volume.

In the treatise on Algebra, great care has been taken to remove irregularities, and other difficulties, of which beginners usually complain; and the demonstrations of the fundamental rules are generalized, and deduced from one principle intimately connected with the nature of abstract quantity. A short Appendix is annexed to this part, which treats of the management of indeterminate problems, of the relations of variable quantities, and of the limits of ratios, with as much of the practice of Fluxions and Fluents as is requisite in this performance.

The Practical Geometry, though short, contains every thing necessary for what follows. Some new methods of operation are introduced, and the lines and angles are generally expressed in numbers.

In the Mensuration, the application of the series for finding the circumference of the circle, of which the diameter is unit, has been taken from Euler, and appears to be as simple as it can be made. New rules are given for approximating to the length of an arc of a circle, and to the area of a segment of it, which are both easier and more accurate than those formerly employed for this purpose. The method of forming the most common solids with pasteboard is introduced, because it renders the reader familiar with their shapes, and illustrates the rules for finding their superficies.

Land-surveying, Gauging, &c., are the application of Trigonometry and Mensuration to practical purposes. Great plainness has therefore been studied in explaining them, and the shortest, easiest, and most approved methods of practice have been adopted.

The Appendix is appropriated to the demonstration of the rules delivered in the preceding parts of the work. Such of the principles of Geometry and of Conic Sections are introduced as are necessary for enabling the reader to understand the demonstration of the rules, without having recourse to other publications. Here accuracy is rigidly adhered to. Many new demonstrations are given, which are more simple than those that were formerly employed. The theory of Parallel Lines has been rendered as plain and concise as possible. The principles of Conic Sections have been deduced from the ratio of the curve, or its relation to the focus and directrix,—a method which has been generally held by mathematicians to be superior to every other. The leading propositions only are delivered; but they comprehend those principles from which the other properties of these curves may be easily derived.

The student who has abundance of time should begin with Algebra, and then read the Appendix to the work and the Practical Geometry together; after which, he should go regularly through the book, in the order in which it is printed. In doing this, he may acquire as much knowledge of Mathematics as will be sufficient for ordinary purposes, and be enabled to prosecute that most extensive science with pleasure and advantage. If his time and other pursuits do not admit of such a regular progress, he may study separately any of the practical branches best adapted to his taste, or the purpose to which he intends to apply them.

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ALGEBRA.

DEFINITIONS.

ALGEBRA is a general method of computation and of investigation, in which quantities are represented by letters, and their relations pointed out by characters.

CHARACTERS EXPLAINED.

1. $+$ *plus*, is the sign of addition; as $a+b$ signifies the quantity represented by b added to that represented by a.

2. $-$ *minus*, is the sign of subtraction; as $a-b$ denotes the quantity b taken from the quantity a.

The sign $\backsim$ is employed to denote the difference between two quantities, when it is not known which is the greater; as $a \backsim b$ signifies the difference between a and b: Also $a \pm b$ signifies the sum or difference of a and b.

3. $\times$ *into*, is the sign of multiplication; as $a \times b$ represents the product of a by b, or of b by a. Instead of this sign we often use a point, or write the letters together as in one word: thus $a.b$ or ab signifies $a \times b$.

4. $\div$ *by*, is the sign of division; but it is generally expressed by placing the dividend above the line and the divisor below it, in the form of a fraction: thus $a \div b$ or $\frac{a}{b}$ signifies a divided by b.

5. $: :: :$ is the sign of proportion; as $a:b::c:d$ is read, As a is to b, so is c to d.

6. $=$ *equal to*, is the sign of equality: thus $a=b$ signifies a is equal to b.

7. $>$ $<$ are signs of greater and less: thus $a > b$, a is greater than b; $a < b$, a is less than b; the opening of the sign being always turned towards the greater quantity, and its angular point towards the less.

8. $7a$. A number prefixed to a letter is called its *coefficient*, and shews how often the letter is to be taken; as here, 7 times a. When no coefficient is expressed, the coefficient 1 is always understood: thus a and $1a$ denote the same thing.

9. $(a+b) \times c$ or $\overline{a+b} \times c$. A parenthesis enclosing letters, or a line drawn over them, is called a *vinculum*, and

points out how many are to be multiplied, divided, &c.; as here, the sum of a and b is to be multiplied by c.

10. *aaa*. When the same letter is repeated twice, or oftener, it is understood to be multiplied as often into itself, and the product is called a power of the quantity represented by that letter: thus aa is the second power or square of a, aaa is the third power of a, &c.; and in relation to these powers the quantity is called the first power of itself.

11. a^3. Instead of repeating the same letter, we generally place a figure above it towards the right hand, to shew how often it is repeated; as a^3 is the third power of a, a^4 the fourth power, a^n the power of a denominated by the number n.

12. The character placed above is called the exponent or index of the power.

13. The *root* of any quantity or power is a quantity which, if multiplied by itself a certain number of times, produces the original quantity or power; and is denoted by *the radical sign* $\surd$: thus $\sqrt{9}$ is the square root of 9, $\sqrt[3]{8}$ is the cube root of 8, $\sqrt[4]{81}$ is the fourth root of 81.

14. A fractional exponent or index is more generally used to express the root, and then the upper figure denotes the power, and the under figure the root: thus $a^{\frac{2}{3}}$ is the third root of the second power of a, $a^{\frac{1}{4}}$ is the fourth root of the first power of a, or of a itself.

15. A *simple* quantity is that which consists of but one term; as a, ab, $4abc$, &c.

16. A *compound* quantity consists of two or more simple terms, connected by the signs + or —; as $a+b$, $4a-3b+6ac$, &c. If a compound quantity consists of only two terms, it is called a *binomial;* if of three, a *trinomial;* if of four, a *quadrinomial;* and if it consists of more than four, a *polynomial* or *multinomial.*

17. *Like* terms are those of which the literal parts are the same, *i. e.* consist of the same letters; as $4ab$, ab, $9ab$, &c.

18. *Unlike* terms are those which consist of different letters; as $2ab$, $3bc$, $5cd$, &c.

19. The sign $\therefore$ is sometimes used to denote the words *therefore* or *consequently.*

Quantities which have the sign + before them are said to be positive or *additive,* and those which have the sign — negative or *subtractive.* A quantity which has no sign prefixed is understood to have +.

The following examples will illustrate these characters, and shew their use, in which any values may be affixed to the letters:—

Let $a=12 \quad c=2 \quad e=5 \quad g=25 \quad i=11$

$b=3 \quad d=4 \quad f=9 \quad h=7 \quad k=1.$

1. $a+b-c+d$ . . . = 17.
2. $4a-5b+4c-7d$ . . . = 13.
3. $ab-2cd+4be-3cf$ . . . = 26.
4. $8a^2-5ab+10ac-4bc+4b^2$ . . = 1224.
5. $6a^3-4a^2b+2ab^2-7b^3$ . . = 8667.
6. $2a^2bc+3ab^2c-5abc^2$ . . = 1656.
7. $(a-b)\times 2c-d\times(b+c)$ . . = 16.
8. $(a+b)\times(g-h)\times(i-k)$ . . = 2700.
9. $2a^2c-\frac{a^2}{c}+\frac{a}{c^2} = 507.$ 10. $\frac{3a}{c}+\frac{4g}{5d}-\frac{6d}{b} = 15.$
11. $\frac{a+b+e}{d+k}+\frac{g}{e} = 9.$ 12. $\frac{3a}{b}\times\frac{2c}{d}\times\frac{4g}{d} = 300.$
13. $\frac{ac}{d}+\frac{gf}{c}-\overline{cd}^{\frac{1}{3}} = 116\frac{1}{2}.$ 14. $\frac{\sqrt{ae+d}}{i-h}$. = 2.
15. $(g+c)^{\frac{1}{3}}+(ai-b^2-c)^{\frac{1}{2}}$ . . . = 14.

NOTE. All the fundamental operations of algebra depend upon this single principle, viz. When a quantity is to be increased or diminished by other quantities, the same result will be obtained in whatever order the procedure is carried on, provided none of the quantities be neglected. This is manifest from the nature of quantity, which has no relation to order. Thus, if we have to add 7 and 5, and to subtract 3, we may first subtract 3 from 7, and add the remainder to 5; or we may subtract 3 from 5, and add the remainder to 7; or we may add 7 to 5, and from the sum 12 subtract 3: the result in every case is 9. Again, if we have to multiply 12 and 6, and to divide by 3; we may first divide 12 by 3, and multiply the quotient by 6; or we may divide 6 by 3, and multiply the quotient by 12; or we may multiply 12 by 6, and divide the product by 3: the result in every case is 24.

ADDITION.

CASE 1. WHEN the quantities are alike; if the signs be the same, add the coefficients, but if not the same, take their difference, and to the sum or difference prefix the sign of the greater, and annex the common letter or letters.

CASE 2. When the quantities are unlike; write them one after another, with their proper signs and coefficients.

NOTE. When there are more than two like quantities, add the coefficients of those which have + into one sum, and of those which have — into another, and subtract the less sum from the greater. The arrangement of the quantities is arbitrary, and must often be altered to bring like quantities under like.

1. $3a-5b+4c-3d-2e$
$6a+2b-7c-4d+8e$
$\overline{9a-3b-3c-7d+6e.}$

2. $8a^2b-5ab^2-8abc+4bc^2$
$-2a^2b+6ab^2-\ abc-4bc^2.$

3. $6ab+2ac-3bc+4bd$
$-7ab-3ac+6bc+5bd.$

4. $8a^{\frac{1}{2}}b^3-7a^2bc^{\frac{1}{2}}-4ab^{\frac{1}{2}}c^2$
$7a^{\frac{1}{2}}b^3+7a^2bc^{\frac{1}{2}}-3ab^{\frac{1}{2}}c^2.$

5. $8a^3b-7a^2b^2+4ab^3-a^4$
$7a^2b^2-8ab^3+4a^4-2a^3b$
$6ab^3-2a^4-7a^3b+5a^2b^2$
$5a^4-6a^3b+5a^2b^2-3ab^3$
$2a^2b^2-2a^3b-ab^3+4a^4.$

6. $a+(a-v)^{\frac{1}{2}}+\ 5$
$2a+(a-v)^{\frac{1}{2}}-10.$

7. $a+(a+v)^{\frac{1}{2}}+\ 5$
$2a+(a-v)^{\frac{1}{2}}-10.$

8. a^3+a^2-a
$a^{\frac{3}{2}}+a^{\frac{2}{3}}-a^{\frac{1}{2}}$
$a^{\frac{3}{2}}+a^2-a^{\frac{1}{3}}.$

9. $10(a+e)^{\frac{1}{2}}+(a-e)^{\frac{1}{2}}$
$-(a+e)^{\frac{1}{2}}-(a-e)^{\frac{1}{2}}.$

10. $a^3+3a^2+5+(a-v)^{\frac{1}{2}}+a+6(a+v)^{\frac{1}{2}}$
$3a^2-2a+6a^3-2(a-v)^{\frac{1}{2}}+10-6(a+v)^{\frac{1}{2}}$
$7a-5a^3-2a^2+4(a+v)^{\frac{1}{2}}-b+8(a-v)^{\frac{1}{2}}$
$8c-6a^2+4a^3-2(a-v)^{\frac{1}{2}}+7-6a$
$7a^2-8a^3+4-5(a+v)^{\frac{1}{2}}+3a-8(a+v)^{\frac{1}{2}}.$

NOTE. If the difference $a-b$ is to be added to $3a$, we may first subtract b from a, and then add the remainder to $3a$; or we may subtract b from $3a$, and add a to the remainder. Here we first add a to $3a$, and then subtract b, and it becomes $4a-b$. If $2a+b$ is to be added to $3a-4b$, we add $2a+b$ to $3a$, and it becomes $5a+b$; from which we take $4b$, and it becomes $5a-3b$.

SUBTRACTION.

CHANGE the signs of the subtrahend from + to —, or from — to +, and then proceed as in Addition.

1. $8ab-2cd+5ac-7ad$
$3ab+4cd+5ac-2ad$
$\overline{5ab-6cd\quad * \quad -5ad.}$

2. $18a^2b-12abc-3ab^2+b^3$
$6a^2b+\ 3abc-4ab^2-3b^3.$

3. $a^2x^2c-5ax^2c^2+2a^2xc^2$
$3a^2x^2c+4ax^2c^2+2a^2xc^2.$

4. $-3a^3b^{\frac{1}{2}}+2a^2bc^{\frac{1}{2}}-5a^{\frac{1}{2}}b^2c$
$4a^3b^{\frac{1}{2}}-2a^2bc^{\frac{1}{2}}-5a^{\frac{1}{2}}b^2c.$

5. $3bd+2a$
$2bd-3a-b.$

6. $\dfrac{(a-b+2)^{\frac{1}{2}}}{a+b}$
$-\left(\dfrac{a-b+2}{a+b}\right)^{\frac{1}{2}}.$

7. $2bc-11a-d$
$d+11a-2bc.$

8. $a^5 + a^{\frac{5}{2}}$
$a^3 - a^{\frac{3}{2}}$.

9. $6a^{\frac{1}{2}} - 4b^{\frac{2}{3}} + x^2$
$4x^2 - 3a^{\frac{1}{2}} + 2b^{\frac{2}{3}}$.

NOTE. If we are to subtract $a - c$ from $3a$, we may first subtract c from a, and then subtract the remainder from $3a$; or we may add c to $3a$, and then subtract a from the sum. Here we subtract the whole a from $3a$, and add c to the remainder. If $a - c$ is to be subtracted from $3a + 2c$, we subtract a as before from $3a$, and then add c, and the remainder becomes $2a + 3c$. Now all this is performed by changing the signs of the quantity $a - c$ into $-a + c$, and then adding it.

These considerations lead us to perceive how we may add or subtract any two terms, without regard to the other terms with which they are connected.

MULTIPLICATION.

MULTIPLY the coefficients, and to the product annex the letters of both factors.

If the sign of the multiplier is $+$, make the sign of the product the same with that of the multiplicand. If the sign of the multiplier is $-$, make the sign of the product contrary to that of the multiplicand.

Hence, like signs produce $+$, and unlike signs $-$.

If the multiplicand is compound, multiply each term of it separately by the multiplier.

If the multiplier is compound, multiply first by one of its terms, then by another, &c. and afterwards add the products.

Powers of the same quantity are multiplied by adding their exponents.

1. Multiply $5a - 4b + 3c - 2d + e - 1$
by $5a$
$\overline{25a^2 - 20ab + 15ac - 10ad + 5ae - 5a.}$
2. Multiply $6a^2 - 7ab + 4ac - b^2 + 2bc - c^2$ by $4ab$.*
3. ...... $3a - 2b$ by $-2a + 4b$.
4. ...... $5a^2 - 3ab + 4b^2$ by $6a - 5b$.
5. ...... $a^2 + ab + b^2$ by $a - b$.
6. ...... $a^4 - x^4$ by $a^4 - x^4$.
7. ...... $2x^2 - 3xy + 6$ by $3x^2 + 3xy - 5$.
8. ...... $5a^2 - 4ax + 3x^2$ by $2a^2 - 3ax - 4x^2$.

* ANSWERS. (2.) $24a^3b - 28a^2b^2 + 16a^2bc - 4ab^3 + 8ab^2c - 4abc^2$. (3.) $-6a^2 + 16ab - 8b^2$. (4.) $30a^3 - 43a^2b + 39ab^2 - 20b^3$. (5.) $a^3 - b^3$. (6.) $a^8 - 2a^4x^4 + x^8$. (7.) $6x^4 - 3x^3y + 8x^2 - 9x^2y^2 + 33xy - 30$. (8.) $10a^4 - 23a^3x - 2a^2x^2 + 7ax^3 - 12x^4$.

9. Multiply $2a^2x^2 - 2ax + 3a^2$ by $3a^2x^2 + 4ax - 5a^2$.*

10. ...... $x^2 - ax + \frac{1}{4}a^2$ by $x^2 + ax - \frac{1}{4}a^2$.

11. ...... $x - \frac{1}{2}a$ by $x + \frac{1}{2}a$.

12. ...... $x^2 + xy + y^2$ by $x^2 - xy + y^2$.

13. ...... $2a^2 - 3ax + 4x^2$ by $5a^2 - 6ax - 2x^2$.

14. ...... $3a - 2b + 2c$ by $2a - 4b + 5c$.

15. ...... $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.

16. ...... $a^3 - 3a^2 + 3a - 1$ by $a^2 - 2a + 1$.

NOTE. Since $1\times b + 1\times b + 1\times b = 3\times b$, if as many units be taken as are in a, and each of them be multiplied by b and the products be added, the sum will be $a\times b$; but b taken as many times as there are units in a produces $b\times a$; therefore $a\times b$ is the same with $b\times a$, or $ab = ba$. In like manner abc, acb, bac, bca, cab, cba, are all the same, so that the factors may be placed in any order.

Again, since $ma = a + a + a$, &c. being repeated m times,
and $mb = b + b + b$, &c. being repeated m times;
therefore $ma + mb = (a+b) + (a+b) + (a+b)$ repeated m times, that is, $ma + mb = m(a+b)$. In like manner $ma - mb = m(a-b)$.

In multiplying $a - b$ by c, we may either first subtract and then multiply, or first multiply and then subtract. The latter is the order in algebra: we first multiply a by c, which makes ac, and then b by c, which makes bc, and subtract the latter product from the former to get the just product $ac - bc$, where the signs are the same with those of the multiplicand.

In multiplying $a - b$ by $c - d$, we first multiply $a - b$ by c as before, and it prodnces $ac - bc$; then we multiply $a - b$ by d, and it produces $ad - bd$, which we subtract from the former product, or change its signs, and it becomes $-ad + bd$, where the signs are contrary to those of the multiplicand.

The first and last terms shew that quantities with like signs produce $+$, and the other two terms shew that those which have unlike signs prodnce $-$.

DIVISION.

WHEN the divisor is a simple quantity, write it under the dividend in the form of a fraction, then cancel like quantities in them, and divide the coefficients by their greatest common measure.

When the signs are alike, the sign of the quotient is $+$; but if they be unlike, it is $-$.†

Powers of the same quantity are divided by subtracting the

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* ANSWERS. (9.) $6a^4x^4 + 2a^3x^3 - a^4x^2 - 8a^2x^2 + 22a^3x - 15a^4$. (10.) $x^4 - a^2x^2 + \frac{1}{2}a^3x - \frac{1}{16}a^4$. (11.) $x^2 - \frac{1}{4}a^2$. (12.) $x^4 + x^2y^2 + y^4$. (13.) $10a^4 - 27a^3x + 34a^2x^2 - 18ax^3 - 8x^4$. (14.) $6a^2 - 16ab + 19ac + 8b^2 - 18bc + 10c^2$. (15.) $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$. (16.) $a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1$.

† This is evident; for the divisor multiplied by the quotient must produce the dividend with its proper sign. The whole operation depends upon this principle, that the value of a quantity is not altered by both multiplying and dividing it by the same quantity.

exponent of the divisor from that of the dividend; the remainder is the exponent of the quotient.

If the dividend be compound, divide each term of it separately by the divisor.

Divide the following:

1. $56a^2b^3c$ by $8ab^3$. . . . . Ans. $7ac$.
2. $54xy^2$ by $36x^2y$. . . . $\frac{3y}{2x}$.
3. $63a^3b^2c^3 - 42a^2b^3c^3$ by $14a^2b^2c^2$. . $\frac{9ac}{2} - 3bc$.
4. $24x^3y - 18x^2y^2 + 15xy^3$ by $30xy^2$. . $\frac{4x^2}{5y} - \frac{3x}{5} + \frac{y}{2}$.

When the divisor is compound, arrange the terms of the dividend and divisor according to the powers of the same letter. Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient, then multiply the whole divisor by this term, and subtract the product from the dividend; bring down as many terms to the remainder as is requisite for a new dividual, with which proceed as before.

NOTE. When the last remainder is a simple quantity, place the divisor below it in the form of a fraction, and annex it with its proper sign to the quotient.

5. Divide $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.

$$a - b)a^3 - 3a^2b + 3ab^2 - b^3(a^2 - 2ab + b^2.$$
$$a^3 - a^2b$$
$$-2a^2b + 3ab^2$$
$$-2a^2b + 2ab^2$$
$$+ab^2 - b^3$$
$$+ab^2 - b^3.$$

6. $8a^3 - 4a^2b - 6ab^2 + 3b^3$ by $2a - b$. Ans. $4a^2 - 3b^2$.
7. $3b^3 + 3ab^2 - 4a^2b - 4a^3$ by $a + b$. . $-4a^2 + 3b^2$.
8. $a^4 - b^4$ by $a - b$. . . $a^3 + a^2b + ab^2 + b^3$.
9. $8a^4 + 2a^2b^2 - 3b^4$ by $2a^2 - b^2$. . $4a^2 + 3b^2$.
10. $2a^2x^2 - 5ax + 2$ by $2ax - 1$. . $ax - 2$.
11. $x^2 - x + \frac{1}{4}$ by $x - \frac{1}{2}$. . . $x - \frac{1}{2}$.
12. $21a^5 - 21b^5$ by $7a - 7b$.
    Ans. $3a^4 + 3a^3b + 3a^2b^2 + 3ab^3 + 3b^4$.
13. $x^4 - y^4 + 2y^2z^2 - z^4$ by $x^2 + y^2 - z^2$. . $x^2 - y^2 + z^2$.
14. $1 + a$ by $1 - a$. . $1 + 2a + 2a^2 + 2a^3 + 2a^4 +$, &c.

15. $8x^2 - 15y^2 + 23yz - 2xy - 8xz - 6z^2$ by $2x - 3y + z$. Ans. $4x + 5y - 6z$.

16. $a^2 - 2ab + b^2$ by $a^{\frac{1}{2}} + b^{\frac{1}{2}}$. . $a^{\frac{3}{2}} - ab^{\frac{1}{2}} - a^{\frac{1}{2}}b + b^{\frac{3}{2}}$.

17. $6x^4 - 96$ by $3x - 6$. . . $2x^3 + 4x^2 + 8x + 16$.

18. $1 + 2x$ by $1 - x$. . . . $1 + 3x + 3x^2 + 3x^3 +$, &c.

FRACTIONS.

A **fraction** is one or more parts of a unit. The denominator expresses the number of parts into which the unit is supposed to be divided, and the numerator expresses the number of these parts of which the fraction consists: thus, in the fraction $\frac{m}{n}$, n denotes the number of parts into which the unit is divided, and m points out the number of these parts of which the fraction consists. If the unit had been divided into $2n$ parts, then the fraction must have consisted of twice the number of these parts, and would have been $\frac{2m}{2n}$. In the same manner it might be expressed by $\frac{3m}{3n}$, $\frac{rm}{rn}$, &c.

Hence, the value of a fraction is not altered by multiplying or dividing both its terms by the same quantity.

REDUCTION.

PROBLEM I.

To reduce an integer or a mixed quantity to the form of a fraction.

If the denominator be given, multiply the integer by it for the numerator, and under the product place the denominator. If no denominator is given, place unit for it.

Hence, a mixed quantity may be reduced to the form of a fraction by multiplying the integer by the denominator of the fraction, and adding the numerator to the product for the numerator, below which place the denominator.

1. Reduce $3a$ to a fraction, of which the denominator is $2b$. Ans. $\frac{6ab}{2b}$.

2. Reduce $a + \frac{b}{c}$ to an improper fraction. . $\frac{ac + b}{c}$.

3. . . . . $x + \frac{a^2}{x}$. . . . . $\frac{x^2 + a^2}{x}$.

4. Reduce $x-\frac{a^2x^2}{x}$ to an imp. frac. Ans. $\frac{x^2-a^2x^2}{x}$.

5. . . . . $5-\frac{3x}{a}$. . . . $\frac{5a-3x}{a}$.

6. . . . . $a-\frac{ab-a^2}{2b}$.* . . . $\frac{ab+a^2}{2b}$.

7. . . . . $a-x-\frac{a^2x^2}{2x}$. . . . $\frac{2ax-2x^2-a^2x^2}{2x}$.

8. . . . . $a+1-\frac{x-1}{b}$. . . . $\frac{ab+b-x+1}{b}$.

9. . . . . $1+3a-\frac{4x-5}{4x}$. . . . $\frac{12ax+5}{4x}$.

PROBLEM II.

To reduce an improper fraction to an integer or a mixed quantity.

Divide the numerator by the denominator, the quotient is the integer, to which annex the remaining terms, with their proper signs, and the result will be the mixed number required.

1. Reduce $\frac{ab+b^2}{a}$ to a mixed quantity. Ans. $b+\frac{b^2}{a}$.

2. . . . . $\frac{ax+2x^2}{a+x}$. . . . $x+\frac{x^2}{a+x}$.

3. . . . . $\frac{x^2-y^2}{x+y}$. . . . $x-y$.

4. . . . . $\frac{x^3-y^3}{x-y}$. . . . x^2+xy+y^2.

5. . . . . $\frac{12x^2-18}{3x}$. . . . $4x-\frac{6}{x}$.

6. . . . . $\frac{4x^2-2x}{2x^2-x+1}$. . . . $2-\frac{2}{2x^2-x+1}$.

PROBLEM III.

To reduce fractions of different denominators to others of the same value which have a common denominator.

Multiply each of the numerators into all the denominators, except its own, for the new numerators, and all the denominators together for the common denominator.

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* When a fraction has the sign — before it, all the signs of the numerator are to be changed. Here $ab-a^2$ becomes $-ab+a^2$.

1. Reduce $\frac{3a}{b}$ and $\frac{2a}{3c}$ to a common denominator.

Ans. $\frac{9ac}{3bc}$ and $\frac{2ab}{3bc}$.

2. .... $\frac{1}{a+b}$ and $\frac{1}{a-b}$. . . $\frac{a-b}{a^2-b^2}$ and $\frac{a+b}{a^2-b^2}$.

3. .... $\frac{a}{1-x}$ and $\frac{b}{1+x}$. . . $\frac{a+ax}{1-x^2}$ and $\frac{b-bx}{1-x^2}$.

4. .... $\frac{x-y}{x+y}$ and $\frac{x+y}{x-y}$. $\frac{x^2-2xy+y^2}{x^2-y^2}$ and $\frac{x^2+2xy+y^2}{x^2-y^2}$.

5. .... $\frac{a+b}{c}$ and $\frac{3d}{m}$. . . . $\frac{am+bm}{cm}$ and $\frac{3cd}{cm}$.

6. .... $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{m}{n}$. . . . $\frac{adn}{bdn}$, $\frac{bcn}{bdn}$, $\frac{bdm}{bdn}$.

7. .... $\frac{2a}{3}$, $\frac{3b}{4}$, and $\frac{5c}{3d}$. . . $\frac{8ad}{12d}$, $\frac{9bd}{12d}$, $\frac{20c}{12d}$.

8. .... $2a$ and $\frac{3b}{4}$. . . . $\frac{8a}{4}$ and $\frac{3b}{4}$.

9. $\frac{7a^2}{x}$, $\frac{a}{4}$, $\frac{a^2-x^2}{a+x}$. . $\frac{28a^3+28a^2x}{4ax+4x^2}$, $\frac{a^2x+ax^2}{4ax+4x^2}$, $\frac{4a^2x-4x^3}{4ax+4x^2}$.

PROBLEM IV.

To reduce a fraction to lower terms.

Divide its numerator and denominator by any quantity which measures both.

The greatest divisor of the coefficients is found as in arithmetic, and the greatest simple divisor of the letters is discovered by inspection.

1. Reduce $\frac{ax^2-x^3}{ax+x^2}$ to lower terms. . Ans. $\frac{ax-x^2}{a+x}$.

2. .... $\frac{6a^2-12x^2}{3a-6x}$. . . . . $\frac{2a^2-4x^2}{a-2x}$.

3. .... $\frac{4a^2x^3}{2ax-2a^2}$. . . . . $\frac{2ax^3}{x-a}$.

4. .... $\frac{36a^2x^2}{24a^3x}$. . . . . $\frac{3x}{2a}$.

5. .... $\frac{9a^2-12ax+4x^2}{3ax-2x^2}$. . . $\frac{3a-2x}{x}$.

To find the greatest compound divisor.

Divide that term which is of the higher dimensions by the other and the divisor by the remainder continually, till nothing remains: the last divisor is the greatest common measure.

NOTE. The several divisors must be first divided by the greatest simple quantity which measures all their terms; and when the first term of a divisor is not contained an exact number of times in the first term of the dividend, the latter must be multiplied by any simple quantity that will make the division succeed. Also any compound quantity in a remainder which does not measure the divisor from which it proceeds, may be taken out of it. And when any of the divisors become negative, they may have all their signs changed without affecting the truth of the result.

What is the greatest common measure of

1. $\frac{a^4-b^4}{a^5+a^3b^2}$? . . . Ans. a^2+b^2.

2. $\frac{x^2-y^2}{x^4-y^4}$? . . . . x^2-y^2.

3. $\frac{x^4-y^4}{x^3-x^2y-xy^2+y^3}$? . . . x^2-y^2.

4. $\frac{6x^3-6x^2y+2xy^2-2y^3}{12x^2-15xy+3y^2}$? . . $x-y$.

5. $\frac{3bcq+30mp+18bc+5mpq}{24ad-7fgq-42fg+4adq}$?* . . $q+6$.

6. $\frac{x^3+ax^2+bx^2-2a^2x+bax-2ba^2}{x^2-bx+2ax-2ab}$? . $x+2a$.

7. Reduce $\frac{x^2-1}{xy+y}$ to its lowest terms. . $\frac{x-1}{y}$.

8. . . . . $\frac{ax+x^2}{ac^2+c^2x}$. . . Divide by $a+x$.

9. . . . . $\frac{x^3-a^2x}{x^2+2ax+a^2}$. . . . by $x+a$.

10. . . . . $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$. . . by a^2-x^2.

11. . . . . $\frac{5a^5+10a^4x+5a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$. . by $a+x$.

12. . . . . $\frac{a^5+a^3b^2}{a^4-b^4}$. . . . by a^2+b^2.

ADDITION AND SUBTRACTION.

REDUCE the fractions to a common denominator, if they have different ones; then add or subtract their numerators, and

* In fractions like this, where a letter is but of one dimension in either the numerator or the denominator, divide it into two parts, one of which has that letter in every term; then find the common measure of these two parts, and try whether it will divide the other quantity. Here the parts of the denominator are $4adq+24ad$ and $-7fgq-42fg$, and the common measure of these is $q+6$, which succeeds.

under the sum or the remainder write the common denominator, for the sum or the difference of the fractions.

1. Add $\frac{3a}{4}$, $\frac{5a}{6}$, $\frac{a}{3}$ together. . Ans. $\frac{23a}{12}$.

2. . . $\frac{x-3}{4}$, $\frac{5x+2}{3}$, $\frac{7x}{5}$. . . . $\frac{199x-5}{60}$.

3. . . $4x$, $\frac{3x^2}{2a}$, $\frac{x+a}{3x}$. . $\frac{9x^3+24ax^2+2ax+2a^2}{6ax}$.

4. . . $2a+\frac{a+3}{5}$, $4a+\frac{2a-5}{4}$. . $6a+\frac{14a-13}{20}$.

5. . . $\frac{x}{a}-\frac{x}{2a}$, $\frac{3x}{a}-\frac{4x}{2a}$, $4x$. . $4x+\frac{3x}{2a}$.

6. . . $x-\frac{a^2}{x}$, $a-\frac{a-x}{c}$. . $a+x+\frac{x^2-ax-a^2c}{cx}$.

7. From $3a-\frac{4x}{a}$, take $a+\frac{5x}{3a}$. . $2a-\frac{17x}{3a}$.

8. . . . $\frac{7x}{v}-\frac{4x^2}{5v}$, take $\frac{3x}{7v}-\frac{2x^2}{17v}$. . $\frac{46x}{7v}-\frac{58x^2}{85v}$.

9. . . . $\frac{x-y}{2a}$, take $\frac{x+y}{3a}$. . . . $\frac{x-5y}{6a}$.

What is the sum and the difference of

10. $\frac{x+y}{2}$ and $\frac{x-y}{2}$? . . Sum x, Diff. y.

11. $\frac{1}{a-b}$ and $\frac{1}{a+b}$? . Sum $\frac{2a}{a^2-b^2}$, Diff. $\frac{2b}{a^2-b^2}$.

12. $2x+\frac{3x}{a}$ and $x-\frac{2x-2a}{3c}$?

Sum $3x+\frac{9cx-2ax+2a^2}{3ac}$, Diff. $x+\frac{9cx+2ax-2a^2}{3ac}$.

MULTIPLICATION AND DIVISION.

Multiply the numerators together for the numerator of the product, and the denominators together for its denominator.

In division, invert the divisor and work as in multiplication.

1. Multiply $\frac{2x}{3}$ by $\frac{5x}{6}$. . . Ans. $\frac{5x^2}{9}$.

2. . . . . $\frac{x+a}{a+c}$ by $\frac{a}{x}$. . . . $\frac{ax+a^2}{ax+cx}$.

3. . . . . $b+\frac{bx}{a}$ by $\frac{a}{x}$. . . $\frac{ab}{x}+b$.

4. . . . . $\frac{ad}{2bc}$ by $\frac{4c}{d}$. . . . $\frac{2a}{b}$.

5. Divide $\frac{x}{3}$ by $\frac{2x}{9}$. . . . Ans. $1\frac{1}{2}$.

6. . . . . $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{x+a}$. . . $\frac{2x(x+a)}{a^3+x^3}$.

7. . . . . $\frac{x}{x-1}$ by $\frac{x}{2}$. . . . $\frac{2}{x-1}$.

8. . . . . $\frac{x^4-a^4}{x^2-2ax+a^2}$ by $\frac{x^2+ax}{x-a}$. . $\frac{x^2+a^2}{x}$.

9. . . . . $\frac{a+x}{b^2+2bx+x^2}$ by $\frac{1}{b+x}$. . . $\frac{a+x}{b+x}$.

NOTE. The four fundamental rules require the aid of those for fractions, when any terms of the given quantities, or of those which arise in the course of the operation, are fractional.

10. Multiply $\frac{a^2}{9}-\frac{ax}{3}+\frac{x^2}{4}$ by $\frac{a}{3}-\frac{x}{2}$. . Ans. $\left(\frac{a}{3}-\frac{x}{2}\right)^3$.

11. . . . . $\frac{a}{b}+\frac{c}{d}$ by $\frac{a}{b}-\frac{c}{d}$. . $\frac{a^2}{b^2}-\frac{c^2}{d^2}$.

12. . . . . $\frac{3a}{4b}+\frac{2c}{3d}$ by $\frac{3a}{4b}-\frac{2c}{3a}$. . . $\frac{9a^2}{16b^2}-\frac{4c^2}{9d^2}$.

13. . . . . $\frac{x^2}{a^2}+\frac{xy}{ac}+\frac{y^2}{c^2}$ by $\frac{x}{a}-\frac{y}{c}$. . $\frac{x^3}{a^3}-\frac{y^3}{c^3}$.

14. Divide a^2+b^2 by $a+b$. . . $a-b+\frac{2b^2}{a+b}$.

15. . . . . $\frac{x^2}{16}-\frac{xy}{6}+\frac{y^2}{9}$ by $\frac{x}{4}-\frac{y}{3}$. . $\frac{x}{4}-\frac{y}{3}$.

16. . . . . $\frac{x^3}{a^3}-\frac{z^3}{c^3}$ by $\frac{x}{a}-\frac{z}{c}$. . . $\frac{x^2}{a^2}+\frac{xz}{ac}+\frac{z^2}{c^2}$.

OF NEGATIVE QUANTITIES.

IF c be the difference between a and b, the algebraical expression for this is $a-b=c$, where a is supposed to be greater than b; if it be less, the expression is $a-b=-c$. As, however, a greater quantity cannot be taken from a less, the expression $-c$ is impossible; so that a negative quantity standing by itself has, strictly speaking, no meaning. But if it be joined to another quantity, as $m-c$, the expression is proper, and may be subjected to all the operations of algebra. The absurdity appears only in the result; and when it does appear, it points out that something impossible has been admitted into the question, some condition inconsistent with its other conditions. We therefore reckon a negative result to be a proper algebraical solution of a problem; for it agrees with the preceding steps of the process, and points out the impossibility of the conditions, and thus it has its use in limiting the terms of the question. It will therefore be necessary in what follows to attend to negative expressions, and the forms which result from them, as well as from the positive ones. But this should create no hesitation in the operations; for it has been shown, not only how whole quantities, but also how single terms of them, may be added together or subtracted from one another, and how they may be multiplied or divided by one another with the signs of the resulting terms. But it is to be remarked, that these signs do not belong to the terms taken as isolated quantities, but to the relation in which they stand to the other terms of the result. When Diophantus of old said, "A defect drawn into a defect produces an excess," he did not by *a defect* mean a simple quantity, without relation to any other quantity: he meant to express by it, what one quantity wanted to make it equal to another, and that after the sum of the products of the wholes by these defects had been subtracted from the product of the wholes, the true product would exceed the remainder by the product of the defects, which must therefore be added to the remainder. And that this is the case, has been proved before, in the note explaining Multiplication. It is therefore improper to apply to simple quantities the rules by which the terms of compound quantities are connected together; and much of the obscurity of algebra has arisen from this confusion.

If $a-x$ be multiplied by itself, the product is $a^2-2ax+x^2$; and if $x-a$ be multiplied by itself, the product is the same; so that from this product it cannot be determined whether a be greater or less than x; that is, if $a-x=c$, whether the product has arisen from $+c$ or from $-c$, for each of these multiplied by itself produces $+c^2$, and therefore the square root of $+c^2$ may be either $+c$ or $-c$, and of course the square root of $-c^2$ is impossible. This expression is in some instances found useful for promoting the investigation of rules.

The formula $a^2-b^2=(a+b)\times(a-b)$ is useful in every branch of the mathematics. Now $a^2+b^2=a^2-b^2\times-1=(a+b\sqrt{-1})\times(a-b\sqrt{-1})$. This latter expression is therefore useful in several investigations.

The algebraist does not consider the solution of a problem to be complete, unless it exhibit all the cases which can occur; and the results which flow from contradictory suppositions can only be exhibited by such expressions as have been just now explained.

In the application of algebra to various sciences, where position and other states must be introduced, quantities are often found in such opposite states, that when in one of them they are to be added, they must be invariably subtracted in the other. These different states may therefore be naturally pointed out by prefixing the sign $+$ to the quantity when it is in one of them, and the sign $-$ when it is in the opposite state; and this use does not appear to alter in the smallest degree the meaning affixed to these signs in the definitions, for here they are prefixed solely for the purpose of subjecting the quantity to algebraical processes.

From the whole it appears, that the meaning of the signs $+$ and $-$ given in the definitions ought to be steadily adhered to, by which means many of the difficulties of beginners would be avoided.

In dividing a^5 by a^2, we either place the quantities in the form of a fraction, $\frac{a^5}{a^2}$, and expunge like quantities, which gives a^3 for the quotient, or else we subtract the exponent of the divisor from that of the dividend, $a^{5-2} = a^3$. These two methods make the quotients to have in some cases different appearances. Suppose a^2 to be divided by a^5. By the former method $\frac{a^2}{a^5} = \frac{1}{a^3}$. By the second $a^{2-5} = a^{-3}$; so that $a^{-3} = \frac{1}{a^3}$. Here the negative exponent does not represent a negative quantity, but only shows that the quantity placed in the numerator ought to be in the denominator; but in either place it can be subjected with equal ease to all the rules of algebra. From this it appears, that any quantity may be removed from the numerator to the denominator, or from the denominator to the numerator, by changing the sign of its exponent. Thus $\frac{a^2b}{c^2} = a^2bc^{-2}$, and $ab^{-3}c^2 = \frac{ac^2}{b^3}$.

INVOLUTION.

Involution is the method of finding the powers of quantities.

When the quantity is simple, multiply the exponent of each letter by the name of the power to which it is to be raised, and prefix the same power of the coefficient.

If the sign of the quantity be $+$, all its powers are positive;

but if the sign be —, its odd powers have —, and all the rest have +.*

In a fraction, raise its terms separately to the power required.

1. Raise $+3ab^2$ to the 3d power. . Ans. $+27a^3b^6$.

2. . . . $-2a^3x$ to the 6th power. . $+64a^{18}x^6$.

3. . . . $+\frac{4a^3bc^2}{3c}$ to the 5th power. . $+\frac{1024a^{15}b^5c^{10}}{243c^5}$.

4. . . . $-\frac{7a^2}{3b^3}$ to the 3d power. . $-\frac{343a^6}{27b^9}$.

5. . . . $+\frac{2a^{\frac{1}{2}}b^{\frac{3}{4}}c}{3x^{\frac{1}{3}}v^{\frac{1}{2}}}$ to the 8th power. . $+\frac{256a^4b^6c^8}{6561x^{\frac{8}{3}}v^4}$.

When the quantity is compound, raise it by actual multiplication.

6. Thus the powers of $a+b$ are,

2d, $= a^2+2ab+b^2$.

3d, $= a^3+3a^2b+3ab^2+b^3$.

4th, $= a^4+4a^3b+6a^2b^2+4ab^3+b^4$.

5th, $= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

6th, $= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

The powers of $a-b$ are the same with those of $a+b$, except that the signs of the even terms are —, all the rest are +.

Hence it appears,

1. That the number of terms is one greater than the name of the power.

2. That the exponent of the leading quantity in the first term is the name of the power, and that it decreases by 1 in each of the following terms to the last, where it is 0.

3. That the second quantity is not found in the first term; in the second its exponent is 1; and it increases by 1 in each of the following terms to the last, in which it is the name of the power.

4. That the coefficient of the first term is 1, that of the second is the name of the power, and in the following terms it

* It was shown in Multiplication, that $-x^m \times -x^m = +x^{2m}$, and $+x^{2m} \times -x^m = -x^{3m}$. Hence x^m raised to the nth power $= x^{mn}$, and $-x^m$ raised to the nth power is either $+x^{mn}$ or $-x^{mn}$, according as n is even or odd.

is got by multiplying the coefficient of the preceding term by the exponent of the leading quantity in that term, and dividing the product by the number of that term.

5. That when the signs of both quantities are alike, all the terms have the sign $+$; but if the signs of the quantities be different, the odd terms have $+$, and the even terms $-$.

7. Raise $x-v$ to the 7th power.

Ans. $x^7-7x^6v+21x^5v^2-35x^4v^3+35x^3v^4-21x^2v^5+7xv^6-v^7$.

8. Raise $m-n$ to the 8th power.

Ans. $m^8-8m^7n+28m^6n^2-56m^5n^3+70m^4n^4-56m^3n^5+28m^2n^6-8mn^7+n^8$.

9. Raise $ab-cd$ to the 5th power.

Ans. $a^5b^5-5a^4b^4cd+10a^3b^3c^2d^2-10a^2b^2c^3d^3+5abc^4d^4-c^5d^5$.

10. Raise $2a-3b$ to the 4th power.

Ans. $(2a)^4-4(2a)^3(3b)+6(2a)^2(3b)^2-4(2a)(3b)^3+(3b)^4=16a^4-96a^3b+216a^2b^2-216ab^3+81b^4$.

NOTE. In this manner care must be taken to distinguish the quantities affected by the different exponents, and to raise them accordingly.

11. Raise $8rs-5vs$ to the 3d power.

Ans. $512r^3s^3-960r^2s^3v+600rs^3v^2-125s^3v^3$.

12. Raise x^2-v^2 to the 5th power.

Ans. $x^{10}-5x^8v^2+10x^6v^4-10x^4v^6+5x^2v^8-v^{10}$.

13. Raise a^2-2ab to the 6th power.

Ans. $a^{12}-12a^{11}b+60a^{10}b^2-160a^9b^3+240a^8b^4-192a^7b^5+64a^6b^6$.

14. Raise $2ac-c^2$ to the 7th power.

Ans. $128a^7c^7-448a^6c^8+672a^5c^2-560a^4c^{10}+280a^3c^{11}-84a^2c^{12}+14ac^{13}-c^{14}$.

15. Raise $3x^2-4xv$ to the 4th power.

Ans. $81x^8-432x^7v+864x^6v^2-768x^5v^3+256x^4v^4$.

16. Raise $5a^2c-3xv^2$ to the 3d power.

Ans. $125a^6c^3-225a^4c^2xv^2+135a^2cx^2v^4-27x^3v^6$.

17. Raise $a + b$ to the nth power.

Ans. $a^n + na^{n-1}b + n.\frac{n-1}{2}a^{n-2}b^2 + n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3$, &c. or dividing by a^n, and putting A, B, C, &c. for the preceding terms with their signs, it becomes $a^n \times \left(1 + \frac{nb}{a} + \frac{n-1}{2}.\frac{bA}{a} + \frac{n-2}{3}.\frac{bB}{a} + \frac{n-3}{4}.\frac{bC}{a}, \text{&c.}\right)$ where the law of continuation is evident.

If the quantity consists of more than two terms, divide the terms into two classes, and raise them as if each class were a simple quantity; after which the classes must be raised according to the exponents placed over them, and then connected with one another, and with the coefficients by multiplication.

18. Raise $a + b - c$ to the 3d power.

Ans. $(a+b)^3 - 3 \times (a+b)^2c + 3(a+b)c^2 - c^3 = a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c - 6abc - 3b^2c + 3ac^2 + 3bc^2 - c^3$.

19. Raise $a^2 + b^2 - c^2$ to the 2d power.

Ans. $a^4 + 2a^2b^2 + b^4 - 2a^2c^2 - 2b^2c^2 + c^4$.

20. Raise $a^2 - 2ab + b^2$ to the 4th power.

Ans. $a^8 - 8a^7b + 28a^6b^2 - 56a^5b^3 + 70a^4b^4 - 56a^3b^5 + 28a^2b^6 - 8ab^7 + b^8$.

21. Raise $a - b + c - d = (a - b) + (c - d)$ to the 3d power.

Ans. $a^3 - 3a^2b + 3ab^2 - b^3 + 3a^2c - 3a^2d - 6abc + 6abd + 3b^2c - 3b^2d + 3ac^2 - 3bc^2 - 6acd + 6bcd + 3ad^2 - 3bd^2 + c^3 - 3c^2d + 3cd^2 - d^3$.

EVOLUTION.

Evolution is the method of finding the roots of quantities, or those from which given powers have been raised.

In simple quantities, divide the exponents of the letters by the name of the root required, and prefix the same root of the coefficients.

If the sign of the given quantity be $+$, the sign of the root is also $+$. If the sign of the quantity be $-$, the sign of its odd roots is $-$; but it can have no even root, for the square of $+a$, and also of $-a$, is $+a^2$.*

* It was shown in the note on Involution, that x^{mn} is the nth power of x^m, therefore, $x^{\frac{mn}{n}}$ is the nth root of x^{mn}, and consequently that $\frac{1}{n}$ is the proper exponent of the nth root; also that the nth power of $-x^m$ is either $+x^{mn}$ or $-x^{mn}$, according as n is even or odd. Therefore, in the first case, $+x^{\frac{mn}{n}}$, when n is even, may be either $+x^m$ or $-x^m$, and that in this case $-x^{\frac{mn}{n}}$ is impossible.

1. Required the 3d root of a^6b^3. . Ans. a^2b.

2. . . . . 4th root of $\frac{16a^4b^6c^8}{81a^8}$. . $\frac{2ab^{\frac{3}{2}}c^2}{3a^2}$.

3. . . . . 5th root of $\frac{32a^{10}b^8c^5}{c^6x^3}$. . $\frac{2a^2b^{\frac{8}{5}}c}{c^{\frac{6}{5}}x^{\frac{3}{5}}}$.

4. . . . . 6th root of $\frac{m^3n^5}{c^6e^7}$. . . $\frac{m^{\frac{1}{2}}n^{\frac{5}{6}}}{ce^{\frac{7}{6}}}$.

TO FIND THE SQUARE ROOT OF A COMPOUND QUANTITY.

Arrange the terms according to the dimensions of some letter in them, and take the square root of the first term for the first term of the root; subtract its square from the given quantity, and bring down the two next terms to the remainder for a resolvend. Double the root for a divisor, by which divide the first term of the resolvend to get another term of the root; annex this term with its proper sign to the divisor, then multiply the divisor thus completed by it, and subtract the product from the resolvend, and proceed in the same way with the remainder, as in common arithmetic.

1. Required the square root of $x^2-2xv+v^2$.

$$\begin{array}{r|l} & x^2-2xv+v^2\,(x-v \text{ root.} \\ & x^2 \\ 2x-v) & \overline{\quad -2xv+v^2} \\ & -2xv+v^2. \end{array}$$

2. $\sqrt{(x^4-2x^2+1)}$ . . $= x^2-1$.

3. $\sqrt{\left(\frac{x^2}{4}-xv+v^2\right)}$ . . $= \frac{x}{2}-v$.

4. $\sqrt{(x^4-4x^3a+6x^2a^2-4xa^3+a^4)} = x^2-2xa+a^2$.

5. $\sqrt{\left(\frac{a^2}{c^2}-\frac{2ax}{c}+x^2\right)}$ . . $= \frac{a}{c}-x$.

6. $\sqrt{(a^2+2ab+b^2+2ac+2bc+c^2)} = a+b+c$.

7. $\sqrt{(4x^4+6x^3+\frac{89x^2}{4}+15x+25)} = 2x^2+\frac{3x}{2}+5$.

8. $\sqrt{(x^6+4x^5+2x^4+9x^2-4x+4)} = x^3+2x^2-x+2$.

TO EXTRACT ANY OTHER ROOT.

Arrange the terms as in Division; take the root of the first term for the first term of the root; raise this root to a power less by one than the given power, and multiply it by the name of the root for a divisor, by which divide the second term of the given quantity to get another term of the root. Raise the whole root thus found to the given power, and subtract it from the given quantity; if there be a remainder, divide its first term, by the divisor got before, to obtain another term of the root, and proceed as before.

1. Required the cube root of $x^3+3x^2v+3xv^2+v^3$.

$$x^3+3x^2v+3xv^2+v^3\left(x+v \text{ root.}\right.$$
$$x^3$$
$$3x^2)\overline{\quad +3x^2v}$$
$$(x+v)^3=x^3+3x^2v+3xv^2+v^3.$$

2. $(27a^3-54a^2c+36ac^2-8c^3)^{\frac{1}{3}} \quad = 3a-2c.$

3. $(m^6+6m^5-40m^3+96m-64)^{\frac{1}{3}} \quad = m^2+2m-4.$

4. $(16x^4-96x^3y+216x^2y^2-216xy^3+81y^4)^{\frac{1}{4}}=2x-3y.$

5. $(81a^4-432a^3c+864a^2c^2-768ac^3+256c^4)^{\frac{1}{4}}=3a-4c.$

6. $\left(x^5-\frac{5x^4v}{2}+\frac{10x^3v^2}{4}-\frac{10x^2v^3}{8}+\frac{5xv^4}{16}-\frac{v^5}{32}\right)^{\frac{1}{5}} \quad = x-\frac{v}{2}.$

7. $\left(x^6-9x^5+\frac{135x^4}{4}-\frac{135x^3}{2}+\frac{1215x^2}{16}-\frac{729x}{16}+\frac{729}{64}\right)^{\frac{1}{6}}=$ $x-1\frac{1}{2}.$

OF IRRATIONAL QUANTITIES OR SURDS.

Irrational Quantities or Surds are expressions of the roots of such quantities as are not complete powers.

Thus $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$ is a surd, because a^2 is not a cube.

TO REDUCE SURDS TO A COMMON EXPONENT.

Express them with fractional exponents, and reduce these exponents to a common denominator. This denominator is the common exponent of the root, and the numerators are the exponents of the powers to which the quantities are to be raised.

Note. An integer may be expressed as a surd by raising it to any power, and then making the name of the power the exponent of the root: thus $a=a^{\frac{2}{2}}=a^{\frac{4}{4}}$, also $2=\sqrt[3]{8}=\sqrt[5]{32}$.

1. Reduce $\sqrt{3}$ and $\sqrt[3]{2}$ to the same exponent; here $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ $=3^{\frac{3}{6}}$ and $2^{\frac{2}{6}}=\sqrt[6]{27}$ and $\sqrt[6]{4}$.

2. Reduce a and $x^{\frac{1}{4}}$. . Ans. $a^{\frac{4}{4}}$ and $x^{\frac{1}{4}}$.

3. . . . . $\sqrt[4]{15}$ and $\sqrt[6]{9}$. . $\sqrt[12]{3375}$ and $\sqrt[12]{81}$.

4. . . . . $(a+b)^{\frac{1}{2}}$ and $(a-b)^{\frac{1}{3}}$. $(a+b)^{\frac{3}{6}}$ and $(a-b)^{\frac{2}{6}}$.

5. . . . . $(4a)^{\frac{1}{3}}$ and $(3b)^{\frac{1}{4}}$. . $\sqrt[12]{256a^4}$ and $\sqrt[12]{27b^3}$.

6. . . . . $x^{\frac{1}{n}}$ and $v^{\frac{1}{m}}$. . . $x^{\frac{m}{mn}}$ and $v^{\frac{n}{mn}}$.

TO REDUCE A SURD TO ITS MOST SIMPLE FORM.

If any power of the same name with the surd, measures the quantity under the radical sign, place the quotient under the radical, and the root of that power before it for the rational part.

If no such power can be found, the surd is already in its most simple form.

1. Reduce $\sqrt{75}$ to its most simple form. Ans. $5\sqrt{3}$.

2. . . . . $\sqrt[3]{81}$. . . $3\sqrt[3]{3}$.

3. . . . . $\sqrt[3]{243}$ and $\sqrt[3]{16}$. . . $3\sqrt[3]{9}$ and $2\sqrt[3]{2}$.

4. . . . . $\sqrt{98a^2x}$. . . . $7a\sqrt{2x}$.

5. . . . . $(x^3-ax^2)^{\frac{1}{2}}$. . . . $x(x-a)^{\frac{1}{2}}$.

6. . . . . $(a^4x+3a^3x^2)^{\frac{1}{3}}$. . . $a\sqrt[3]{x}\times(a+3x)^{\frac{1}{3}}$.

7. . . . . $(32a^6-96a^5x)^{\frac{1}{5}}$. . . $2a(a-3x)^{\frac{1}{5}}$.

TO ADD AND SUBTRACT SURDS.

Reduce them to the same exponent, and to their most simple forms: then, if the quantity under the radical sign be the same in them all, add or subtract the rational parts, and to the sum or difference annex the common surd. But if the quantities under the radical be different, the surds must be added or subtracted as unlike quantities.

1. Add $3\sqrt{2}$ and $2\sqrt{2}$. . Ans. $5\sqrt{2}$.

2. . . . $ab^{\frac{1}{3}}$ and $\frac{3ab^{\frac{1}{3}}}{2}$. . . $\frac{5ab^{\frac{1}{3}}}{2}$.

3. . . . $\sqrt[3]{48a^7}$ and $\sqrt[3]{6a}$. . . $(2a^2+1)\sqrt[3]{6a}$.

4. . . . $2\sqrt{a^2x}$ and $3\sqrt{64x^3}$. . $(2a+24x)\sqrt{x}$.

5. From $9a\sqrt{3}$ take $2a\sqrt{3}$. . $7a\sqrt{3}$.

6. . . . $\sqrt[3]{81a}$ take $\sqrt[3]{24a}$. . $\sqrt[3]{3a}$.

7. From $2\sqrt{50}$ take $\sqrt{18}$. . Ans. $7\sqrt{2}$.

8. . . . $\sqrt{80a^4x}$ take $\sqrt{20a^2x^3}$. . $(4a^2 - 2ax)\sqrt{5x}$.

9. Add and subtract $3\sqrt{\frac{5}{27}}$, $4\sqrt{\frac{3}{5}}$. . $\frac{17}{15}\sqrt{15}$ and $\frac{7}{15}\sqrt{15}$.

TO MULTIPLY AND DIVIDE SURDS.

Reduce them to a common exponent, if they have different ones, and then find the product or quotient of the rational parts, and also of the surds; and the two joined together, with the common radical sign between them, will give the whole product or quotient required.

NOTE. When the quantities under the radical signs are alike, the product or quotient of the surds is found by adding or subtracting their exponents.

1. Multiply $\sqrt{2}$ by $\sqrt[3]{2}$. . Ans. $\sqrt[6]{32}$.
2. . . . . $\sqrt[3]{4}$ by $\sqrt[3]{5}$. . . $\sqrt[3]{20}$.
3. . . . . $a^{\frac{1}{3}}$ by $a^{\frac{3}{4}}$. . . $a\sqrt[12]{a}$.
4. . . . . $a^{\frac{1}{2}}$ by $b^{\frac{2}{3}}$. . . $\sqrt[6]{a^3b^4}$.
5. . . . . $2\sqrt{3}$ by $3\sqrt[3]{4}$. . . $6\sqrt[6]{432}$.
6. Divide $\sqrt{7}$ by $\sqrt[4]{7}$. . . $\sqrt[4]{7}$.
7. . . . . $\sqrt[3]{8}$ by $\sqrt[3]{2}$. . . $\sqrt[3]{4}$.
8. . . . . $a^{\frac{3}{4}}b^{\frac{1}{2}}$ by $a^{\frac{2}{3}}b^{\frac{1}{3}}$. . . $a^{\frac{1}{12}}b^{\frac{1}{6}}$.
9. . . . . $2\sqrt[3]{bc}$ by $3\sqrt{ac}$. . . $\frac{2}{3}\sqrt[6]{\frac{b^2}{a^3c}}$.
10. . . . . $10\sqrt[3]{108}$ by $5\sqrt[3]{84}$. . $\frac{2}{7}\sqrt[3]{441}$.

INVOLUTION AND EVOLUTION OF SURDS.

The powers and roots of surds are found as those of other quantities, by multiplying or dividing their exponents by the name of the power or root.

In some cases it is preferable to raise the quantity under the radical to the power or root required, and then to place the radical sign over it.

1. The 4th power of $\sqrt{3a}$ $= 9a^2$.
2. The 3d power of $(a - b)^{\frac{1}{3}} = a - b$.
3. The 4th power of $\frac{1}{6}\sqrt{6}$ $= \frac{1}{36}$.

4. The 5th power of $\frac{2\sqrt{a}}{3\sqrt[5]{c}}$ $= \frac{32\sqrt{a^5}}{243c}$.

5. The 3d root of $a^{\frac{1}{2}}b^{\frac{3}{4}}c$ $= a^{\frac{1}{6}}b^{\frac{1}{4}}c^{\frac{1}{3}}$.

6. The 4th root of $\frac{ab^{\frac{1}{2}}c^4}{a^3}$ $= \frac{a^{\frac{1}{4}}b^{\frac{1}{8}}c}{a^{\frac{3}{4}}}$.

7. The 3d root of $\frac{1}{8}\sqrt{2}$ $= \frac{1}{2}\sqrt[6]{2}$.

8. The 5th root of $\frac{b^{\frac{2}{3}}}{32a^{\frac{1}{3}}}$ $= \frac{1}{2}\sqrt[15]{\frac{b^2}{a}}$.

TO FIND THE SQUARE ROOT OF A COMPOUND SURD.

When a quantity consists of two terms, a rational and a surd; if it has a root, the rational part is the sum of the squares of its terms, and the surd is the double of their product.

From the square of the rational term subtract the quantity affected by the radical sign, and take the square root of the remainder; add it to the rational term, and also subtract it from that term, and take the halves of the sum and remainder for the squares of the two terms of the root.

1. $(6-\sqrt{20})^{\frac{1}{2}} = \sqrt{5}-1$, for $\sqrt{36-20} = \sqrt{16} = 4$, and $\sqrt{\frac{6\pm4}{2}} = \sqrt{5}$ and 1.

2. $(136-96\sqrt{2})^{\frac{1}{2}} = 6\sqrt{2}-8$.

3. $(51-10\sqrt{2})^{\frac{1}{2}} = 5\sqrt{2}-1$.

4. $(14-6\sqrt{5})^{\frac{1}{2}} = 3-\sqrt{5}$.

5. $(5-2\sqrt{6})^{\frac{1}{2}} = \sqrt{3}-\sqrt{2}$.

6. $(76-42\sqrt{3})^{\frac{1}{2}} = 7-3\sqrt{3}$.

7. $(19+8\sqrt{3})^{\frac{1}{2}} = 4+\sqrt{3}$.

8. $(12-2\sqrt{35})^{\frac{1}{2}} = \sqrt{7}-\sqrt{5}$.

9. $(7+4\sqrt{3})^{\frac{1}{2}} = 2+\sqrt{3}$.

10. $(7-2\sqrt{10})^{\frac{1}{2}} = \sqrt{5}-\sqrt{2}$.

11. $(39-6\sqrt{30})^{\frac{1}{2}} = \sqrt{30}-3$.

EQUATIONS.

WHEN two expressions are equal to one another, they are written with the sign $=$ of equality between them, and the whole is called an equation. Thus $x-a=b+c$ is an equation; $x-a$ is called the left side, and $b+c$ the right side of the equation.

An equation which contains only the first power of the unknown quantity or quantities is called a simple equation.

RESOLUTION OF SIMPLE EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.

The resolution of simple equations containing one unknown quantity consists in separating the unknown quantity from the other quantities with which it is connected, and making it stand alone upon one side of the equation, and the known quantities upon the other side. This is performed by the following rules taken in their order:

RULE 1. If a term be divided by any quantity, multiply every term by the divisor.

In this way the equation may be cleared of fractions.

RULE 2. Any term may be transposed from one side of the equation to the other, by changing its sign from $+$ to $-$, or from $-$ to $+$.

In this way the terms containing the unknown quantity may be brought to one side of the equation, and the known terms to the other; after which they may be collected by addition.

COR. If a term be found on both sides with the same sign, it may be erased from both.

RULE 3. If the unknown quantity be multiplied by any other, divide both sides by the multiplier.

In this way the value of the unknown quantity is found, when there are no surds nor powers.

RULE 4. If the equation have a surd in it, after bringing it to one side by itself, take away the radical sign, and raise the other side to the corresponding power.

RULE 5. If one side of the equation be a complete power, take the corresponding root of both sides.*

* It is evident that the operations prescribed in these rules do not render the two sides of the equation unequal, for they are both increased or diminished in the same degree. Thus, in the first operation, both sides are multiplied by the same quantity; in transposition the same quantity is subtracted from both sides; in the third both sides are divided by the same quantity; in the fourth they are both raised to the same power; and in the last the same root is taken of both sides.

Let the equation be $2x-\frac{19}{4}=\frac{3x}{4}+4$

Multiply by 4, . . $8x-19=3x+16$
Add $19-3x$ to both sides, $8x-3x=16+19$
And collecting, . . . $5x=35$
Divide by 5, . . . $x=7$
So that 7 is the value of x.

In the second line the equation is cleared of fractions, and in the third line the quantities 19 and $3x$ are transposed with their signs changed; and it is evident that the two sides of the equation have been kept equal to one another in every line.

Let the equation be $(3x+1)^{\frac{1}{2}}+5=10$

By transposing 5, $(3x+1)^{\frac{1}{2}}=10-5=5$
Square by rule 4, $3x+1=25$
Transposing 1, . . $3x=25-1=24$
And dividing by 3, . $x=8$.

The removal of the sign from the radical is equivalent to the raising of it to the power.

Let the equation be . $9x^2+9=3x^2+63$
By transposing, . $9x^2-3x^2=63-9$
Collecting, . . . $6x^2=54$
Dividing by 6, . . $x^2=9$
Taking the square root, . $x=3$.

Any analogy or proportion may be changed into an equation by making the product of the first and last terms equal to the product of the two mean terms.

Let $2+x:6-x::15:9$
Then $9(2+x)=15(6-x$
Or $18+9x=90-15x$
Transposing, . $9x+16x=90-18$
And collecting, . $24x=72$
Dividing by 24, . $x=3$.

Again, let $x-5:2x::5:20$
Then $20\times(x-5)=5\times 2x$
Or $20x-100=10x$
Transposing, $20x-10x=100$
And collecting, . $10x=100$
Dividing by 10, . $x=10$.

RESOLVE THE FOLLOWING EQUATIONS:

EQUATIONS.	ANSWERS.
1. $5x+3=2x+15$.	$x=4$.
2. $24-2x=3x-6$.	$x=6$.

EQUATIONS.	ANSWERS.
3. $15x-26=12x+16$.	$x=14$.
4. $\frac{x}{2}-3=5$.	$x=16$.
5. $6-x=4-\frac{2x}{3}$.	$x=6$.
6. $4x-8=3x+20$.	$x=28$.
7. $40-6x-16=120-14x$.	$x=12$.
8. $x+\frac{x}{2}+\frac{x}{3}=11$.	$x=6$.
9. $ax+2ab=3c^2$.	$x=\frac{3c^2}{a}-2b$.
10. $5ax-3b=2dx+c$.	$x=\frac{3b+c}{5a-2d}$.
11. $2x-\frac{x}{2}+1=5x-2$.	$x=\frac{6}{7}$.
12. $x^{\frac{1}{2}}-2=6$.	$x=64$.
13. $(4x+16)^{\frac{1}{2}}=12$.	$x=32$.
14. $5x-15=2x+6$.	$x=7$.
15. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=10$.	$x=9\frac{3}{13}$.
16. $3x^2-x=8x+x^2$.	$x=4\frac{1}{2}$.
17. $x-a=\frac{x^2}{x-a}$.	$x=\frac{a}{2}$.
18. $(2x+3)^{\frac{1}{3}}+4=8$.	$x=30\frac{1}{2}$.
19. $\left(\frac{2x}{3}\right)^{\frac{1}{2}}+5=7$.	$x=6$.
20. $\frac{x-3}{2}+\frac{x}{5}=20-\frac{x-19}{2}$.	$x=25\frac{5}{6}$.
21. $\frac{a}{1+x}+\frac{a}{1-x}=b$.	$x=\left(\frac{b-2a}{b}\right)^{\frac{1}{2}}$.
22. $x+(a^2+x^2)^{\frac{1}{2}}=\frac{2a^2}{(a^2+x^2)^{\frac{1}{2}}}$.	$x=\frac{a}{\sqrt{3}}$.
23. $x^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}=\frac{2a}{(a+x)^{\frac{1}{2}}}$.	$x=\frac{a}{3}$.
24. $(12+x)^{\frac{1}{2}}=2+x^{\frac{1}{2}}$.	$x=4$.

EQUATIONS.	ANSWERS.
25. $(a^2+x^2)^{\frac{1}{2}}=(b^4+x^4)^{\frac{1}{4}}$.	$x=\left(\frac{b^4-a^4}{2a^2}\right)^{\frac{1}{2}}$.
26. $bx^2+c+3=2bx^2+1$.	$x=\left(\frac{c+2}{b}\right)^{\frac{1}{2}}$.
27. $4x-\frac{x-1}{2}=x+\frac{2x-2}{5}+24$.	$x=11$.
28. $a+x=[a^2+x(b^2+x^2)^{\frac{1}{2}}]^{\frac{1}{2}}$.	$x=\frac{b^2}{4a}-a$.
29. $\frac{3x}{a}-c+\frac{x}{b}=4x+\frac{2x}{d}$.	$x=\frac{abcd}{(3b+a)d-2ab(2d+1)}$.
30. $3x-\frac{a}{b}-cx=\frac{a+x}{3}-\frac{b-x}{a}$.	$x=\frac{a^2b-3b^2+3a^2}{8ab-3abc-3b}$.
31. $5ax-2b+4bx=2x+5c$.	$x=\frac{5c+2b}{5a+4b-2}$.

RESOLUTION OF SIMPLE EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

When there are several unknown quantities, there must be as many independent equations involving them: and from these an equation must be deduced, which contains only one of the unknown quantities.

This may be performed by any of the following rules:

Rule I. Find a value of one of the unknown quantities in each of the equations, supposing all the rest to be known. Make these values equal to one another, and from them find the value of another unknown quantity. Make again these values equal, and find another unknown quantity, and so on, until an equation be obtained containing only one unknown quantity, which is to be resolved by the preceding rules.

Rule II. Find a value of one of the unknown quantities in that equation in which it is least involved; substitute this value and its powers for that unknown quantity and its powers in all the other equations, and proceed in the same way with these equations to get rid of other unknown quantities.

Rule III. Multiply the equations by such quantities as will make the coefficients of one of the unknown quantities, or of its highest power, the same in all the equations; then, if the signs of these equal terms be like, subtract the equations, but if the signs be unlike, add them, and new equations will arise, wanting that unknown quantity or its highest power, and these equations are to be treated in the same way.

NOTE. The first method seems to be the most regular; the second is shorter than the first, but the reductions are more intricate; the third is the most simple and expeditious.

Let the equations be $x + y = 12$, and $5x + 3y = 50$ } To find the values of x and y.

By RULE I.

From the 1st equation $x = 12 - y$, and from the 2d $x = \frac{50-3y}{5}$

$\therefore \quad 12 - y = \frac{50-3y}{5}$, clearing this equation of fractions $60 - 5y = 50 - 3y$, transposing and collecting $10 = 2y$ or $y = 5$, and $x = 12 - y = 7$.

By RULE II.

From the 1st equation $x = 12 - y$; substituting this value for x in the 2d equation, we have

$$5(12 - y) + 3y = 50$$

Or $\quad 60 - 5y + 3y = 50$

$\therefore \quad 10 = 2y$, or $y = 5$, and $x = 12 - y = 7$.

By RULE III.

1st Equation multiplied by 5,	$5x + 5y = 60$
2d Equation, . . .	$5x + 3y = 50$
By subtraction, . . .	$2y = 10$
	$\therefore \ y = 5$.

Let the equations be $\left.\begin{aligned} x + y + z &= 53 \\ x + 2y + 3z &= 105 \\ x + 3y + 4z &= 134 \end{aligned}\right\}$ To find the values of x, y, and z.

By RULE I.

From the 1st equation $x = 53 - y - z$, from the 2d $x = 105 - 2y - 3z$, and from the 3d $x = 134 - 3y - 4z$; whence

$$53 - y - z = 105 - 2y - 3z$$
$$53 - y - z = 134 - 3y - 4z,$$

and these equations, by transposing and collecting, become

$$y + 2z = 52$$
$$2y + 3z = 81.$$

Now from the 1st of these $y = 52 - 2z$, and from the 2d $y = \frac{81 - 3z}{2}$; whence $52 - 2z = \frac{81 - 3z}{2}$, an equation containing only one unknown quantity, which, by clearing of fractions, transposing and collecting, gives $z = 23$; hence $y = 52 - 2z = 52 - 46 = 6$, and $x = 53 - y - z = 53 - 29 = 24$.

By Rule II.

From the 1st equation $x = 53 - y - z$, and this value substituted for x in the 2d and 3d equations gives

$$53 - y - z + 2y + 3z = 105, \text{ or } y + 2z = 52 \text{ (A)},$$
$$\text{and } 53 - y - z + 3y + 4z = 134, \text{ or } 2y + 3z = 81 \text{ (B)}.$$

Now from equation (A) $y = 52 - 2z$, and this substituted for y in equation (B) gives $2(52 - 2z) + 3z = 81$, an equation containing only one unknown quantity; whence, by transposing and collecting, we obtain $z = 23$, and the values of x and y as before.

By Rule III.

2d Equation, . .	$x + 2y + 3z = 105$
1st Equation, . .	$x + y + z = 53$
By subtraction, .	$y + 2z = 52$ (A).
3d Equation, .	$x + 3y + 4z = 134$
2d Equation, . .	$x + 2y + 3z = 105$
By subtraction, .	$y + z = 29$ (B).
Equation (A), . .	$y + 2z = 52$
Equation (B), . .	$y + z = 29$
By subtraction, . .	$z = 23$.

EQUATIONS.	ANSWERS.
2. $\left.\begin{array}{l} 5x + 8y = 124 \\ 3x - 2y = 20 \end{array}\right\}$	$\left\{\begin{array}{l} x = 12 \\ y = 8. \end{array}\right.$
3. $\left.\begin{array}{l} 5x - 3y = 90 \\ 2x + 5y = 160 \end{array}\right\}$	$\left\{\begin{array}{l} x = 30 \\ y = 20. \end{array}\right.$
4. $\left.\begin{array}{l} x - y = 2 \\ 8y + 5x - 6y = 120 \end{array}\right\}$	$\left\{\begin{array}{l} x = 17\frac{5}{7} \\ y = 15\frac{5}{7}. \end{array}\right.$
5. $\left.\begin{array}{l} \frac{x}{2} + \frac{y}{3} = 16 \\ \frac{x}{5} - \frac{y}{9} = 2 \end{array}\right\}$	$\left\{\begin{array}{l} x = 20 \\ y = 18. \end{array}\right.$
6. $\left.\begin{array}{l} x + y = a \\ x^2 - y^2 = b \end{array}\right\}$	$\left\{\begin{array}{l} x = \frac{1}{2}a + \frac{b}{2a} \\ y = \frac{1}{2}a - \frac{b}{2a}. \end{array}\right.$
7. $\left.\begin{array}{l} 4x + 3y = 31 \\ 3x + 2y = 22 \end{array}\right\}$	$\left\{\begin{array}{l} x = 4 \\ y = 5. \end{array}\right.$
8. $\left.\begin{array}{l} 5x - 4y = 19 \\ 4x + 2y = 36 \end{array}\right\}$	$\left\{\begin{array}{l} x = 7 \\ y = 4. \end{array}\right.$

EQUATIONS.	ANSWERS.
9. $\left.\begin{array}{l}3x+7y=79\\ 2y-\frac{x}{2}=9\end{array}\right\}$	$\left\{\begin{array}{l}x=10\\ y=7.\end{array}\right.$
10. $\left.\begin{array}{l}\frac{x+y}{3}+1=6\\ \frac{x-y}{7}+3=4\end{array}\right\}$	$\left\{\begin{array}{l}x=11\\ y=4.\end{array}\right.$
11. $\left.\begin{array}{l}\frac{x+y}{3}-2y=2\\ \frac{2x-4y}{5}+y=\frac{23}{5}\end{array}\right\}$	$\left\{\begin{array}{l}x=11\\ y=1.\end{array}\right.$
12. $\left.\begin{array}{l}\frac{3x-7y}{3}=\frac{2x+y+1}{5}\\ 8-\frac{x-y}{5}=6\end{array}\right\}$	$\left\{\begin{array}{l}x=13\\ y=3.\end{array}\right.$
13. $\left.\begin{array}{l}x+y=13\\ x+z=14\\ y+z=15\end{array}\right\}$	$\left\{\begin{array}{l}x=6\\ y=7\\ z=8.\end{array}\right.$
14. $\left.\begin{array}{l}2x+3y+4z=29\\ 3x+2y+5z=32\\ 4x+3y+2z=25\end{array}\right\}$	$\left\{\begin{array}{l}x=2\\ y=3\\ z=4.\end{array}\right.$
15. $\left.\begin{array}{l}x+100=y+z\\ y+100=2x+2z\\ z+100=3x+3y\end{array}\right\}$	$\left\{\begin{array}{l}x=9\frac{1}{11}\\ y=45\frac{5}{11}\\ z=63\frac{7}{11}.\end{array}\right.$
16. $\left.\begin{array}{l}x+y=90-z\\ 2x+40=3y+20\\ x+20=2z+5\end{array}\right\}$	$\left\{\begin{array}{l}x=35\\ y=30\\ z=25.\end{array}\right.$
17. $\left.\begin{array}{l}x+y=a\\ x+z=b\\ y+z=c\end{array}\right\}$	$\left\{\begin{array}{l}x=\frac{b+a-c}{2}\\ y=\frac{a+c-b}{2}\\ z=\frac{c+b-a}{2}.\end{array}\right.$
18. $\left.\begin{array}{l}\frac{1}{2}x+\frac{1}{3}v+\frac{1}{4}z=62\\ \frac{1}{3}x+\frac{1}{4}v+\frac{1}{5}z=47\\ \frac{1}{4}x+\frac{1}{5}v+\frac{1}{6}z=38\end{array}\right\}$	$\left\{\begin{array}{l}x=24\\ v=60\\ z=120.\end{array}\right.$

QUADRATIC EQUATIONS.

If, after all the unknown quantities, except one, are exterminated from an equation, both that unknown quantity and its square are found in it, the equation is called a Quadratic.

RESOLUTION OF QUADRATIC EQUATIONS.

Having cleared the equation, and brought the terms involving the unknown quantity to one side of it by themselves, divide by the coefficient of the square of the unknown quantity, if it have one; then add to both sides the square of half the coefficient of the unknown quantity, which will complete the square of the side containing the unknown quantity; after which extract the square root of both sides, and the equation will be reduced to a simple one, which may be resolved as before.

NOTE 1. Since the square root of $x^2 - 2ax + a^2$ is either $a - x$ or $x - a$, the root of the known side of the equation must have both the signs + and — before it. Sometimes both these give proper solutions, and at other times only one of them.

NOTE 2. The root of the side involving the unknown quantity consists of that quantity, and of ½ its coefficient with its sign.*

Let the equation be $3x^2 + 12x = 96$
By dividing by 3, . $x^2 + 4x = 32$
Add the square of 2, $x^2 + 4x + 4 = 36$
And taking the root, . $x + 2 = \pm 6$
And transposing, . . $x = \pm 6 - 2 = +4$ or -8.
Here the positive value of the root only is proper.

Let the equation be $2x^2 - 8x = 90$
Dividing by 2, . $x^2 - 4x = 45$
Completing the square, $x^2 - 4x + 4 = 49$
Taking the root, . $x - 2 = \pm 7$
Transposing, . . . $x = \pm 7 + 2 = +9$ or -5.
Here also the root 7 is greater than ½ the coefficient of x; therefore the positive value only is proper.

Let the equation be $15x - x^2 = 54$
Or $x^2 - 15x = -54$
Completing the square, $x^2 - 15x + \frac{225}{4} = \frac{225}{4} - 54 = +\frac{9}{4}$
Taking the root, . . $x - \frac{15}{2} = \pm \frac{3}{2}$
Transposing, . . . $x = +\frac{15}{2} \pm \frac{3}{2} = +9$ or $+6$.

* Quadratic equations assume one of these three forms, viz. $x^2 + ax = +b$; $x^2 - ax = +b$; or $x^2 - ax = -b$; and when they are resolved by the rule, the value of x assumes one of these forms, $x = \frac{-a \pm \sqrt{a^2 + 4b}}{2}$; $x = \frac{+a \pm \sqrt{a^2 + 4b}}{2}$; or $x = \frac{+a \pm \sqrt{a^2 - 4b}}{2}$.

If a positive answer is required, the sign of the radical in the first two forms

Here both the roots are proper. But it is to be remarked, that if 54 had been greater than $\frac{225}{4}$, the known side would have been negative, and its root impossible; in which case x would have had no value in numbers.

NOTE. To avoid fractions, instead of dividing by the coefficient of x^2, and then adding the square of $\frac{1}{2}$ the coefficient, multiply the equation by 4 times the coefficient of x^2, and then add the square of the coefficient, which x had before multiplying.

Let the equation be	$7x^2 - 20x$	$= 32$
Multiplying by $4 \times 7 = 28$,	$196x^2 - 560x$	$= 896$
Adding $400 = 20^2$, .	$196x^2 - 560x + 400$	$= 1296$
Taking the root, . . .	$14x - 20$	$= \pm 36$
Whence	x	$= +4$ or $-1\frac{1}{7}$.

EQUATIONS.	ANSWERS.
1. $x^2 + 6x = 27$.	$x = +3$.
2. $x^2 + 10x = 56$.	$x = 4$.
3. $x^2 - 4x = 60$.	$x = 10$.
4. $x^2 - 6x = 72$.	$x = 12$.
5. $8 + x^2 - 6x = 80$.	$x = 12$.
6. $8x - 20 = 70 - 2x^2$.	$x = 5$.
7. $3x^2 + 6 = 3x + 5\frac{1}{3}$.	$x = \frac{2}{3}$ or $\frac{1}{3}$.
8. $\frac{x}{3} + 42\frac{2}{3} = \frac{x^2}{2} + 20\frac{1}{2}$.	$x = 7$.
9. $3x^2 - 9 = 76 - 2x$.	$x = 5$.
10. $x^2 - x = 210$.	$x = 15$.
11. $\frac{1}{2}x^2 + 7\frac{3}{8} = \frac{1}{3}x + 8$.	$x = 1\frac{1}{2}$.
12. $4x^2 - 3x = 85$.	$x = 5$.
13. $\frac{4x^2}{3} - 11 = \frac{x}{3}$.	$x = 3$.
14. $5x^2 + 4x = 273$.	$x = 7$.
15. $\frac{7}{x+1} + \frac{2}{x} = 5$.	$x = \frac{2 + \sqrt{14}}{5}$.

must be +, but in the third it may be either + or —. There is, however, a limitation in this case, for $4b$ must not be greater than a^2, otherwise the quantity below the radical sign would be negative, and its root impossible. This happens when the absolute term b is greater than $\frac{1}{4}a^2$, the square of $\frac{1}{2}$ the coefficient of x.

EQUATIONS.	ANSWERS.
16. $\frac{x}{5}+\frac{5}{x}=5\frac{1}{5}$.	$x=25$ or 1.
17. $\frac{3x}{x+2}-\frac{x-1}{6}=x-9$.	$x=10$.
18. $x^2+6ax=c^2$.	$x=(c^2+9a^2)^{\frac{1}{2}}-3a$.
19. $\frac{x}{a}+\frac{a}{x}=\frac{2}{a}$.	$x=1\pm\sqrt{1-a^2}$.

NOTE. If the equation contain two powers of the unknown quantity, and the exponent of the one is double that of the other, it may be resolved like a quadratic.

20. Let the equation be $x^6-6x^3 \quad =16$
Completing the square, $x^6-6x^3+9=25$
Taking the root, $x^3-3=\pm 5$
Transposing, $x^3=3\pm 5=8$
Taking the cube root, $x=2$.

EQUATIONS.	ANSWERS.
21. $2x^4-x^2=496$.	$x=4$.
22. $x^4+2ax^2=b$.	$x=(\sqrt{a^2+b}-a)^{\frac{1}{2}}$.
23. $x-8x^{\frac{1}{2}}=9$.	$x=81$ or 1.
24. $x-x^{\frac{1}{2}}=a$.	$x=a+\frac{1}{2}\pm\sqrt{a+\frac{1}{4}}$.
25. $\frac{1}{2}x-\frac{1}{3}x^{\frac{1}{2}}=22\frac{1}{2}$.	$x=49$ or $40\frac{1}{2}$.
26. $(1+x)^{\frac{1}{2}}-2(1+x)^{\frac{1}{4}}=4$.	$x=55\pm 24\sqrt{5}$.
27. $3x^{2n}-2x^n=25$.	$x=\left(\frac{1\pm 2\sqrt{19}}{3}\right)^{\frac{1}{n}}$.
28. $x^n-6x^{\frac{n}{2}}=e$.	$x=(18+e\pm 6\sqrt{e+9})^{\frac{1}{n}}$.
29. $4ax^4-bx^2=c$.	$x=\left(\frac{b\pm\sqrt{16c+b^2}}{8a}\right)^{\frac{1}{2}}$.

SOLUTION OF QUESTIONS.

WHEN a question is proposed, the analyst ought to form a clear idea of its nature, and then attempt to express its terms, and the relations of its parts, in algebraical characters, putting the letters x, y, z, &c. for the unknown quantities in it; and in this way he must deduce as many independent equations from the conditions of the question as there are unknown quantities in it, which he can always do when the question is properly limited; after which, these equations being resolved by the preceding rules, will give the answer or answers.

Put x for the greatest unknown quantity, y for the next, z, v, &c. for the lesser ones in their order.

Suppose it to be a condition of the question, that

The two quantities together, or their sum, amounts to 18. This condition may be expressed thus, . $x+y=18$

Their excess, difference, &c. is 6, . $x-y=6$

Their product, rectangle, the one into the other, or multiplied by it, is 72, . . $xy=72$

One of them taken out of the other, divided by it, applied to it, or their quotient, is 2, $\frac{x}{y}=2$

The greater is to the less, or their ratio is as 4 to 2, ... $x:y::4:2$

And this proportion, by multiplying the means together, and also the extremes, becomes an equation, . . . $2x=4y$

The sum of their squares is 180, . $x^2+y^2=180$

The difference of their squares is 108, $x^2-y^2=108$

And in a similar way may any other relations of the quantities be expressed in equations.

When the relation of one unknown quantity to another is simple, a letter may be taken for one of them, and an expression for the other deduced from the relation between them, which will abridge the work, and render it more elegant. Thus, if their difference be 3, take y for the less, and $y+3$ will be the greater.

It will often abridge the work, if letters are taken not for the unknown quantities themselves, but for their sum, difference, or any other relation from which the quantities may be easily found.

QUESTIONS PRODUCING SIMPLE EQUATIONS.

1. To find such a number, that, if it be multiplied by 5, and also by 3, the former product shall exceed the latter by 26. Let $x=$ the number required, then the first product is $5x$, the second $3x$, and their difference $5x-3x=26$, or $2x=26 \therefore x=13$.

2. To find a number, to which if 27 be added, the sum shall be 10 times the number required. Let $x=$ the number required, then $10x=x+27$, or $9x=27 \therefore x=3$.

3. To find a number, from which if 4 be taken, and the remainder multiplied by 3, the product shall be twice the

number sought. Let $x=$ the number required, then $(x-4)3 = 2x$, or $3x-12=2x$; whence $3x-2x=12$, or $x=12$.

4. To find a number of which the fourth part exceeds the fifth part by 13.

$\frac{x}{4}-\frac{x}{5}=13.$ Ans. 260.

5. To find a number, to the half of which if 7 be added, the sum shall be equal to twice the number with 20 taken from it. Ans. 18.

6. To find a number, of which the square shall be equal to 4 times the number, together with 5 times the same number. Ans. 9.

7. To find a number, to which if its half, its third, and its fourth parts be added, the sum shall be equal to the square of that number.

$x^2=x+\frac{x}{2}+\frac{x}{3}+\frac{x}{4}.$ Ans. $2\frac{1}{12}$.

8. To find a number, from which if 3 be taken, and the remainder multiplied by 3, and then 4 added to the product, the sum divided by 5 shall give half the number sought. Ans. 10.

9. To find a number of pounds, to which if 3 be added, and the sum multiplied by 12, the product shall be equal to the number of shillings in the value of the pounds, diminished by as many crowns as there are pounds required.

$(x+3)12=20x-5x.$ Ans. £12.

10. To find two numbers, of which the sum is 133, and their difference 47. Ans. 90 and 43.

11. To find two numbers, of which the sum is 84, and their quotient 13. Ans. 78 and 6.

12. To find two numbers, of which the difference is 104, and their quotient 9. Ans. 117 and 13.

13. To find two numbers, so that 3 times the greater added to twice the less shall make 54, and 4 times the greater with 3 times the less shall make 75. Ans. 12 and 9.

14. To find two numbers, so that the greater with half the less shall make 25, and the less with half the greater shall make 23. Ans. 18 and 14.

15. To find two numbers in the ratio of 4 to 3, so that if one be added to each of them, the sums shall be in the ratio of 9 to 7.

$3x=4y,\ (x+1)\times 7=(y+1)\times 9.$ Ans. 8 and 6.

16. To find two numbers, of which the difference shall be 9, and the difference of their squares 351.

Ans. 24 and 15.

17. To divide the number 36 into two parts, so that the square of the greater part shall exceed that of the less by 360. Ans. 23 and 13.

18. To divide the number 72 into two parts, so that three times the greater shall exceed twice the less by 121. Ans. 53 and 19.

19. To divide the number 56 into two parts, which shall be to one another as 4 to 3. Ans. 32 and 24.

20. To find a number, so that its half added to its third part shall be greater by $6\frac{1}{2}$ than its double divided by 5. Ans. 15.

21. To find a number, from the double of which if 22 be taken, the remainder shall exceed 100 as much as the number itself is below 100.

$$2x - 22 - 100 = 100 - x.$$ Ans. 74.

22. A person being asked his age, replied, that $\frac{1}{5}$ of his age, multiplied by $\frac{1}{6}$ of his age, would produce his age. How old was he? Ans. 30.

23. A general sends out $\frac{1}{3}$ of his army, and 1500 men more, and he retains $\frac{1}{2}$ of his army, and 1200 men more. How many men had he in his army? Ans. 16200.

24. A gentleman distributing money among some poor people, found that he wanted 10s. to be able to give 5s. to each of them; he therefore gave each 4s., and then he had 5s. left. How much money had he, and how many poor were there? Ans. 65s., 15 poor.

25. To find two numbers in the ratio of 3 to 2, so that their sum shall be the sixth part of their product. Ans. 15 and 10.

26. There were 6 children in a family, whose ages differed by 2 years, and each received a guinea for every year of his age, the money they received amounted to 72 guineas. Required their ages? Ans. 7 youngest, 17 eldest.

27. A and B inherited equal estates; but A spent annually £60 more than his income, while B saved £80 annually; in consequence of which, at the end of 12 years, B was twice as rich as A. Required the value of their estates?

$$(x - 60 \times 12)2 = x + 80 \times 12.$$ Ans. £2400.

28. A says to B, If you will give me £25, I shall have as much money as you shall have left. Says B, If you give me £30, I shall then have twice as much as you will have remaining. How much had each? Ans. B £190, A £140.

29. A farmer has 15 more cows than horses, and as many

scores of sheep as horses and cows together; the number of all the three is 651. How many has he of each kind?

Ans. 8 horses, 23 cows, 31 scores sheep.

30. Two merchants join in company with a capital of £2000. A's share was 11 months in trade, and B's 9 months, and their shares of the gain were equal. What was the stock of each? Ans. B's £1100, A's £900.

31. A field was sown with wheat at 35s. per boll, and produced 9 returns: the crop was sold at 30s. per boll, and, after paying for the seed, there remained £293, 15s. How much wheat was sown? Ans. 25 bolls.

32. A merchant laid aside £200 annually for his expenses, and increased his capital annually by $\frac{1}{3}$ of what was not thus expended. At the end of three years his capital was double of what he began with. What was it at first?

$$x+\frac{x-200}{3}+\frac{4x-800}{9}+\frac{16x-3200}{27}=2x.$$ Ans. £740.

33. Five persons have money divided among them. The share of the first was £10 more than that of the second; the share of the second was £16 less than that of the third; the share of the third was £5 more than that of the fourth; and the share of the fourth £15 less than that of the fifth: also the shares of the two last were together equal to the sum of the shares of the other three. What was the share of each?

Ans. £21, £11, £27, £22, £37.

34. Two travellers set out at the same time to meet one another, from two places distant 390 miles: the first travels 30 miles in a day, and the other 22 miles. In what time will they meet? Ans. $7\frac{1}{2}$ days.

35. A privateer, sailing at the rate of 9 miles in an hour, discovers a merchant vessel 18 miles distant, sailing at the rate of 7 miles in an hour. In what time will the privateer overtake the other vessel? Ans. 9 hours.

36. A woman bought some apples at 3 for a penny, and as many at 2 for a penny, and sold them all again at 5 for two-pence, and found that she had lost sixpence. How many of each kind did she buy? Ans. 180.

37. A hare, 40 of her leaps before a hound, takes 4 leaps for the hound's 3, but 2 of the hound's leaps are equal to 3 of the hare's. How many leaps must the hound take before he catch the hare?

$$\frac{3x}{2}-\frac{4x}{3}=40.$$ Ans. 240 hound's leaps.

38. A son asked his father's age. The father replied,

7 years ago I was 3 times as old as you were; but if we live together 7 years longer, my age will be the double of yours. What were their ages? Ans. 49 and 21.

39. An army being drawn up in a square, there were 79 men over; but in attempting to enlarge each side of the square by one man, there were 80 men too few. Required the number of men? Ans. 6320 men.

40. The paving of a square court, at 8d. per square yard, cost as much as the enclosing of it at 5s. the yard. Required its extent? Ans. 30 yards each side.

41. A person lost $\frac{1}{5}$ of his money by gaming, and then won 4s. Again he lost $\frac{1}{4}$ of what he then had, and afterwards won 3s. The third time he lost $\frac{1}{5}$ of what he then had; and after that, he had remaining $\frac{1}{2}$ of what he began with. How much money had he?

$$\frac{4x}{5}+4-\frac{4x}{20}-1+3-\frac{2x}{10}-2=\frac{x}{2}. \qquad \text{Ans. 40s.}$$

42. A cistern can be filled with water by one cock in 12 hours, and by another in 8 hours. In what time will it be filled if both run together? Ans. $4\frac{4}{5}$ hours.

43. The tail of a fish weighed 9 lb., the head weighed as much as the tail and half the body, and the weight of the body was equal to that of the head and tail. What was the weight of the fish? Ans. 72 lb.

44. A gentleman's two horses with the harness cost him £120; the value of the worst horse with the harness was double that of the best horse, and the value of the best horse with the harness was triple that of the worst horse. What was the value of each?

Ans. £50 harness, £40 and £30 horses.

45. A master with his apprentice can perform a piece of work in 8 days, which the master alone could do in 12 days. In what time could the apprentice do it?

$$\frac{x}{8}-\frac{x}{12}=1. \qquad \text{Ans. 24 days.}$$

46. Three men can do a piece of work, the first in 50 hours, the second in 60 hours, and the third in 75 hours. In what time will they do it, all working together? Ans. 20 hours.

47. A and B together can do a piece of work in 12 hours, A and C together in 20 hours, and B and C together in 15 hours. In what time will they do it, all working together, and in what time will each do it separately?

$\frac{x}{12}+\frac{x}{20}+\frac{x}{15}=2.$ Ans. Together in 10 hours, A alone in 30 hours, B alone in 20 hours, and C alone in 60 hours.

48. A labourer engages to work 160 days, on condition that he should receive half-a-crown for every day that he wrought, and should forfeit 10d. for every day he was absent from work. At the end of the stipulated time he had nothing to receive nor to pay. How many days did he work?

Ans. Wrought 40 days.

49. To find three numbers, so that the first with $\frac{1}{2}$ of the other two, the second with $\frac{1}{3}$ of the other two, and the third with $\frac{1}{4}$ of the other two, shall each be equal to 34.

Ans. 10, 22, and 26.

50. To find a number consisting of three places, of which the digits have equal differences in their order, and if the number be divided by the sum of its digits, the quotient shall be 48; and if 198 be subtracted from the number, the digits shall be inverted. $100x + 10y + z$ the number.

$x + z = 2y$, $48 \times 3y =$ number, $99x - 99z = 198$. Ans. 432.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

51. To divide the number 100 into two parts, so that their product shall be 2100. Ans. 70 and 30.

52. To find two numbers, of which the difference shall be 8, and their product 240. Ans. 20 and 12.

53. To find two numbers, of which the difference shall be 12, and the sum of their squares 1424. Ans. 32 and 20.

54. To find two numbers, of which the sum shall be 30, and their product 224. Ans. 16 and 14.

55. To find two numbers, of which the product shall be 108, and the sum of their squares 225. Ans. 12 and 9.

56. A gardener and his lad digged each a square piece of ground, of which the side was as many feet long as the worker was years old. The difference of their ages was 12 years, and the number of square feet digged by both was 1040. Required their ages? Ans. 28 and 16.

57. An oblong pond was surrounded by a terrace-walk 7 yards broad, the pond measured 15000 square yards, and the walk 3696 square yards. Required the length and breadth of the pond?

$$xy = 15000, \text{ and } 14x + 14y + 196 = 3696.$$

Ans. 150 and 100 yards.

58. To find two numbers of which the sum is 13, and the sum of their cubes 637. Ans. 8 and 5.

59. To find two numbers, of which the product shall be 120, and the product of the greater, increased by 8, multiplied by the less, increased by 5, shall be 300.

Ans. 12 and 10, or 16 and $7\frac{1}{2}$.

60. To divide 125 into two parts, so that the sum of their square roots shall be 15.

$$\sqrt{y}+(125-y)^{\frac{1}{2}}=15. \qquad \text{Ans. 100 and 25.}$$

61. A grazier bought a number of sheep for £60, and, reserving 15 to himself, he sold the remainder for £54, and gained 2s. on each of them. How many sheep did he buy, and what did each cost? Ans. 75 sheep at 16s.

62. Sold an ox for £24, and gained as much per cent. as the ox cost. What was paid for him?

$$x+\frac{x^2}{100}=24. \qquad \text{Ans. £20.}$$

63. A person bought some oxen for £80: if he had got 4 oxen more for the same money, each of them would have cost him £1 less. How many did he buy? Ans. 16.

64. A number of bees alighted upon a tree: at the first flight the square root of $\frac{1}{2}$ of them went away, and at the next $\frac{8}{9}$ of them, and then only two bees remained. How many alighted on the tree?

$$\sqrt{\tfrac{1}{2}x}+\frac{8x}{9}+2=x. \qquad \text{Ans. 72 bees.}$$

65. A person bought cloth for £33, 15s., which he sold again at £2, 8s. per piece, and gained as much as a piece cost him. Required the number of pieces? Ans. 15 pieces.

66. A and B set out at the same time for a place at the distance of 150 miles. A travels 3 miles an hour faster than B, and arrives at his journey's end $8\frac{1}{3}$ hours before him. At what rate per hour did each person travel?

Ans. A 9 miles, B 6 miles.

67. There are two numbers, of which the product is 120: if 2 be added to the less, and 3 subtracted from the greater, the product of the sum and difference will be also 120. What are the numbers? Ans. 15 and 8.

68. A and B distribute each £1200 among some poor persons: A relieves 40 persons more than B, and B gives £5 a-piece to each person more than A. How many persons were relieved by A and B? Ans. 120 by A, 80 by B.

69. A person bought some sheep for £57, but he lost 8 of them, and then sold the remainder at 8s. a-head profit; and thus he neither gained nor lost by the bargain. How many sheep did he buy? Ans. 38.

70. To divide the number 18 into two factors, so that the sum of their cubes shall be 243. Ans. 6 and 3.

71. There is a number consisting of two digits, the left-hand digit is 3 times the other; and if 12 be subtracted from the

number, the remainder will be the square of the left-hand digit. What is the number? Ans. 93.

72. A, travelling to London, overtook at the 50th milestone a flock of sheep, proceeding at the rate of 3 miles in 2 hours; and 2 hours afterwards met a waggon moving at the rate of 9 miles in 4 hours. B, travelling at the same rate, overtook the sheep at the 45th milestone, and met the waggon 40 minutes before he came to the 31st milestone. Where would B be when A reached London? $x =$ distance between them, $y =$ rate of their travelling per hour, $\frac{10y}{3} - 5 = x$, $50 - 2y - \frac{32y^2}{27} + \frac{76y}{9} = 31 + \frac{2y}{3} - x$. Ans. $x = 25$, $y = 9$.

OF RATIOS.

Ratio is the relation which one quantity bears to another of a *similar kind* with respect to its magnitude.

The *magnitude* or *value* of a ratio is estimated by stating how often one quantity *contains* or is *contained* in another. Thus, in comparing the number 16 with 2, we observe that it has a certain magnitude with respect to 2 which it contains 8 times; and if we compare 16 with 4 we observe that it has a different relative magnitude, for it contains 4 only 4 times. Hence 16 is less when compared with 4, than it is when compared with 2.

The general method of expressing the ratio which one quantity bears to another is by placing two points between them. Thus

The ratio of 12 to 4 is expressed by 12 : 4
. . . . . . of 17 to 9 . . . . by 17 : 9
. . . . . . of a to b . . . . by $a : b$.

The first term of a ratio is called the *Antecedent*, and the last term the *Consequent*. The antecedents in the preceding ratios are therefore 12, 17, and a, and the consequents 4, 9, and b. Ratios may also be represented in the form of fractions, by making the antecedents the numerators, and the consequents the denominators: Thus $\frac{12}{4}$, $\frac{17}{9}$, and $\frac{a}{b}$ express the ratios of 12 to 4, of 17 to 9, and of a to b.

A ratio is said to be a ratio of *greater inequality* when the antecedent is greater than the consequent, a ratio of *equality* when it is equal to the consequent, and a ratio of *less inequality* when it is less than the consequent: Thus

The ratio of 8 : 4 or $a+b:a$ is a ratio of greater inequality.
. . . . . . of 8 : 8 or a : a . . . . . of equality.
. . . . . . of 8 : 12 or a : $a+b$ . . . of less inequality.

NOTE. It is evident that a ratio of equality may always be represented by unity.

COMPARISON OF RATIOS.

1. If the terms of a ratio are both multiplied or both divided by the same quantity, the value of the ratio is not altered.

The ratio of $a:b$ is expressed by the fraction $\frac{a}{b}$. Let both terms of this fraction be multiplied by n, and it becomes $\frac{na}{nb}$; now since the value of a fraction is not altered by multiplying both the numerator and denominator by the same quantity $\frac{a}{b}=\frac{na}{nb}$, or the ratio of $a:b$ is the same as the ratio of $na:nb$ where n may be any number either integral or fractional: Thus

The ratio of 16 : 12 (divid. by 4) is the same as the ratio of 4 : 3.
. . . . . of 5 : 8 (mult. by 3) . . . . of 15 : 24.
. . . . . of $a^2:ab$ (divid. by a) . . . . of $a:b$.

II. Ratios are compared together by reducing the fractions which represent them to a common denominator.

Thus the ratios of 7 : 9 and 10 : 13 are represented by the fractions $\frac{7}{9}$ and $\frac{10}{13}$, which are equivalent to $\frac{91}{117}$ and $\frac{90}{117}$; and since $\frac{91}{117}$ is greater than $\frac{90}{117}$, we infer that the ratio of 7 : 9 is greater than that of 10 : 13.

When the antecedents or consequents are the same in two or more ratios, we may immediately compare those ratios together, by expressing them in a fractional form: Thus since $\frac{17}{5}$ is greater than $\frac{17}{9}$, the ratio of 17 : 5 is greater than that of 17 : 9; and since $\frac{a}{a+b}$ is less than $\frac{a}{b}$, the ratio of $a:a+b$ is less than that of $a:b$.

III. A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same quantity to both of its terms.

Let $\frac{a}{b}$ represent any ratio, and add n to each of its terms, then these two ratios will be $\frac{a}{b}$ and $\frac{a+n}{b+n}$, which are equivalent to $\frac{ab+an}{b(b+n)}$ and $\frac{ab+bn}{b(b+n)}$; now if $a > b$, then $\frac{a}{b}$ is a ratio of greater inequality, and $\frac{ab+an}{b(b+n)} > \frac{ab+bn}{b(b+n)} \therefore \frac{a}{b}$ is diminished by adding n to each of its terms; again, if $a < b$, then $\frac{a}{b}$ is a ratio of less inequality, and $\frac{ab+an}{b(b+n)} < \frac{ab+bn}{b(b+n)} \therefore \frac{a}{b}$ is increased by the addition of n to both its terms.

COMPOSITION OF RATIOS.

I. Ratios are compounded by multiplying their antecedents together to form a new antecedent, and their consequents to form a new consequent, and the resulting ratio is called the *sum* of the compounding ratios. Thus

The ratio of $a:b$ is compounded with the ratio of $c:d$ by multiplying the antecedents a and c together for a new antecedent, and the consequents b and d together for a new consequent, and the resulting ratio of $ac:bd$ is the sum of the compounding ratios $a:b$ and $c:d$.

If the ratios $4:7$, $6:11$, and $7:9$, are compounded together, the resulting ratio is $4 \times 6 \times 7 : 7 \times 11 \times 9$ or $168:693$, which, reduced to its lowest terms by dividing both terms by 21, becomes the ratio of $8:33$.

II. When any ratio $a:b$ is compounded with itself twice, thrice, or any number of times, denoted by n, then the resulting ratios are $a^2:b^2$, $a^3:b^3$, $a^n:b^n$, or twice, thrice, and n times the ratio of $a:b$.

The ratios $a^2:b^2$, $a^3:b^3$, $a^4:b^4$, &c. are also called the *duplicate, triplicate, quadruplicate,* &c. ratios of the primitive.

As the indices or exponents 2, 3, and n, express the number of times the ratio of $a:b$ is compounded with itself, they are called the measures of these ratios.

III. Since the index may be any quantity either integral or fractional, let it be a fraction, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{m}$, &c., then

The ratio of $a^{\frac{1}{2}}:b^{\frac{1}{2}}$ is $\frac{1}{2}$ the ratio of $a:b$.
. . . . $a^{\frac{1}{3}}:b^{\frac{1}{3}}$ is $\frac{1}{3}$ . . . . . of $a:b$.
. . . . $a^{\frac{1}{4}}:b^{\frac{1}{4}}$ is $\frac{1}{4}$ . . . . . of $a:b$.
. . . . $a^{\frac{1}{m}}:b^{\frac{1}{m}}$ is $\frac{1}{m}$th . . . . of $a:b$.

The ratios of $a^{\frac{1}{2}} : b^{\frac{1}{2}}$, $a^{\frac{1}{3}} : b^{\frac{1}{3}}$, $a^{\frac{1}{4}} : b^{\frac{1}{4}}$, &c. are also called the *subduplicate, subtriplicate, subquadruplicate,* &c. ratios of the primitive.

IV. The sum of any number of ratios, of which the consequent of the preceding ratio is the antecedent of the succeeding one, is the ratio of the first antecedent to the last consequent.

Let the ratios be $a : b$, $b : c$, $c : d$, $d : e$, $e : f$, &c., then the resulting ratio is $a \times b \times c \times d \times e : b \times c \times d \times e \times f$, or the ratio of $abcde : bcdef$, which, reduced to its least terms, by dividing both its terms by $bcde$, becomes the ratio of $a : f$, or first antecedent : last consequent.

V. Any ratio compounded with a ratio of greater inequality is increased, and compounded with a ratio of less inequality is diminished.

Let $a+b : a$ represent the ratio of greater inequality,
and $a : a+b$ . . . . of less inequality.

Then the ratio of $a+b : a$, compounded with that of $c : d$, gives $ac+bc : ad$, which is evidently greater than the ratio of $c : d$; and the ratio of $a : a+b$, compounded with that of $c : d$, gives $ac : ad+bd$, which is evidently less than the ratio of $c : d$. Hence the ratio of $c : d$ is increased by compounding it with the ratio of $a+b : a$, and diminished by compounding it with the ratio of $a : a+b$.

APPROXIMATION OF RATIOS.

The ratio of the powers or roots of two quantities, whose difference is small with respect to themselves, is found very nearly by multiplying that difference by the index or exponent of the power or root.

Let $x+z$ and x be two quantities, whose difference is z; then $(x+z)^n = x^n + nx^{n-1}z + \frac{n(n-1)}{2}x^{n-2}z^2 + \frac{n(n-1)(n-2)}{2\times 3} \times x^{n-3}z^3 +$ &c.; hence the ratio of $(x+z)^n : x^n$ is that of $x^n + nx^{n-1}z + \frac{n(n-1)}{2}x^{n-2}z^2 + \frac{n(n-1)(n-2)}{2\times 3}x^{n-3}z^3 +$ &c.$: x^n$, and dividing by x^{n-1} this ratio becomes that of $x + nz + \frac{n(n-1)z^2}{2x} + \frac{n(n-1)(n-2)z^3}{2\times 3x^2} +$ &c.$: x$.

Now if z is very small with respect to x, the fractions $\frac{z^2}{x}$ and $\frac{z^3}{x^2}$ must also be very small; and when n is not very large, the fractional terms of the series which forms the antecedent will also be very small with respect to the integral part $x+nz$; hence the ratio of $x+nz:x$ will be a near approximation to the ratio of $(x+z)^n:x^n$, which is the rule.

Let it be required to approximate to the ratio of $1000^3:999^3$.

Here $x=999$, $z=1$, and $n=3$ $\therefore$ the ratio of $x+nz:x$ is that of $1002:999$, which is true to *three* places of decimals very nearly; since $1000^3:999^3::1002:998{\cdot}997$.

Let it be required to approximate to the ratio of $\sqrt[3]{480}:\sqrt[3]{477}$.

Here $x=477$, $z=3$, and $n=\frac{1}{3}$ $\therefore$ the ratio of $x+nz:x$ is that of $478:477$, which is true to *three* places of decimals very nearly; since $\sqrt[3]{480}:\sqrt[3]{477}::478:477{\cdot}002$.

EXERCISES.

1. Reduce the ratio of $375:420$ to its lowest terms. Ans. $25:28$.

2. Reduce the ratio of $a^4-a^3b+a^2:a^2$ to its lowest terms. Ans. $a^2-ab+1:1$.

3. Show that the ratio of $18:13$ is the same as that of $162:117$.

4. Show that the ratio of $17:21$ is less than that of $153:168$.

5. Which is the greater ratio, that of $a+6:\frac{1}{2}a+9$, or that of $a+9:\frac{1}{2}a+10$? Ans. That of $a+9:\frac{1}{2}a+10$.

6. Is the ratio of $13:18$ increased or diminished by adding 7 to each of its terms? Ans. Increased.

7. Is the ratio $a:a-b$ increased or diminished by adding c to each of its terms? Ans. Diminished.

8. Approximate to the ratios of $740^2:738^2$ and of $\sqrt{740}:\sqrt{738}$, and show to how many places of decimals the approximation is true. Ans. $742:738$ and $739:738$, the former being true to *three* and the latter to *four* places very nearly.

9. What is the sum of the ratios of $8:11$, $9:13$, $5:7$, and $21:23$? Ans. $1080:3289$.

10. What is the sum of the ratios of $a^3-x^3:a^2$, $a^2+x^2:b^2$, $a-x:b$, and $b:a-x$?

Ans. $a^5+a^3x^2-a^2x^3-x^5:a^2b^2$.

11. Is the sum of the ratios of $5a-1:4a+1$, $3a+6:2a+4$, and $2a+1:3a-1$, a ratio of greater or less inequality? Ans. A ratio of greater inequality.

12. Show that the sum of the ratios of $a+b:x$, $a-b:y$, and $y:\frac{a^2-b^2}{x}$, is a ratio of equality.

13. Find in its lowest terms the sum of the duplicate ratio of $8:11$, the triplicate ratio of $5:4$, and the ratio of $22:15$. Ans. $50:33$.

14. Tell extempore the sum of the ratios of $4:3^2$, $3^2:5^3$, $5^3:7^2$, and $7^2:9$.

15. Of what two simple ratios is the ratio of $9:24$ compounded? Ans. Of the ratios of $1:2$ and $3:4$.

OF PROPORTION.

Proportion consists in the *equality of ratios.*

Thus if the ratio of $a:b$ is equal to that of $c:d$, or $\frac{a}{b}=\frac{c}{d}$; then a, b, c, d, are said to be proportionals. The numbers 3, 12, 4, 16, are proportionals for $\frac{3}{12}=\frac{1}{4}$, and $\frac{4}{16}=\frac{1}{4}$.

This equality of ratios is expressed by writing the four quantities, thus $a:b::c:d$, and read a is to b as c is to d. In Algebraic investigations the quantities are generally expressed like fractions, thus $\frac{a}{b}=\frac{c}{d}$.

In the proportion $a:b::c:d$ or $\frac{a}{b}=\frac{c}{d}$, a and d are the *extremes,* and b and c the *means.* The first term is likewise called the first antecedent, the second term the first consequent, the third term the second antecedent, and the fourth term the second consequent.

If, in a series of proportional quantities each consequent be identical with the next antecedent, these quantities are said to be in *continued* proportion: Thus $a:b::b:c::c:d::d:e::e:f$, &c.; the quantities a, b, c, d, e, f, &c. are said to be in *continued* proportion; when the second and third terms of a proportion are identical, as in the proportion $a:b::b:c$; then b is said to be a *mean proportional* between the extremes a and c, and c is called a *third proportional* to a and b.

Prop. I. If four quantities are proportional, the product of the extremes is equal to the product of the means, and conversely.

Let $a:b::c:d$ or $\frac{a}{b}=\frac{c}{d}$. Multiplying both by bd, we obtain $ad=bc$.

Conversely. If the product of any two quantities is equal to the product of any other two, these four quantities constitute a proportion, the factors of either of the products being made the extremes, and the factors of the other the means.

Let $ad=bc$. Dividing both by bd, we obtain $\frac{a}{b}=\frac{c}{d}$ or $\frac{c}{d}=\frac{a}{b}$; whence $a:b::c:d$ or $c:d::a:b$.

PROP. II. If three quantities are in continued proportion, the product of the extremes is equal to the square of the mean, and conversely.

Let $a : b :: b : c$; then $a \times c = b \times b$ or $ac = b^2$.

Conversely. If the product of any two quantities is equal to the square of a third, the third is a mean proportional between the other two.

Let $ac = b^2$, and, dividing both by bc, we obtain $\frac{a}{b} = \frac{b}{c}$ or $a : b :: b : c$.

PROP. III. Of four proportionals, any three being given, the fourth may be found.

Let $a : b :: c : d$; then $ad = bc$.

Hence $a = \frac{bc}{d}$ $\therefore$ if b, c, d, are known, a is also known.

. . . . $b = \frac{ad}{c}$ . . $a, d, c,$ . . . . . . b . . . .

. . . . $c = \frac{ad}{b}$ . . $a, d, b,$ . . . . . . c . . . .

. . . . $d = \frac{bc}{a}$ . . $a, b, c,$ . . . . . . d . . . .

Hence of three proportionals, any two being given, the third may be found; for $ad = b^2$ $\therefore b = \sqrt{ad}$, $a = \frac{b^2}{d}$, and $d = \frac{b^2}{a}$.

PROP. IV. Quantities which have the same ratio to the same quantity are equal to one another, and conversely.

Let $a : b :: c : b$; then $\frac{a}{b} = \frac{c}{b}$, and, multiplying each by b, we obtain $a = c$.

Conversely. Quantities which are equal to one another have the same ratio to the same quantity.

Let $a = c$, and let b be a third quantity; then, dividing both by b, we obtain $\frac{a}{b} = \frac{c}{b}$ $\therefore a : b :: c : b$.

PROP. V. Ratios that are equal to the same ratio are equal to one another.

Let $a : b :: e : f$, and $c : d :: e : f$; then also $a : b :: c : d$.

Since $\frac{a}{b} = \frac{e}{f}$ and $\frac{c}{d} = \frac{e}{f}$ $\therefore \frac{a}{b} = \frac{c}{d}$ or $a : b :: c : d$.

PROP. VI. If four quantities are proportionals, they will also be proportionals *invertendo*, that is, the second will have the same ratio to the first that the fourth has to the third.

Let $a:b::c:d$; then also $b:a::d:c$.

Since (Prop. I.) $bc=ad$, and, dividing by ac, we get $\frac{b}{a}=\frac{d}{c}$; hence $b:a::d:c$.

PROP. VII. If four quantities are proportionals, they will also be proportionals *alternando*, or the first will have the same ratio to the third that the second has to the fourth.

Let $a:b::c:d$; then also $a:c::b:d$; since $\frac{a}{b}=\frac{c}{d}$, multiply each by $\frac{b}{c}$, and we obtain $\frac{a}{c}=\frac{b}{d}$ $\therefore a:c::b:d$.

PROP. VIII. If four quantities are proportionals, they will also be proportionals *componendo*, or the sum of the first and second will have the same ratio to the second that the sum of the third and fourth has to the fourth.

Let $a:b::c:d$; then also $a+b:b::c+d:d$; since $\frac{a}{b}=\frac{c}{d}$, add 1 to each of these, and we obtain $\frac{a}{b}+1=\frac{c}{d}+1$ or $\frac{a+b}{b}=\frac{c+d}{d}$ $\therefore a+b:b::c+d:d$.

PROP. IX. If four quantities are proportionals, they will also be proportionals *dividendo*, or the difference between the first and second will have the same ratio to the second that the difference between the third and fourth has to the fourth.

Let $a:b::c:d$; then also $a-b:b::c-d:d$; since $\frac{a}{b}=\frac{c}{d}$, subtract 1 from each of these, and we obtain $\frac{a}{b}-1=\frac{c}{d}-1$ or $\frac{a-b}{b}=\frac{c-d}{d}$ $\therefore a-b:b::c-d:d$.

PROP. X. If four quantities are proportionals, they will also be proportionals *convertendo*, or the first will have the same ratio to the sum or difference of the first and second that the third has to the sum or difference of the third and fourth.

Let $a:b::c:d$; then also $a:a\pm b::c:c\pm d$; since $\frac{a}{b}=\frac{c}{d}$, and by Prop. VIII. and IX. $\frac{a\pm b}{b}=\frac{c\pm d}{d}$, invert these fractions, and we have $\frac{b}{a\pm b}=\frac{d}{c\pm d}$; and, multiplying the one by $\frac{a}{b}$ and the other by $\frac{c}{d}$, we obtain $\frac{b}{a \pm b} \times \frac{a}{b} = \frac{d}{c \pm d} \times \frac{c}{d}$ or $\frac{a}{a \pm b} = \frac{c}{c \pm d} \therefore a : a \pm b :: c : c \pm d$.

PROP. XI. If four quantities are proportionals, the sum of the first and second has the same ratio to their difference that the sum of the third and fourth has to their difference.

Let $a : b :: c : d$; then also $a+b : a-b :: c+d : c-d$. For taking VIII. and IX. *alternando*, $a+b : c+d :: b : d$ and $a-b : c-d :: b : d$; hence (V.), $a+b : c+d :: a-b : c-d$ (and *alternando*), $a+b : a-b :: c+d : c-d$.

PROP. XII. In any number of proportionals any antecedent has the same ratio to its consequent that the sum of all the antecedents has to the sum of all the consequents.

Let $a : b :: c : d :: e : f :: g : h$, &c.; then also $a : b :: a+c+e+g+$ &c.$: b+d+f+h+$ &c.
Since $ab = ba$, $ad = bc$, $af = be$, $ah = bg$, &c., we have $a(b+d+f+h+ \text{\&c.}) = b(a+c+e+g+ \text{\&c.})$; whence $\frac{a}{b} = \frac{a+c+e+g+\text{\&c.}}{b+d+f+h+\text{\&c.}} \therefore a : b :: a+c+e+g+$ &c.$: b+d+f+h+$ &c. In like manner it may be shown that $c : d :: a+c+e+g+$ &c.$: b+d+f+h+$ &c.

PROP. XIII. In two or more ranks of proportionals the products of the corresponding terms are also proportionals.

Let $\left.\begin{array}{l} a : b :: c : d \\ e : f :: g : h \\ i : k :: l : m \end{array}\right\}$ Then also $aei : bfk :: cgl : dhm$.

Since $\frac{a}{b} = \frac{c}{d}$, $\frac{e}{f} = \frac{g}{h}$, $\frac{i}{k} = \frac{l}{m}$; then $\frac{aei}{bfk} = \frac{cgl}{dhm} \therefore aei : bfk :: cgl : dhm$.

PROP. XIV. If there are any number of quantities more than two, and as many others, which, taken two and two in order, are proportionals; then *ex æquo* are the extreme terms in the former series, proportional to the extreme terms in the latter.

Let a, b, c, d, be any number of quantities,
and e, f, g, h, as many others.

Let $\left.\begin{array}{l} a : b :: e : f \\ b : c :: f : g \\ c : d :: g : h \end{array}\right\}$ Then also $a : d :: e : h$.

Since $\frac{a}{b}=\frac{e}{f}$, $\frac{b}{c}=\frac{f}{g}$, and $\frac{c}{d}=\frac{g}{h}$, we obtain, by multiplying the alternate fractions together, $\frac{abc}{bcd}=\frac{efg}{fgh}$ or $\frac{a}{d}=\frac{e}{h}\therefore a:d::e:h$.

PROP. XV. If there are any number of quantities more than two, and as many others, which, taken two and two in a cross order, are proportionals; then *ex æquo inversely* are the extreme terms in the first rank, proportional to the extreme terms in the second.

Let a, b, c, d, be any number of terms,
and e, f, g, h, as many others;

and let $\left.\begin{array}{l} a:b::g:h \\ b:c::f:g \\ c:d::e:f \end{array}\right\}$ Then also $a:d::e:h$.

Since $\frac{a}{b}=\frac{g}{h}$, $\frac{b}{c}=\frac{f}{g}$, and $\frac{c}{d}=\frac{e}{f}$; by multiplying the alternate fractions together, we obtain $\frac{abc}{bcd}=\frac{gfe}{hgf}$ or $\frac{a}{d}=\frac{e}{h}\therefore a:d::e:h$.

PROP. XVI. When four quantities are proportionals, if the first and second are multiplied or divided by any quantity, and also the third and fourth by the same or any other quantity, the resulting quantities will be proportionals.

Let $a:b::c:d$; then also $ma:mb::nc:nd$.

Since $\frac{a}{b}=\frac{c}{d}$, multiply both terms of the first by m, and both terms of the last by n, and we obtain $\frac{ma}{mb}=\frac{nc}{nd}\therefore ma:mb::nc:nd$, where m and n may be any quantities either integral or fractional.

PROP. XVII. When four quantities are proportionals, if the first and third are multiplied or divided by the same quantity, and also the second and fourth by the same quantity, the resulting quantities will be proportionals.

Let $a:b::c:d$; then also $ma:nb::mc:nd$.

Since $\frac{a}{b}=\frac{c}{d}$, multiply both these by $\frac{m}{n}$, and we obtain $\frac{ma}{nb}=\frac{mc}{nd}$ $\therefore ma:nb::mc:nd$, where m and n may be any quantities either integral or fractional.

PROP. XVIII. If four quantities are proportionals, the like powers or roots of these quantities are also proportionals.

Let $a:b::c:d$; then also $a^m:b^m::c^m:d^m$.

Since $\frac{a}{b}=\frac{c}{d}$ raise each of these fractions to the power expressed by m; then $\left(\frac{a}{b}\right)^m=\left(\frac{c}{d}\right)^m$ or $\frac{a^m}{b^m}=\frac{c^m}{d^m}$ $\therefore$ $a^m:b^m::c^m:d^m$, where m may be any quantity either integral or fractional.

PROP. XIX. Of any number of quantities in continued proportion, the first has to the third the *duplicate* ratio, to the fourth the *triplicate* ratio, to the fifth the *quadruplicate* ratio, &c. of that which it has to the second, or of that which the second has to the third, &c.

Let $a:b::b:c::c:d::d:e::e:f::$ &c. &c.
Then $a:c::a^2:b^2$ or in the duplicate ratio of $a:b$.
$a:d::a^3:b^3$ . . . . . triplicate ratio of $a:b$.
$a:e::a^4:b^4$ . . . . . quadruplicate ratio of $a:b$.
&c. &c. &c. &c.

1st, $a:b::b:c$, or by Prop. XVIII., $a^2:b^2::b^2:c^2$; but by Prop. II., $b^2=ac$ $\therefore a^2:b^2::ac:c^2$ or $a^2:b^2::a:c$; hence $a:c::a^2:b^2$; also $a^2:ac::b^2:c^2$ $\therefore a:c::b^2:c^2$.

2d, $a:c::a^2:b^2$;
but $c:d::a\ :b$
$\therefore\ a:d::a^3:b^3::b^3:c^3::c^3:d^3$.

3d, $a:d::a^3:b^3$.
and $d:e::a\ :b$.
$\therefore\ a:e::a^4:b^4::b^4:c^4::c^4:d^4::d^4:e^4$.

EXERCISES.

1. There are two numbers which are to each other as 5 : 4, and if 5 is added to the greater, and 1 subtracted from the less, the sum will be to the remainder as 5 : 3. What are the numbers? Ans. 20 and 16.

2. Divide the number 120 into two such parts that their product shall be to the difference of their squares as 2 : 3. Ans. 80 and 20.

3. The number 45 is divided into two parts, which are to each other in the triplicate ratio of 4 : 2. Find a mean proportional between them. Ans. 14·14213.

4. The product of two numbers is 48, and the difference of their cubes is to the cube of their difference as 37 : 1. What are the numbers? Ans. 8 and 6.

5. Divide the number 100 into two such parts that 6 times their product shall be to the sum of their squares as 24 : 17. Ans. 80 and 20.

6. There are two numbers whose product is 15, and the difference of their squares is to the square of their difference as 4 : 1. What are the numbers? Ans. 5 and 3.

7. Let $x^2 : y^2 :: 49 : 36$ and $2x - y : x + 6$, in a ratio compounded of the ratios of $2^5 : 2^2$ and $2 : 5$. Required the values of x and y. Ans. $x = 14$, and $y = 12$.

8. There are two numbers in the triplicate ratio of 4 : 1 whose mean proportional is 32. What are the numbers?

Ans. 256 and 4.

9. If $dx = cy$ and $x : y$ in the triplicate ratio of $a : b$; show that the ratio of $a : b$ is that of $\sqrt{c+x} : \sqrt{d+y}$.

OF VARIABLE QUANTITIES.

Quantities which alter their values are called Variable Quantities, and they are often so related to one another, that when one of them is increased the others are increased or diminished according to a constant rule.

Thus if a body moves uniformly, the space it describes increases in the same ratio with the time.

Let S and s be two spaces, T and t the times in which they are described, then $S : T :: s : t$ or $S : s :: T : t$ where S is said to *vary directly*, as T, and this relation is written $S \propto T$.

If the relation between S and T is such, that whilst S by increasing becomes s, and T by diminishing becomes t, in such a manner that in all cases $S : s :: t : T$ or $S : s :: \frac{1}{T} : \frac{1}{t}$; then S is said to *vary inversely*, as T, and is expressed $S \propto \frac{1}{T}$.

If three quantities, S, T, V, are so related to one another, that when S is increased to s, $T \times V$ is also increased to $t \times v$, so that in all cases $S : s :: TV : tv$; then S is said to *vary*, as T and V *jointly*, and is written $S \propto TV$.

If the three variable quantities are so related to one another, that when V is increased to v, S is also increased to s, and T diminished to t, so that in all cases $V : v$ in the ratio compounded of the ratios of $S : s$ and $\frac{1}{T} : \frac{1}{t}$ or $V : v :: \frac{S}{T} : \frac{s}{t}$; then V is said to *vary directly*, as S, and *inversely*, as T, and is written $V \propto \frac{S}{T}$. In this case, if S is constant, $V \propto \frac{1}{T}$ or V is said to vary *inversely*, as T.

These are called *general proportions;* and if the values of the variable quantities can be determined at a given period of their increase or decrease, they may be reduced to determined proportions.

PROP. I. If $S \propto T$, and $T \propto V$, then $S \propto V$. For $S : s :: T : t$ and $T : t :: V : v \therefore S : s :: V : v$; hence $S \propto V$.

PROP. II. If $S \propto T$, and $T \propto \frac{1}{V}$, then $S \propto \frac{1}{V}$. For $S : s :: T : t$ and $T : t :: v : V :: \frac{1}{V} : \frac{1}{v} \therefore S : s :: \frac{1}{V} : \frac{1}{v}$; hence $S \propto \frac{1}{V}$.

PROP. III. If $S \propto V$ and $T \propto V$, then $S \pm T \propto V$. For $S : s :: V : v$ and $T : t :: V : v \therefore S : s :: T : t$, or alternando, $S : T :: s : t$, componendo, $S \pm T : T :: s \pm t : t$, and alternando, $S \pm T : s \pm t : T : t$, but $T : t :: V : v \therefore S \pm T : s \pm t :: V : v$; hence $S \pm T \propto V$.

PROP. IV. If $S \propto V$, and $T \propto V$, then $V \propto \sqrt{ST}$. For $S : s :: V : v$ and $T : t :: V : v \therefore ST : st :: V^2 : v^2$; hence $\sqrt{ST} : \sqrt{st} :: V : v$ or $V \propto \sqrt{ST}$.

PROP. V. If $S \propto T$, and $V \propto X$, then $SV \propto TX$. For $S : s :: T : t$ and $V : v :: X : x \therefore SV : sv :: TX : tx$; hence $SV \propto TX$.

PROP. VI. If $S \propto T$, then $S \propto nT$, where n may be any number either integral or fractional. For $S : s :: T : t \therefore$ multiplying the last ratio by n, we get $S : s :: nT : nt$; hence $S \propto nT$.

Cor. If $S \propto T$, then $S = T$ multiplied by some constant quantity. For $S : s :: T : t$, or alternando, $S : T :: s : t$; hence in every state of the quantities the ratio of $S : T$ is the same. Let it be that of $n : 1$, then $S : T :: n : 1 \therefore S = nT$ or $n = \frac{S}{T}$; hence the value of n will be known, if the corresponding values of S and T at any period of their variation be known.

PROP. VII. If $S \propto T$, then $S^n \propto T^n$, where n may be any number either integral or fractional. For $S : s :: T : t \therefore$ by Prop. XVIII. of Proportion, $S^n : s^n :: T^n : t^n$; hence $S^n \propto T^n$.

PROP. VIII. If $(V+T)^2 \propto (V-T)^2$, then $V^2+T^2 \propto VT$. For $(V+T)^2 : (v+t)^2 :: (V-T)^2 : (v-t)^2$ or $(V+T)^2 : (V-T)^2 :: (v+t)^2 : (v-t)^2$, and by Prop. XI. of Proportion, $2V^2+2T^2 : 4VT :: 2v^2+2t^2 : 4vt$, dividing by 2, we get $V^2+T^2 : 2VT :: v^2+t^2 : 2vt$ or $V^2+T^2 : v^2+t^2 :: 2VT : 2vt :: VT : vt$; hence $V^2+T^2 \propto VT$.

Prop. IX. If $V \propto T$, then $SV \propto ST$, and $\frac{V}{S} \propto \frac{T}{S}$. For $V : v :: T : t$ and $S : s :: S : s$ $\therefore$ $SV : sv :: ST : st$, also $\frac{V}{S} : \frac{v}{s} :: \frac{T}{S} : \frac{t}{s}$; hence $SV \propto ST$ and $\frac{V}{S} \propto \frac{T}{S}$.

Prop. X. If there are two ranks of quantities, S, T, V, &c. and X, Y, Z, &c. related in such a manner, that $S \propto X$, $T \propto Y$, $V \propto Z$, &c.; then will STV, &c. $\propto$ XYZ, &c. For $S : s :: X : x$, $T : t :: Y : y$, $V : v :: Z : z$, &c. $\therefore$ by Prop. XIII. of Proportion, STV, &c. : stv, &c. :: XYZ, &c. : xyz, &c.; hence STV, &c. $\propto$ XYZ, &c.

Prop. XI. If S depends upon T, V, X, in such a manner, that $S \propto T$, when V and X are constant; $S \propto V$, when T and X are constant; and $S \propto X$, when T and V are constant; then $S \propto TVX$, when they all vary. For let the respective values of S be s, a, b; then when all the quantities vary, we have

$$\left.\begin{array}{r} S : s :: T : t \\ s : a :: V : v \\ a : b :: X : x \end{array}\right\}$$

Hence, by composition of ratios, $S : b :: TVX : tvx$ or $S \propto TVX$, which must be true, whatever be the number of quantities.

LITERAL ANALYSIS.

When the known quantities are expressed in numbers, these numbers disappear during the progress of the operation, and the answer, when obtained, does not exhibit the process by which it has been deduced from the assumed data. This mode, though generally adopted in the solution of practical exercises, does not exhibit sufficiently the true difference between arithmetic and algebra, but rather confounds them. The essential character of algebra, taken in its most extensive meaning, is, that the results of its operations do not give the particular values of the quantity or quantities sought; they only represent the operations which ought to be made upon the given quantities, for obtaining the values of those sought, according to the conditions of the problem; so that the principal object of algebra is the investigation of theorems and the exhibition of rules for the arithmetical or geometrical solution of problems. For accomplishing these purposes, it is necessary to represent the known quantities by letters, as well as the unknown ones. The former are represented by the first letters of the alphabet, a, b, c, &c. and the unknown ones by the last letters, x, y, z, &c. The question is translated into equations, and these equations are resolved by the preceding rules; and then the values of the unknown quantities will be expressed in a general way, from their relations to those which are given in the question. Consequently, if this general expression be transferred from algebraical characters into common language, it will give a general rule for the solution of all questions of the same kind. But the expressions will answer the same purpose as accurately in algebraical characters, and then they are called Theorems, or Formulæ.

1. Given the sum s, and the difference d, of two quantities x and y; to find the quantities. $x+y=s$, and $x-y=d$: by adding these equations we get $2x=s+d$, whence $x=\frac{s+d}{2}$; and by subtracting the equations we get $2y=s-d$, and $y=\frac{s-d}{2}$. These values, expressed in common language, give the following rules, viz.

To find the greater, add the difference to the sum, and divide by 2.

To find the less, subtract the difference from the sum, and divide by 2.

2. Given the sum s, of two quantities x and y, and the difference of their squares D; to find the quantities. $x+y=s$, and $x^2-y^2=D$; and dividing the latter by the former, we get $x-y=\frac{D}{s}$; whence, as before, $x=\frac{s}{2}+\frac{D}{2s}$ and $y=\frac{s}{2}-\frac{D}{2s}$, or $x=\frac{s^2+D}{2s}$ and $y=\frac{s^2-D}{2s}$.

3. As exercises, the student may investigate the following, viz. Of two quantities, their sum, difference, product, quotient, sum and difference of their squares, any two being given; to find all the rest. The operations will be similar to those used in the two last questions; and the results, except for the sum and difference of their squares, are given in the following Table, in which x and y are the quantities, $s=$ their sum, $d=$ their difference, $p=$ their product, $q=$ their quotient, $Z=$ the sum of their squares, and $D=$ the difference of their squares.

TABLE.

Given.	Greater $= x$.	Less $= y$.	Sum $= s$.	Difference $= d$.	Product $= p$.	Quotient $= q$.
s and d	$\frac{s+d}{2}$	$\frac{s-d}{2}$			$\frac{s^2-d^2}{4}$	$\frac{s+d}{s-d}$
s and p	$\frac{s+(s^2-4p)^{\frac{1}{2}}}{2}$	$\frac{s-(s^2-4p)^{\frac{1}{2}}}{2}$		$(s^2-4p)^{\frac{1}{2}}$		$\frac{s+(s^2-4p)^{\frac{1}{2}}}{s-(s^2-4p)^{\frac{1}{2}}}$
s and q	$\frac{sq}{q+1}$	$\frac{s}{q+1}$		$\frac{q-1}{q+1}s$	$\frac{s^2q}{(q+1)^2}$	
d and p	$\frac{d+(d^2+4p)^{\frac{1}{2}}}{2}$	$\frac{d-(d^2+4p)^{\frac{1}{2}}}{2}$	$(d^2+4p)^{\frac{1}{2}}$			$\frac{d+(d^2+4p)^{\frac{1}{2}}}{d-(d^2+4p)^{\frac{1}{2}}}$
d and q	$\frac{dq}{q-1}$	$\frac{d}{q-1}$	$\frac{q+1}{q-1}\times d$		$\frac{qd^2}{(q-1)^2}$	
p and q	$(pq)^{\frac{1}{2}}$	$\sqrt{\frac{p}{q}}$	$(q+1)\sqrt{\frac{p}{q}}$	$(q-1)\sqrt{\frac{p}{q}}$		
d and D	$\frac{d^2+D}{2d}$	$\frac{D-d^2}{2d}$	$\frac{D}{d}$		$\frac{D^2-d^4}{4d^2}$	$\frac{D+d^2}{D-d^2}$
Z and D	$\left(\frac{Z+D}{2}\right)^{\frac{1}{2}}$	$\left(\frac{Z-D}{2}\right)^{\frac{1}{2}}$	$(Z+\sqrt{Z^2-D^2})^{\frac{1}{2}}$	$(Z-\sqrt{Z^2-D^2})^{\frac{1}{2}}$	$\frac{\sqrt{Z^2-D^2}}{2}$	$\frac{\sqrt{Z^2-D^2}}{Z-D}$

The use of this Table is plain. Suppose the sum of two numbers to be 277, and their difference to be 115; then the greater number is $\left(\frac{s+d}{2}\right)=\left(\frac{277+115}{2}\right)=\frac{392}{2}=196$.

Suppose again the difference of two numbers to be 10, and their product 119.

The greater number is $\frac{d+(d^2+4p)^{\frac{1}{2}}}{2}=\frac{10+(100+476)^{\frac{1}{2}}}{2}$ $=\frac{10+\sqrt{576}}{2}=\frac{10+24}{2}=17$.

Suppose the sum of their squares to be 250, and the difference of their squares to be 88.

The greater number is $\left(\frac{Z+D}{2}\right)^{\frac{1}{2}}=\left(\frac{250+88}{2}\right)^{\frac{1}{2}}=\sqrt{169}$ $=13$.

The less is $\left(\frac{Z-D}{2}\right)^{\frac{1}{2}}=\left(\frac{250-88}{2}\right)^{\frac{1}{2}}=\sqrt{81}=9$.

4. Given the sum s, of the products of two quantities, by known multipliers m and n, and also the sum of their products c, by other known multipliers p and q, to find the quantities.

Here $mx+ny=s$, and $px+qy=c$; multiplying the former equation by p, and the latter by m, they become $pmx+pny=ps$, and $mpx+mqy=mc$; subtracting, we get $npy-mqy=ps-mc$; and dividing by $np-mq$, we obtain $y=\frac{ps-mc}{np-mq}$; in the same way we find $x=\frac{qs-nc}{mq-np}$.

5. Given the sum s of the quotients of two quantities by known divisors m and n, and also the sum c, of their quotients by other known divisors p and q; to find the quantities.

Here $\frac{x}{m}+\frac{y}{n}=s$, and $\frac{x}{p}+\frac{y}{q}=c$, whence $nx+my=mns$, and $qx+py=pqc$; which, resolved as the last, give $x=\frac{pm(ns-qc)}{pn-qm}$, and $y=\frac{nq(ms-pc)}{qm-pn}$

6. Given the values m and n, of two ingredients; to find the quantities which must be taken of each, to form a given quantity a, of a compound of a given value e.

Here $x+y=a$, and $mx+ny=ae$.

Ans. $x=a\frac{e-n}{m-n}$, and $y=a\frac{e-m}{n-m}$.

7. Given the times m and n, in which two agents could

produce the same effect separately; to find the time in which they could do it jointly.

Here $\frac{x}{m}+\frac{x}{n}=1$. Ans. $x=\frac{mn}{m+n}$.

8. Given the times m, n, and r, in which three agents can perform the same work separately; to find the time in which they can do it jointly.

Here $\frac{x}{m}+\frac{x}{n}+\frac{x}{r}=1$. Ans. $x=\frac{mnr}{mn+mr+nr}$.

9. Given the times m, n, and r, in which every two of three agents can perform the same work; to find the time x, in which they can do it jointly, and also the times y, z, and v, in which each of them can do it separately.

Here $\frac{x}{m}+\frac{x}{n}+\frac{x}{r}=2$. Ans. $x=\frac{2mnr}{mn+mr+nr}$, $y=\frac{2mnr}{(m+n)r-mn}$,

$z=\frac{2mnr}{(m+r)n-mr}$, and $v=\frac{2mnr}{(n+r)m-nr}$.

10. Given the specific gravities m and n, of two ingredients, and the quantity a, of the mixture, with its specific gravity r; to find the quantities of the ingredients.

Ans. $x=\frac{ma(r-n)}{r(m-n)}$, and $y=\frac{na(m-r)}{r(m-n)}$.

11. Given the first distance d, of two moving bodies, and their velocities m and n; to find the time of their conjunction.

Ans. $x=\frac{d}{n\pm m}$, where the upper sign must be used when they move in opposite directions, and the under when they move in the same direction.

12. Given the sum $2s$, of two numbers, and also the sum of their squares, of their cubes, of their fourth, or of their fifth powers, &c.; to find the numbers.

NOTE. If their difference be $2x$, the numbers will be $s+x$ and $s-x$; and then the sum of their squares will be $2s^2+2x^2$, the sum of their cubes $2s^3+6sx^2$, the sum of their fourth powers $2s^4+12s^2x^2+2x^4$, and the sum of their fifth powers $2s^5+20s^3x^2+10sx^4$, all of which are of the quadratic or simple form, and may be resolved as before; but the sums of the higher powers exceed the quadratic.

Let $z=$ sum of their squares, $c=$ sum of their cubes, $q=$ sum of their fourth powers, and $p=$ sum of their fifth powers;

then $x=\left(\frac{z-2s^2}{2}\right)^{\frac{1}{2}} \quad =\left(\frac{c-2s^3}{6s}\right)^{\frac{1}{2}}$

$=\left(-3s^2\pm\sqrt{\tfrac{1}{2}q+8s^4}\right)^{\frac{1}{2}} \quad =\left(-s^2\pm\sqrt{\frac{p}{10s}+\frac{4s^4}{5}}\right)^{\frac{1}{2}}$

13. To find two numbers of which the product is given p, and also the product P, of the sums when each is increased by a given number (a and b).

$$\text{Ans. } x = \frac{P - p - ab}{2b} \pm \sqrt{\left(\left(\frac{P - p - ab}{2b}\right)^2 - \frac{ap}{b} \right)}.$$

14. To find two numbers such, that their sum, their product, and the difference of their squares, shall be all equal.

$$\text{Ans. } y = \frac{1 \pm \sqrt{5}}{2}, \text{ and } x = \frac{3 \pm \sqrt{5}}{2}.$$

15. Given the sum a, of two numbers, and the sum of their square roots b; to find the numbers.

$$\text{Ans. } x = \tfrac{1}{2}a \pm \tfrac{1}{2}b\sqrt{2a - b^2}.$$

16. Given the excess of the product of two numbers above their sum a, and also the sum of their squares b; to find the numbers.

Ans. The greater $= \frac{s + \sqrt{(s^2 - 4p)}}{2}$, and the less $= \frac{s - \sqrt{(s^2 - 4p)}}{2}$, where s and p are their sum and product, and can easily be obtained from the question.

17. Given the sum s, of three numbers, of which the square of the greatest is equal to the squares of the other two, and also the continued product p, of the three numbers; to find the numbers.

Ans. The greatest is $\frac{s^2 \pm \sqrt{s^4 - 16sp}}{4s}$; the sum of the two less is $\frac{3s^2 \pm \sqrt{s^4 - 16sp}}{4s}$; and their product is $\frac{s^2 \pm \sqrt{s^4 - 16sp}}{4}$.

18. Let p be the given product of the two lesser numbers, the rest as before; to find the numbers.

Ans. The greatest is $\frac{s^2 - 2p}{2s}$, the sum of the two less is $\frac{s^2 + 2p}{2s}$, and their difference is $\frac{(s^4 - 12s^2p + 4p^2)^{\frac{1}{2}}}{2s}$.

19. Let, as before, the square of the greatest be equal to the squares of the other two, and the square of the middle one equal to the product of the greatest and least, and let the sum s of the three be given; to find each of them.

$$\text{Ans. The greatest} = \tfrac{s}{4}(\sqrt{5} + 1 - \sqrt{2\sqrt{5} - 2}).$$

20. Suppose still the square of the greatest equal to the

squares of the other two, and let the difference of the squares of the two least be equal to the product of the greatest by a given multiplier m, also the difference of the two least is given $= d$; to find the numbers.

Ans. The greatest is $= \frac{d^2}{\sqrt{2d^2 - m^2}}$, or putting $n^2 = 2d^2 - m^2$, it is $= \frac{d^2}{n}$, the next is $= \frac{(m+n)d}{2n}$, and the least is $= \frac{(m-n)d}{2n}$.

PROGRESSIONS.

A SERIES of quantities, which increase or decrease by a common difference, is called an Arithmetical Progression; as, 2, 5, 8, 11, &c., or 88, 85, 82, &c.

A series of quantities, which increase by a constant multiplier, or decrease by a common divisor, is called a Geometrical Progression; as, 2, 8, 32, 128, &c., or 567, 189, 63, &c.

The greatest and least terms are called the Extremes, and the other terms the Means.

ARITHMETICAL PROGRESSION.

If a represent the least term, y the greatest, d the common difference, and n the number of terms, any arithmetical progression may be expressed thus: $a, a+d, a+2d, a+3d$, &c. ascending; or $y, y-d, y-2d, y-3d$, &c. descending.

From these expressions it appears that the coefficient of d in any term is less by 1 than the number of that term.

PROP. I. The difference between the extremes is equal to the common difference, multiplied by the number of terms *minus* one. For the coefficient of d in the nth term is $n-1$.

Cor. Hence $y = a + (n-1)d$, and $a = y - (n-1)d$.

PROP. II. The sum of the extremes is equal to the sum of any two terms equally distant from them.

For any term exceeds the least, as much as its corresponding term is less than the greatest. Thus, if the series ascend from a to y, the whole will be $a, a+d, a+2d$, &c., $y-2d, y-d, y$; where the sum of any two corresponding terms is $a+y$.

Cor. The double of any term is equal to the sum of any two terms equally distant from it.

PROP. III. The sum of any number of terms is equal to the sum of the extremes multiplied by half the number of terms.

For by adding the extremes, and every two equally distant from them, we obtain equal sums, of which the number is half the number of terms of the series.

Cor. 1. Hence if $s =$ sum of the series, $s = (a+y)\frac{n}{2}$.

Cor. 2. If the number of terms be odd, and m the middle one, then $s = nm$; for $2m = a + y$.

Cor. 3. In a series of natural numbers, 1, 2, 3, &c. n, the sum $s = n \times \frac{n+1}{2}$; for n is the greatest term, and $n+1$ the sum of the extremes.

Cor. 4. In a series of even numbers, 2, 4, 6, &c., $s = n(n+1)$; for this series is $2 \times (1+2+3+$ &c.)

Cor. 5. In a series of odd numbers, beginning at 1, as 1, 3, 5, &c., $s = n^2$; for the sum of the extremes is double the number of terms.

1. Required the 12th term of the series 5, 8, 11, &c.

Here $n = 12$, $a = 5$, $d = 3$; therefore $y = 5 + 11 \times 3 = 38$.

2. Required the 7th term of the series 182, 178, 174, &c.

Here $n = 7$, $y = 182$, $d = 4$; therefore $a = 182 - 6 \times 4 = 158$.

3. Required the sum of 12 terms of the series 3, 8, 13, &c.

Here $a = 3$, $d = 5$, $n = 12$, $y = 3 + 11 \times 5 = 58$, and $s = (58 + 3)6 = 366$.

4. Required the sum of 14 terms of the series 89, 85, 81, &c.

Here $a = 89 - 13 \times 4 = 37$, and $s = (89 + 37)7 = 882$.

From these propositions any two of the five things mentioned may be found, if the other three be given. The theorems for finding them are expressed in the following Table:—

USE OF THE TABLE.

1. Let the least term be 7, the common difference 2, and the sum of the series 567. Required the greatest, and the number of terms.

$\sqrt{(567 \times 8 \times 2 + \overline{14-2}|^2)} = \sqrt{(9072 + 144)} = \sqrt{9216} = 96$, and $\frac{96-2}{2} = 47$, the greatest term; and $\frac{96-14+2}{2 \times 2} = 21$, the number of terms.

2. Given the least term 5, the number of terms 30, and the sum of the series 1455; to find the greatest term and the common difference.

$\frac{1455 \times 2}{30} - 5 = 92$ the greatest, $\frac{1455 - 5 \times 30}{15 \times 29} = 3$ the difference.

FORMULÆ FOR FINDING THE OTHER QUANTITIES.

Given.	Least $= a$.	Greatest $= y$.	Difference $= d$.	Number of Terms $= n$.	Sum $= s$.
a, y, n			$\frac{y-a}{n-1}$		$\frac{1}{2}n(y+a)$
a, d, n		$a+(n-1)d$			$\frac{1}{2}n(2a+(n-1)d)$
a, n, s		$\frac{2s}{n}-a$	$\frac{s-an}{\frac{1}{2}n(n-1)}$		
y, n, s	$\frac{2s}{n}-y$		$\frac{ny-s}{\frac{1}{2}n(n-1)}$		
y, n, d	$y-(n-1)d$				$\frac{1}{2}n(2y-(n-1)d)$
d, n, s	$\frac{s}{n}-\frac{n-1}{2}d$	$\frac{s}{n}+\frac{n-1}{2}d$			
a, y, s			$\frac{y^2-a^2}{2s-y-a}$	$\frac{2s}{y+a}$	
a, y, d				$\frac{y-a}{d}+1$	$\frac{(y+d)\times(y+d-a)}{2d}$
a, d, s		$\frac{(8ds+\overline{2a-d}\vert^2)^{\frac{1}{2}}-d}{2}$		$\frac{(8ds+\overline{2a-d}\vert^2)^{\frac{1}{2}}-2a+d}{2d}$	
y, d, s	$\frac{d\pm(\overline{2y+d}\vert^2-8ds)^{\frac{1}{2}}}{2}$			$\frac{2y+d\pm(\overline{2y+d}\vert^2-8ds)^{\frac{1}{2}}}{2d}$	

GEOMETRICAL PROGRESSION.

If a be the least term of a geometrical progression, y the greatest, r the common multiplier or divisor, called the common ratio, and n the number of terms, such a series, if ascending, may be expressed thus, a, ar, ar^2, ar^3, &c., or if descending, thus, y, $\frac{y}{r}$, $\frac{y}{r^2}$, $\frac{y}{r^3}$, &c.; where the exponent of r is one less than the number of the term.

PROP. I. The greatest term of a geometrical progression is equal to the least term, multiplied by that power of the common ratio, of which the exponent is the number of terms *minus* one.

For in the nth term, the exponent of r is $n-1$.

Therefore $y=ar^{n-1}$, and $a=\frac{y}{r^{n-1}}$.

Hence if $a=1$, $y=r^{n-1}$.
Required the 8th term of the series 2, 6, 18, &c.
Here $a=2$, $r=3$, $n=8$; therefore $2\times 3^7=4374$.

PROP. II. The product of the extremes is equal to the product of any two terms equally distant from them.

For $a\times y=ar\times\frac{y}{r}=ar^2\times\frac{y}{r^2}$, &c.

Cor. 1. The square of any term is equal to the product of any two terms equally distant from it.

Cor. 2. If there be four terms, the product of the means, divided by either extreme, gives the other; and if there be three terms, the square of the mean, divided by either extreme, gives the other.

1. Required a third proportional to 85 and 425. Ans. 2125.
2. . . . . . . a fourth proportional to 18, 54, 162. . 486.

PROP. III. If the sum of a geometrical progression be multiplied by the common ratio, and the series be subtracted from the product, the remainder will be equal to the excess of the product of the highest term by the ratio, above the least term.

For the whole series, except the least term, will be included in the product. Thus, if $a+ar+ar^2$, &c. $+\frac{y}{r^2}+\frac{y}{r}+y=s$ be multiplied by r, it becomes $ar+ar^2$, &c. $+\frac{y}{r}+y+yr=sr$; and subtracting the original series, we obtain $yr-a=sr-s$.

Whence $s = \frac{yr - a}{r - 1} = \frac{a(r^n - 1)}{r - 1}$.*

Cor. 1. The difference between any two adjacent terms is equal to the less multiplied by the ratio, wanting one.

Thus, $ar^3 - ar^2 = ar^2 \times (r - 1)$. Wherefore, if the difference of the extremes be multiplied by the greatest term but one, and divided by the difference between the two greatest terms, the quotient will be the sum of all the terms except the greatest. For the divisor is the product of the multiplier by $r - 1$.

Cor. 2. If the common ratio be 2, the difference of the extremes is the sum of all the terms except the greatest.

Cor. 3. If a descending series be interminate, the least term may be considered $= 0$, and the sum $= \frac{yr}{r - 1}$.

1. Required the 8th term of the series 4, 8, 16, &c.
$$4 \times 2^7 = 4 \times 128 = 512.$$

2. Required the sum of 12 terms of the series 1, 3, 9, 27, &c. $\frac{3^{12} - 1}{3 - 1} = \frac{531441 - 1}{2} = 265720.$

3. Required the sum of 8 terms of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, &c. $\frac{1 - \overline{\frac{1}{3}}|^8}{1 - \frac{1}{3}} = \left(1 - \frac{1}{6561}\right) \times \frac{3}{2} = \frac{6560}{6561} \times \frac{3}{2} = \frac{3280}{2187}.$

4. Given the extremes a and y, and the sum of the series s, to find the common ratio and the number of terms.

Ans. $r = \frac{s - a}{s - y}$. Having found r, $r^{n-1} = \frac{y}{a}$. And in logarithms, where R, Y, and A represent the logarithms of r, y, and a, $(n - 1)R = Y - A$, and $n = \frac{Y - A + R}{R}$.

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* In this formula r may represent any quantity, integral or fractional, except unity. If $r = 1$, there could be no progression; for every power of 1 is 1, and therefore the formula would be $\frac{a(1-1)}{1-1} = \frac{a \times 0}{0}$, a very improper expression. When a is multiplied by a quantity less than 1, the product is less than the multiplicand; and the less that the multiplier is taken, the less will the product be; so that $a \times 0 = 0$, or less than any quantity. Again, when a is divided by a quantity less than 1, the quotient is greater than a; and the less that the divisor is taken, the greater will the quotient be: therefore $\frac{a}{0}$ will be infinitely great, or greater than any quantity. To avoid this absurdity, divide first by the denominator, and then affix values to the quantities. If $ar^n - a$ be divided by $r - 1$, the quotient is $ar^{n-1} + ar^{n-2} + ar^{n-3} +$ &c; and if

QUESTIONS ON PROGRESSIONS.

1. To find four numbers in arithmetical progression, such, that their sum shall be 56, and the sum of their squares 864. Let the numbers be x, $x+y$, $x+2y$, $x+3y$, then their sum $4x+6y=56$, or $2x+3y=28$, and the sum of their squares $4x^2+12xy+14y^2=864$, from which subtract $\overline{2x+3y}|^2=28^2$, or $4x^2+12xy+9y^2=784$; the remainder gives $5y^2=80$, or $y=4$, and $x=8$; and the numbers are 8, 12, 16, 20.

2. To find three numbers in arithmetical progression, such, that their sum shall be 9, and the sum of their cubes 153. Let the numbers be $x-y$, x, $x+y$, then their sum $3x=9$, and the sum of their cubes $3x^3+6xy^2=153$.

Ans. The numbers are, 1, 3, 5.

3. To find three numbers in arithmetical progression, such, that their sum shall be 15, and the sum of the squares of the extremes 58. Ans. 3, 5, 7.

4. To find four numbers in arithmetical progression, such, that the sum of the extremes shall be 8, and the product of the means 15. Ans. 1, 3, 5, 7.

5. To find four numbers in arithmetical progression, such, that the sum of the squares of the means shall be 52, and the sum of the squares of the extremes 68. Ans. 2, 4, 6, 8.

6. A traveller goes 9 miles a-day: after 7 days another sets out after him, and travels 4 miles the first day, 5 miles the second, 6 miles the third, and so on. In what time will he overtake the first?

Here $\frac{8+x-1}{2}x=(x+7)9$. Ans. 18 days.

7. To find three numbers in geometrical progression, such, that their sum shall be 7, and the sum of their squares 21. Let x, y, z, be the numbers.

Then $xz=y^2$, $x+y+z=7$, $x^2+y^2+z^2=21$.

Ans. 1, 2, 4.

8. To find four numbers in geometrical progression, such, that their sum shall be 30, and that the greatest shall be equal to the sum of the means multiplied by $1\frac{1}{3}$.
Let x, xy, xy^2, xy^3, be the numbers. Ans. 2, 4, 8, 16.

$r=1$, it will be $a(1+1+1+1+\&c.)=na$, which, though not a geometrical progression, is a determined quantity. In like manner $\frac{x^2-a^2}{x-a}$ would be $\frac{0}{0}$, if x were $=a$; but if we divide first, the quotient will be $x+a$, which is $=2a$, when $x=a$. And many other cases may occur like these.

9. To find three numbers in geometrical progression, such, that their product shall be 64, and the sum of their cubes 584. Let x, xy, xy^2, be the numbers.

Then $x^3y^3 = 64$, $x^3 \times (1+y^3+y^6)=584$. Ans. 2, 4, 8.

10. To find three numbers in geometrical progression, such, that the sum of the first and third shall be 52, and their product 100. Ans. 2, 10, 50.

11. To find two mean proportionals between 4 and 256.

Ans. 16 and 64

12. Given the sum of the squares a, of three numbers in arithmetical progression, and the excess of the square of the mean above the product of the extremes b; to find the numbers.

Ans. Comm. diff. $\sqrt{b}$, mean $\sqrt{\left(\frac{a-2b}{3}\right)}$.

13. Given the product of the extremes a, and the product of the means b, of four numbers in arithmetical progression; to find the numbers.

Ans. Com. diff. $\sqrt{\left(\frac{b-a}{2}\right)}$, least $\frac{1}{2}\left\{\sqrt{\left(\frac{9b-a}{2}\right)}-3\sqrt{\left(\frac{b-a}{2}\right)}\right\}$.

14. Given the number of terms n, of an arithmetical progression, their sum a, and the sum of their squares b; to find the terms. Let the terms be $x+y, x+2y, x+3y \ldots x+ny$.

Then $y=\left(\frac{12nb-12a^2}{n^2(n^2-1)}\right)^{\frac{1}{2}}$, and $x=\frac{a}{n}-\frac{n+1}{n}\left(\frac{3nb-3a^2}{n^2-1}\right)^{\frac{1}{2}}$.

15. Suppose two travellers set out at the same time from two places of which the distance is given, p. The miles travelled by the first per day form a decreasing arithmetical progression, of which the first term is given, a, and the common difference d. Those travelled by the second form an increasing series, of which the first term is b, and the common difference c. In what time will they meet?

Let $a+b=m$, and $c-d=n$.

Ans. $\frac{1}{2}-\frac{m}{n}\pm\sqrt{\left(\frac{2p}{n}+\left(\frac{m}{n}-\frac{1}{2}\right)^2\right)}$, or $\frac{p}{m}$, (if $n=0$).

16. Given the sum s of five numbers in geometrical progression, and the sum of their squares a; to find the numbers.

Suppose $v=$ sum of the first and third, then $v=\frac{s}{2}-\frac{a}{2s}$, and the second $=\sqrt{\left(v^2+\left(\frac{s-v}{2}\right)^2\right)}-\frac{s-v}{2}$.

INTEREST AND ANNUITIES.

In Simple Interest, the interest is computed on the principal only. Let $p =$ principal or money lent, $t =$ time, $r =$ rate or interest of £1 for the time one, $i =$ interest for the whole time, $a =$ amount or sum of principal and interest; then $rt =$ interest of £1 for the time t, and $1 + rt$ the amount of £1, and $p \times (1 + rt) = p + prt = p + i = a$ the amount of the whole; from which equations the value of any of the quantities concerned may be found in terms of the others.

In Compound Interest, the interest at each term of payment is added to the principal, and the amount is the principal for the next term. Let $R = 1 + r$ the amount of £1 for the first term, it will be the principal for the next term, and the interest upon it will be Rr, and the amount $Rr + R = R(r + 1) = R^2$ will be the principal for the next term. In like manner we find that the amounts at the end of the following terms will be R^3, R^4, &c.; and at the end of the time t it will be R^t, and for the principal p it will be pR^t the amount, and the interest will be $pR^t - p = i = a - p$; from which equations any of the quantities may be expressed in terms of the rest.

Of Annuities. If $m =$ principal, which yields £1 of annual interest at the given rate, then $mR^t - m =$ interest of this principal for the time t, which will therefore be the amount of an annuity of £1 for that time. But $m = \frac{1}{r}$, and therefore the amount will be $\frac{R^t - 1}{r}$; and for any annuity n, it will be $\frac{nR^t - n}{r} = a$. And if p be equal to the present value of this annuity, then $\frac{nR^t - n}{r} = pR^t$, and $p = \dfrac{n - \frac{n}{R^t}}{r}$ where $\frac{1}{R^t}$ is the present worth of £1.

Of Reversions. When the annuity does not commence till some time after this, it is said to be in reversion. The amount, if it were to commence just now, would be $n \times \frac{R^t - 1}{r}$; but if it commence s years after this, it will be $\frac{n}{R^s} \times \frac{R^t - 1}{r} = a$, and the present worth $p = \frac{n}{R^s} \times \dfrac{1 - \frac{1}{R^t}}{r}$.

From these equations any of the quantities may be expressed in terms of the others.

In a Freehold Estate, the value $y = \frac{1}{r}$ when the rent is £1, and it commences just now; and $\frac{1}{R^s r}$ is its value, when it does not commence till s years after this, y is called the year's purchase or perpetuity, and $ay = v$ the value of the estate, of which the rent is a, and $\frac{ay}{R^s}$ is the value in reversion.

Annuities on Lives. Adopting Mr De Moivre's hypothesis, that of a certain number born at one time, one dies every year until the whole is extinct, a supposition which agrees nearly with observation, for ages between 10 and 60. An annuity of £1 for a given life will be the sum of the series $\frac{n-1}{nr} + \frac{n-2}{nr^2} + \frac{n-3}{nr^3} +$ &c., continued to $\frac{n-n}{nr^n}$, where n is the complement of the age, or what it wants of the age at which the oldest dies, which he supposed to be 86, and r the amount of £1 for a year. This sum is $\frac{\left(n-1+\frac{1}{r^n}\right)r-n}{n(r-1)^2} = \frac{n-1-q}{n(r-1)}$, supposing q to be the present worth of an annuity of £1 for $n-1$ years.

Again, the value of an annuity for two joint lives, of which the complements are n and m (the greatest m), will be $\frac{n-1}{n} \times \frac{m-1}{mr} + \frac{n-2}{n} \times \frac{m-2}{mr^2} + \frac{n-3}{n} \times \frac{m-3}{mr^3} +$ &c. continued to $\frac{n-n}{n} \times \frac{m-n}{mr^n}$, of which the sum is $\frac{1}{r-1} + \frac{(m-n)\frac{1}{r^n}-(m+n)}{mn} \times \frac{r}{(r-1)^2} + \frac{\left(1-\frac{1}{r^n}\right)(r+1)r}{mn \times (r-1)^3}$; or if $s =$ value of the oldest life, the value of the two lives is $\frac{(n-1)p - s \times (2p+1-(m-n))}{m}$, where $p =$ perpetuity.

If a question occur which involves both interest and annuities, an equation may be found answering to it by comparing with one another the values of the quantities found separately.

1. What will £1000 amount to in 10 years, at 5 per cent. compound interest? Ans. £1628, 17s. 9½d.

2. What principal will, in 15 years, amount to £2000, at 4 per cent. compound interest? Ans. £1110, 10s. 7½d.

3. In what time will £200 amount to £318, 16s., at 6 per cent. compound interest? Ans. 8·0016 years.

4. In what time will a sum of money double itself, at 4 per cent. compound interest? $(1{\cdot}04)^t = 2$.
Ans. 17·673 years.

5. Required the amount of £20 annuity for 41 years, at 5 per cent.? Ans. £2556, 15s. 11d.

6. What annuity will, in 7 years, amount to £79, at 4 per cent.? Ans. £10·0022.

7. What is the value of an annuity of £20, for a life of 54 years of age, at 4 per cent.? Ans. £209·55469.

8. What is the value of an annuity of £20, during the joint lives of two persons, whose ages are 35 and 25 years, at 4 per cent.? Ans. £221·9176.

9. When 12 years of a lease of 21 years were expired, a renewal for the same term was granted for £1000. Eight years of that lease are now expired, and it is required what sum should be paid for a corresponding renewal of the lease, reckoning 5 per cent. compound interest.

From the first transaction, find the annuity $n =$ £175·029955, and from it find p, the present worth of the annuity in reversion $=$ £599·9294.

OF SERIES.

A SERIES is an assemblage of terms, which continually increase or decrease according to a certain law, as the arithmetical and geometrical series treated of before.

A Converging Series is that of which the terms continually decrease, and a Diverging Series is one of which the terms continually increase.

Series are obtained by division, by the extraction of roots, and by various other operations.

Thus, $\frac{ax}{a-x} = x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} +$ &c., where the exponents increase by one.

Also $\sqrt{a^2+x^2} = a + \frac{x^2}{2a} - \frac{x^4}{2.4a^3} + \frac{3x^6}{2.4.6a^5} - \frac{3.5x^8}{2.4.6.8a^7} + \frac{3.5.7x^{10}}{2.4.6.8.10a^9} +$ &c.

OF THE BINOMIAL THEOREM.

The Binomial Theorem is a general formula, discovered by Sir Isaac Newton, whereby any power or root of a binomial may be obtained without performing the involution or extraction. The power or root found by this theorem is called the development or expansion of the binomial.

The following is the form in which it was first proposed by Newton:—

$$(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}}\left\{1+\frac{m}{n}Q+\frac{m}{n}.\frac{m-n}{2n}Q^2+\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}Q^3 + \frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}.\frac{m-3n}{4n}Q^4+ \&c.\right\}$$

Or

$$(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ+\frac{m-3n}{4n}DQ+\&c.$$

Where P is the first term of the binomial, Q the second term divided by the first, $\frac{m}{n}$ the exponent of the power or root, and A, B, C, D, &c. the terms immediately preceding those in which they are first found, including their signs + or —.

This theorem may be applied to any particular case, by substituting the quantities in the given example for P, Q, m, and n, in the formula, and then finding the result.

Note. When the exponent of the binomial is a whole number, the series will terminate, but when it is a negative or fractional number, the series will not terminate, but proceed on, and become more convergent the smaller the second term is with respect to the first.

Required the development of $\frac{x^2}{(x^2-y)^{\frac{1}{2}}}$ in a series.

Here $\frac{x^2}{(x^2-y)^{\frac{1}{2}}} = x^2(x^2-y)^{-\frac{1}{2}}$, $P = x^2$, $Q = -\frac{y}{x^2}$, $m = -1$, and $n = 2$; hence

$$P^{\frac{m}{n}} = (x^2)^{\frac{m}{n}} = (x^2)^{-\frac{1}{2}} = \frac{1}{x} = A.$$

$$\frac{m}{n}AQ = -\frac{1}{2}\times\frac{1}{x}\times -\frac{y}{x^2} = \frac{y}{2x^3} = B.$$

$$\frac{m-n}{2n}BQ = \frac{-1-2}{4}\times\frac{y}{2x^3}\times -\frac{y}{x^2} = \frac{3y^2}{2.4x^5} = C.$$

$$\frac{m-2n}{3n}CQ = \frac{-1-4}{6}\times\frac{3y^2}{2.4x^5}\times -\frac{y}{x^2} = \frac{3.5y^3}{2.4.6x^7} = D.$$

$$\frac{m-3n}{4n}DQ = \frac{-1-6}{8}\times\frac{3.5y^3}{2.4.6x^7}\times -\frac{y}{x^2} = \frac{3.5.7y^4}{2.4.6.8x^9} = E.$$

&c. &c. &c.

$\therefore \dfrac{1}{(x^2-y)^{\frac{1}{2}}} = \dfrac{1}{x} + \dfrac{y}{2x^3} + \dfrac{3y^2}{2.4x^5} + \dfrac{3.5y^3}{2.4.6x^7} + \dfrac{3.5.7y^4}{2.4.6.8x^9} + \text{\&c.}$

and $\dfrac{x^2}{(x^2-y)^{\frac{1}{2}}} = x + \dfrac{y}{2x} + \dfrac{3y^2}{2.4x^3} + \dfrac{3.5y^3}{2.4.6x^5} + \dfrac{3.5.7y^4}{2.4.6.8x^7} + \text{\&c.}$

Required the value of $9^{\frac{1}{3}}$ in an infinite series.

Here $9^{\frac{1}{3}} = (8+1)^{\frac{1}{3}} \therefore \mathrm{P}=8, \mathrm{Q}=\frac{1}{8}, m=1$, and $n=3$; whence

$$\mathrm{P}^{\frac{m}{n}} = 8^{\frac{m}{n}} = 8^{\frac{1}{3}} \qquad = 2 \qquad = \mathrm{A}.$$

$$\frac{m}{n}\mathrm{AQ} = \frac{1}{3} \times 2 \times \frac{1}{2^3} \qquad = \frac{1}{3.2^2} \qquad = \mathrm{B}.$$

$$\frac{m-n}{2n}\mathrm{BQ} = \frac{1-3}{6} \times \frac{1}{3.2^2} \times \frac{1}{2^3} \qquad = -\frac{1}{3.6.2^4} \qquad = \mathrm{C}.$$

$$\frac{m-2n}{3n}\mathrm{CQ} = \frac{1-6}{9} \times -\frac{1}{3.6.2^4} \times \frac{1}{2^3} = \frac{5}{3.6.9.2^7} \qquad = \mathrm{D}.$$

$$\frac{m-3n}{4n}\mathrm{DQ} = \frac{1-9}{12} \times \frac{5}{3.6.9.2^7} \times \frac{1}{2^3} = -\frac{5.8}{3.6.9.12.2^{10}} = \mathrm{E}.$$

&c. &c. &c.

$$\therefore 9^{\frac{1}{3}} = 2 + \frac{1}{3.2^2} - \frac{1}{3.6.2^4} + \frac{5}{3.6.9.2^7} - \frac{5.8}{3.6.9.12.2^{10}} + \text{\&c.}$$

1. Expand $(y^2 - x^2)^{\frac{3}{4}}$ into an infinite series.

Ans. $\dfrac{1}{\sqrt{y}}\left(y^2 - \dfrac{3x^2}{2^2} - \dfrac{3x^4}{2^5y^2} - \dfrac{5x^6}{2^7y^4} - \dfrac{5.9x^8}{2^{11}y^6} - \text{\&c.}\right)$

2. Expand $\left(\dfrac{x^3}{x^3+y^3}\right)^{\frac{1}{3}}$ into an infinite series.

Ans. $1 - \dfrac{y^3}{3x^3} + \dfrac{2y^6}{3^2x^6} - \dfrac{2.7y^9}{3^4x^9} + \text{\&c.}$

3. Develop $\left(\dfrac{x-y}{x+y}\right)^{\frac{1}{2}}$ in an infinite series.

Ans. $1 - \dfrac{y}{x} + \dfrac{y^2}{2x^2} - \dfrac{y^3}{2x^3} + \dfrac{3y^4}{2.4x^4} - \dfrac{3y^5}{2.4x^5} + \dfrac{3.5y^6}{2.4.6x^6} - \text{\&c.}$

4. Develop $\dfrac{x}{(x \mp y)^{\frac{1}{3}}}$ in an infinite series.

Ans. $x^{\frac{2}{3}}\left(1 \pm \dfrac{y}{3x} + \dfrac{4y^2}{3.6x^2} \pm \dfrac{4.7y^3}{3.6.9x^3} + \dfrac{4.7.10y^4}{3.6.9.12x^4} \pm \text{\&c.}\right)$

5. Required the value of $\sqrt[3]{7}$ in an infinite series.

Ans. $2 - \dfrac{1}{3.2^2} - \dfrac{1}{3.6.2^4} - \dfrac{5}{3.6.9.2^7} - \dfrac{5.8}{3.6.9.12.2^{10}} - \text{\&c.}$

6. Expand $(1-a)^{\frac{2}{5}}$ into an infinite series.

$$\text{Ans. } 1-\frac{2a}{5}-\frac{2.3a^2}{5.10}-\frac{2.3.8a^3}{5.10.15}-\frac{2.3.8.13a^4}{5.10.15.20}-\&c.$$

7. Required the development of $(b^2+x)^{\frac{1}{2}}$ in a series.

$$\text{Ans. } b+\frac{x}{2b}-\frac{x^2}{2.4b^3}+\frac{3x^3}{2.4.6b^5}-\frac{3.5x^4}{2.4.6.8b^7}+\frac{3.5.7x^5}{2.4.6.8.10b^9}-\&c.$$

OF THE METHOD OF INDETERMINATE COEFFICIENTS.

This is a general method of obtaining series from fractional and other expressions without either performing the division or extracting the root.

Assume a series with unknown, but constant, coefficients, and having the exponents of x increasing or decreasing in the same way as if the operation was performed at length; then make this series equal to the given expression, and, clearing the equation of fractions, bring all the terms to one side, so as to make the equation $=0$; next make the first term and the coefficients of the several powers of x each $=0$,* and there will arise as many independent equations as there are unknown coefficients, from which their values may be found and substituted for them in the assumed series.

Let it be required to expand $\frac{a}{b+x}$ into a series.

Assume $\frac{a}{b+x}=A+Bx+Cx^2+Dx^3+$ &c.; then multiplying both sides by $b+x$, and transposing a, we obtain $Ab-a+(Bb+A)x+(Cb+B)x^2+(Db+C)x^3+$ &c. $=0$, an equation which must be true whatever be the value of x.

Now, making the first term and the coefficients of the several powers of x each $=0$, we have $Ab-a=0$, or $A=\frac{a}{b}$; $Bb+A=0$, or $B=\frac{A}{b}=-\frac{a}{b^2}$; $Cb+B=0$, or $C=\frac{B}{b}=+\frac{a}{b^3}$; $Db+c=0$, or $D=\frac{C}{b}=-\frac{a}{b^4}$, &c. And substituting these values of A, B, C, D, &c. in the assumed series,

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* If the series $(a+b)x+(c+d)x^2+(e+f)x^3$, &c. continued indefinitely, be always $=$ nothing, whatever be the value of x, then the coefficient of any one power of x is $=0$; that is, $a+b=0$, $c+d=0$, &c. For if the equation be divided by x, then $a+b+(c+d)x+(e+f)x^2=0$. Here let $x=0$, then $a+b=0$; therefore $(c+d)x+(e+f)x^2=0$, whatever be the value of x; and proceeding in the same way we find $c+d=0$, and so on.

we get $\frac{a}{b+x}=\frac{a}{b}-\frac{ax}{b^2}+\frac{ax^2}{b^3}-\frac{ax^3}{b^4}+$ &c., in which it is obvious that the signs are alternately $+$ and $-$, and the exponents, both in the numerator and denominator, increase continually by 1, that of x in the numerator being always 1 less than that of b in the denominator.

2. Expand $\frac{a^2}{a^2+2ax-x^2}$ into a series.

Ans. $1-\frac{2x}{a}+\frac{5x^2}{a^2}-\frac{12x^3}{a^3}+$ &c.

3. Expand $\sqrt{(a^2-x^2)}$ into a series.

Ans. $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-$ &c.

4. Expand $\frac{1+2x}{1-x-x^2}$ into a series.

Ans. $1+3x+4x^2+7x^3+11x^4+18x^5+$ &c.

This is a recurring series, in which each of the coefficients after the second is the sum of the two preceding ones.

5. Expand $\sqrt{(1-a)}$ into a series.

Ans. $1-\frac{a}{2}-\frac{a^2}{2.4}-\frac{3a^3}{2.4.6}-\frac{3.5a^4}{2.4.6.8}-\frac{3.5.7a^5}{2.4.6.8.10}-$ &c.

OF THE SUMMATION AND INTERPOLATION OF SERIES.

The summation of series is the method of finding a terminated expression equal to the whole series, and interpolation is the method of finding any term of an infinite series without producing all the rest.

OF THE DIFFERENTIAL METHOD.

The differential method consists in finding from the successive differences of the terms of a series any intermediate term or the sum of the whole series.

PROB. I. To find the several orders of differences.

Let $a+b+c+d+e+$ &c. be any series; subtract each term from the one following it, and the differences $-a+b$, $-b+c$, $-c+d$, $-d+e$, &c. will form a new series, called the first order of differences. Again, subtract each term of this new series from the one that follows it, and the differences $a-2b+c$, $b-2c+d$, $c-2d+e$, &c. will form another series, called the second order of differences. Proceed in like manner for the third, fourth, fifth, &c. order of differences, until they at last become equal to 0, or are carried as far as is required.

NOTE. When the several terms of the series continually increase, the differences will be all positive; but when they decrease, the differences will be alternately negative and positive.

Required the several orders of differences of the series 1, 6, 20, 50, 105, 196, &c.

1, 6, 20, 50, 105, 196, &c.	the given series.
5, 14, 30, 55, 91, &c.	1st differences.
9, 16, 25, 36, &c.	2d do.
7, 9, 11, &c.	3d do.
2, 2, &c.	4th do.
0, &c.	5th do.

2. Required the several orders of differences of the series 1, 2^2, 3^2, 4^2, 5^2, &c. Ans. 1st diff. 3, 5, 7, 9, 11, &c.; 2d diff. 2, 2, 2, 2, &c.; 3d diff. 0.

3. Required the several orders of differences of the series of cubes 1^3, 2^3, 3^3, 4^3, 5^3, &c. Ans. 1st diff. 7, 19, 37, 61, &c.; 2d diff. 12, 18, 24, &c.; 3d diff. 6, 6, &c.; 4th diff. 0.

PROB. II. To find the first term of any order of differences.

Let d', d'', d''', d^{IV}, &c. represent the first terms of the 1st, 2d, 3d, 4th, &c. orders of differences; then $d' = -a + b$, $d'' = a - 2b + c$, $d''' = -a + 3b - 3c + d$, $d^{\text{IV}} = a - 4b + 6c - 4d + e$, &c. from which it is obvious that the coefficients of the several terms of any order of differences are respectively the same as those of the terms of an expanded binomial, and are obtained in the same manner; for the terms which are subtracted are actually added, but with contrary signs. Hence we infer that d^n, or the first difference of the nth order of differences, is

$$\pm a \mp nb \pm n . \frac{n-1}{2} c \mp n . \frac{n-1}{2} . \frac{n-2}{3} d \pm \text{\&c. to } n+1$$

terms, in which formula the upper signs must be taken when n is an even number, and the under when n is an odd number.

1. Required the first of the fifth order of differences of the series 6, 9, 17, 35, 63, 99, 148, &c.
Here $a, b, c, d, e, f,$ &c. $= 6, 9, 17, 35, 63, 99,$ &c. and $n = 5$

$$\therefore -a + nb - \frac{n(n-1)}{2} c + \frac{n(n-1)(n-2)}{2.3} d - \frac{n(n-1)(n-2)(n-3)}{2.3.4} e$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{2.3.4.5} f = -a + 5b - \frac{5.4}{2} e + \frac{5.4.3}{2.3} d$$

$-\frac{5.4.3.2}{2.3.4}e+\frac{5.4.3.2.1}{2.3.4.5}f=-6+45-170+350-315+99$
$=494-491=+3$.

2. Required the first of the sixth order of differences of the series 3, 6, 11, 17, 24, 36, 50, 72, &c. Ans. -14.

3. Required the first of the eighth order of differences of the series 1, 3, 9, 27, 81, &c. Ans. 256.

4. Required the first of the fifth order of differences of the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c. Ans. $-\frac{1}{32}$.

Prob. III. To find the nth term of the series a, b, c, d, e, f, &c.

Since $d'=-a+b$, therefore $b=a+d'$, and, in the same manner, we find $c=a+2d'+d''$, $d=a+3d'+3d''+d'''$, $e=a+4d'+6d''+4d'''+d^{\text{iv}}$, &c.; whence the nth term is $=a+\frac{n-1}{1}d'+\frac{n-1}{1}.\frac{n-2}{2}d''+\frac{n-1}{1}.\frac{n-2}{2}.\frac{n-3}{3}d'''+$ &c.

1. Required the 7th term of the series 3, 5, 8, 12, 17, &c.

Here $d'=2$, $d''=1$, $d'''=0$, and $n=7$ $\therefore$ $a+\frac{n-1}{1}d'$ $+\frac{n-1}{1}.\frac{n-2}{2}d''=3+\frac{7-1}{1}.2+\frac{7-1}{1}.\frac{7-2}{2}.1=3+12+15$ $=30=$ the 7th term.

2. Required the 9th term of the series 1, 5, 15, 35, 70, &c. Ans. 495.

3. Required the 10th term of the series 1, 3, 6, 10, 15, 21, &c. Ans. 55.

Prob. IV. To find the sum of n terms of the series a, b, c, d, e, &c.

If we add the values of a, b, c, &c. as found in the last problem, we obtain $2a+d'=a+b$, $3a+3d'+d''=a+b+c$, $4a+6d'+4d''+d'''=a+b+c+d$, &c.; whence it is manifest that the sum of n terms must be $na+n.\frac{n-1}{2}d'$ $+n.\frac{n-1}{2}.\frac{n-2}{3}d''+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}d'''+$ &c.

Note. When the differences become at last $=0$, any term, or the sum of any number of terms, can be accurately found; but when the differences do not vanish, the formulæ in this and the preceding problem give only an approximation, which will come nearer the truth as the differences diminish.

1. Required the sum of 8 terms of the series 2, 5, 10, 17, &c. Here $n=8$, $a=2$, $d'=3$, $d''=2$, and $d'''=0$; hence

$na+n.\frac{n-1}{2}d'+n.\frac{n-1}{2}.\frac{n-2}{3}d''=8.2+8.\frac{7}{2}.3+8.\frac{7}{2}.\frac{6}{3}.2$ $=16+84+112=212=$ the sum of 8 terms.

2. Required the sum of 12 terms of the series 21, 56, 126, 252, 462, 792, &c. Ans. 27125.

3. Required an expression for the sum of n terms of the fourth order of figurate numbers, 1, 4, 10, 20, 35, &c.

Here $d'=3$, $d''=3$, $d'''=1$, and $d^{\text{IV}}=0$; hence $s=n+n.\frac{n-1}{2}.3+n.\frac{n-1}{2}.\frac{n-2}{3}.3+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.1$, which, reduced, gives $s=n.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}$; where it may be observed that the number of factors in the formula, and the order of differences which become $=0$, are the same with the order of the figurates.

4. Required the sum of 12 terms of the fourth order of figurates 1, 4, 10, 20, 35, &c. Ans. 1365.

5. Required an expression for the sum of n terms of the series of squares $(m\pm a)^2+(m\pm 2a)^2+(m\pm 3a)^2$, &c. $+(m\pm na)^2$.

Ans. $nm^2\pm n.\frac{n+1}{2}.2ma+n.\frac{n+1}{2}.\frac{2n+1}{3}.a^2$.

6. Required the sum of 12 terms of the series $3^2+5^2+7^2+9^2+$ &c. Ans. 2924.

7. Required an expression for the sum of n terms of the series of cubes $(m\pm a)^3+(m\pm 2a)^3+(m\pm 3a)^3$, &c. $+(m\pm na)^3$.

Ans. $nm^3\pm n.\frac{n+1}{2}.3m^2a+n.\frac{n+1}{1}.\frac{2n+1}{2}.ma^2\pm n^2\left(\frac{n+1}{2}\right)^2a^3$.

8. Required the sum of 9 terms of the series $3^3+6^3+9^3+12^3+$ &c. Ans. 54675.

9. Required an expression for the sum of n terms of the series of products $pq+(p-1)\times(q-1)+(p-2)\times(q-2)+(p-3)\times(q-3)+$ &c.

Ans. $\frac{3pq^2+3pq-q^3+q}{6}$, when $n=q+1$, and the series is complete; but if the number of terms n, be less than q, the expression will be $npq-n.\frac{n-1}{2}(p+q)+n.\frac{n-1}{2}.\frac{2n-1}{3}$.

10. Required the sum of 6 terms of the series $9\times 8+8\times 7+7\times 6+6\times 5$, &c. Ans. 232.

PROB. V. To find by interpolation any intermediate term of the series a, b, c, d, e, &c. whose terms are equidistant from each other.

Let x be the place in the series, of any term y that is to be interpolated, and the first terms of the several orders of differences as before; then will

$$y = a + xd' + x.\frac{x-1}{2}.d'' + x.\frac{x-1}{2}.\frac{x-2}{3}.d''' + \text{\&c.}$$

NOTE. In finding the differences, each term is taken from the one which follows it, so that, when the former is the greater, the difference is *negative*; hence, in applying the formula to practice, the *signs* of the differences must be carefully attended to.

1. Required the logarithmic sine of 1° 1′ 40″, having given the log. sines of 1° 0′, 1° 1′, 1° 2′, and 1° 3′.

Series.	Log. Sines.	1st Diff.	2d Diff.	3d Diff.
1° 0′	8·241855			
		7178		
1 1	8·249033		—117	
		7061		+4
1 2	8·256094		—113	
		6948		
1 3	8·263042			

Here $a = 8·241855$, $x = (1° 1' 40'' - 1° 0') = 1' 40'' = 1\frac{2}{3} = \frac{5}{3}$, $d' = 7178$, $d'' = -117$, and $d''' = +4$; whence $y = a + xd' + x.\frac{x-1}{2}d'' + x.\frac{x-1}{2}.\frac{x-2}{3}.d''' = a + \frac{5}{3}d' + \frac{5}{3}.\frac{2}{6}.d'' - \frac{5}{3}.\frac{2}{6}.\frac{1}{9}d''' = 8·241855 + 011963 - 000065 - 000000 = 8·253753 =$ log. sine of 1° 1′ 40″.

2. Given the log. sines of 2° 4′, 2° 5′, 2° 6′, and 2° 7′, to find the log. sine of 2° 6′ 30″. Ans. 8·565719.

3. Given the series $\frac{1}{40}, \frac{1}{41}, \frac{1}{42}, \frac{1}{43}, \frac{1}{44}$, &c. to find the term which falls in the middle between $\frac{1}{42}$ and $\frac{1}{43}$. Ans. $\frac{2}{85}$.

4. Given the natural signs of 88° 54′, 88° 55′, 88° 56′, 88° 57′, 88° 58′, and 88° 59′, to find the natural sine of 88° 57′ 30″. Ans. ·999837.

PROB. VI. To find any intermediate term of the series a, b, c, d, e, &c. by interpolation, when the first differences of any order are small, or become $= 0$.

Find the value of the unknown quantity in the equation which stands opposite the given number of terms in the following table, and it will be the term required.

$$\begin{array}{l|l}
1. & a - b = 0 \\
2. & a - 2b + c = 0 \\
3. & a - 3b + 3c - d = 0 \\
4. & a - 4b + 6c - 4d + e = 0 \\
5. & a - 5b + 10c - 10d + 5e - f = 0. \\
 & \text{\&c.} \qquad \text{\&c.} \qquad \text{\&c.} \\
n. & a - nb + n.\frac{n-1}{2}.c - n.\frac{n-1}{2}.\frac{n-2}{3}.d + \\
 & \qquad n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.e - \text{\&c.} = 0.^*
\end{array}$$

1. Given the logarithms of 201, 202, 203, and 205, to find that of 204.

Here the given number of terms is 4, and opposite 4 in the table stands $a - 4b + 6c - 4d + e = 0$, or $d = \frac{a + 6c + e - 4b}{4}$.

Now, using the logarithms of the given terms, we have

$$\left.\begin{array}{r} \text{Log. } a = 2{\cdot}303169 \\ b = 2{\cdot}305351 \\ c = 2{\cdot}307496 \\ e = 2{\cdot}311754 \end{array}\right\} \quad \begin{array}{r} \log.\ a = 2{\cdot}303196 \\ 6 \log.\ c = 13{\cdot}844976 \\ \log.\ e = 2{\cdot}311754 \\ \hline 18{\cdot}459926 = a + 6c + e \\ 9{\cdot}221404 = 4 \log.\ b \\ 4\ \underline{)9{\cdot}238522} \\ \text{Log. } d \text{ or log. } 204 = 2{\cdot}309630 \end{array}$$

2. Given the cube roots of 45, 46, 47, 48, and 49, respectively equal to 3·556893, 3·583048, 3·608826, 3·634241, and 3·659306, to find the cube root of 50. Ans. 3·684031.

3. Given the logarithms of 60, 61, 62, 64, 65, and 66, to find that of 63. Ans. 1·799341.

4. Given the logarithms of 101, 102, 104, and 105, to find that of 103. Ans. 2·012837.

REVERSION OF SERIES.

When an equation is given of this form, $x = az + bz^2 + cz^3 + dz^4 +$ &c., and it is required to find z in terms of x, the method of doing this is called the Reversion of the Series.

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* It is obvious that this table is composed of the first terms of the 1st, 2d, 3d, &c. nth orders of differences; and when any of these orders become $= 0$, any intermediate term may be accurately found; but if the differences do not vanish, the result is only an approximation which will come nearer the truth the more terms there are in the given series.

Assume the equation $z = Ax + Bx^2 + Cx^3 + Dx^4 +$ &c., substitute this series and its powers instead of z and its powers in the given equation, then make the coefficients of the like powers of x each $= 0$, and they will give equations for finding the values of A, B, C, D, &c.

Let $x = v + \frac{1}{6}v^3 + \frac{3}{40}v^5 + \frac{15}{336}v^7 + \frac{105}{3456}v^9 +$ &c., and let it be required to find v in terms of x.

Here the assumed equation is $v = Ax + Bx^3 + Cx^5 + Dx^7 + Ex^9 +$ &c. Therefore,

$$\frac{1}{6}v^3 = +\frac{1}{6}x^3 + \frac{3}{6}Bx^5 + \frac{B^2+C}{2}x^7 + \left(\frac{1}{2}D + AB + A^3\right)x^9, \text{ \&c.}$$

$$\frac{3}{40}v^5 = +\frac{3}{40}x^5 + \frac{15}{40}Bx^7 + \left(\frac{3}{4}B^2 + \frac{3}{8}C\right)x^9, \text{ \&c.}$$

$$\frac{15}{336}v^7 = +\frac{15}{336}x^7 + \frac{5}{16}Bx^9, \text{ \&c.}$$

$$\frac{105}{3456}v^9 = +\frac{105}{3456}x^9, \text{ \&c.}$$

And equating the coefficients of the like powers of x, we have $B + \frac{1}{6} = 0$ or $B = -\frac{1}{6}$, $C + \frac{3}{6}B + \frac{3}{40} = 0$ or $C = +\frac{1}{120}$, $D + \frac{1}{2}B^2 + \frac{1}{2}C + \frac{3}{8}B + \frac{5}{112} = 0$ or $D = -\frac{1}{5040}$, &c. Therefore $v = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \text{\&c.} = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.5.6.7} +$ &c., where the law of continuation is evident.

REVERT THE FOLLOWING SERIES.

1. $x = y - y^2 + y^3 - y^4 +$ &c.

   Ans. $y = x + x^2 + x^3 + x^4 +$ &c.

2. $x = y + \frac{1}{2}y^2 + \frac{1}{3}y^3 + \frac{1}{4}y^4 +$ &c.

   Ans. $y = x - \frac{x^2}{2} + \frac{x^3}{2.3} - \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} -$ &c.

3. $x = \frac{y}{a} - \frac{y^2}{2a^2} + \frac{y^3}{3a^3} - \frac{y^4}{4a^4} +$ &c.

   Ans. $y = a \times \left(x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \text{\&c.}\right)$

4. $x = y - \frac{y^3}{2.3a^2} + \frac{y^5}{2.3.4a^4} - \frac{y^7}{2.3.4.5.6a^6} +$ &c.

   Ans. $y = x + \frac{x^3}{2.3a^2} + \frac{x^5}{2.3.4a^4} + \frac{x^7}{2.3.4.5.6a^6} +$ &c.

5. $x=\frac{y^2}{2}+\frac{y^3}{3}+\frac{y^4}{4}+$ &c. (put $v=2x$).

Ans. $y=v^{\frac{1}{2}}-\frac{v}{3}+\frac{v^{\frac{3}{2}}}{36}-\frac{v^2}{170}+$ &c.

6. $x=y^{-\frac{1}{2}}-\frac{y^{\frac{1}{2}}}{2}-\frac{y^{\frac{3}{2}}}{8}-\frac{y^{\frac{5}{2}}}{16}-\frac{y^{\frac{7}{2}}}{121}-$ &c.

Ans. $y=x^{-2}-x^{-4}+x^{-6}-x^{-8}+$ &c.

OF LOGARITHMS.

LOGARITHMS are a set of artificial numbers invented and formed into tables for the purpose of facilitating arithmetical computations. They are adapted to the natural numbers in such a manner, that, by their aid, Addition supplies the place of Multiplication, Subtraction that of Division, Multiplication that of Involution, and Division that of the Extraction of Roots.

Logarithms may be considered as the exponents of the powers to which a given number must be raised, in order to produce all the natural numbers.

Thus, let r be any given number, and let such values be successively assigned to x as will make $r^x=a$, $r^{x'}=b$, $r^{x''}=c$, &c.; then x, x', x'', &c. are the logarithms of a, b, c, &c. respectively.

If $x=0$, then $r^x=1$, whatever be the value of r; hence in every system of logarithms the logarithm of 1 is $=0$. Hence, also, when $x=1$, it is obvious a will be equal to r. The constant quantity r is called the *radix* or *base* of the system, and in every system it is that number whose logarithm is 1.

Since r may be assumed of any value greater or less than unity, it is evident that there may be innumerable systems of logarithms answering to the natural numbers; but since 10 is the *base* of our system of arithmetic, it has accordingly been assumed as the *base* of our common tables of logarithms; therefore,

Let $r=10$, and we have $10^{-3}=\frac{1}{1000}$, $10^{-2}=\frac{1}{100}$, $10^{-1}=\frac{1}{10}$, $10^{\circ}=1$, $10^1=10$, $10^2=100$, $10^3=1000$, &c. that is, the log. of $\frac{1}{1000}$ or ·001 is -3, of $\frac{1}{100}$ or ·01 is -2, of $\frac{1}{10}$ or ·1 is -1, of 1 is 0, of 10 is 1, of 100 is 2, of 1000 is 3, &c. Hence it is evident that the logarithm of any number falling between ·001 and ·01 will be $-3+$ some fraction; that of a number between ·01 and ·1 will be $-2+$ some fraction; that of a number between ·1 and 1 will be $-1+$ some fraction; that of a number between 1 and 10 will be a proper fraction; that of a number between 10 and 100 will be $1+$ some fraction; that of a number between 100 and 1000 will be $2+$some fraction, and so on.

It is therefore manifest that in this system the logarithm of any number, and that of another 10, 100, 1000, &c. times greater or less, consist of the same decimal fraction, and differ only in the integral part; so that all numbers, whether they are integers, decimals, or partly integral and partly decimal, have the same positive quantity for the decimal part of their logarithm: Thus,

The logarithm of	2746	is 3·438701
. . . .	274·6	is 2·438701
. . . .	27·46	is 1·438701
. . . .	2·746	is 0·438701
. . . .	·2746	is $\bar{1}$·438701
. . . .	·02746	is $\bar{2}$·438701
. . . .	·002746	is $\bar{3}$·438701.*

PROPERTIES OF LOGARITHMS.

1. Let a and b be any two numbers, and let $r^x = a$, and $r^{x'} = b$; then x is the log. of a, and x' that of b. Now $a \times b = r^x \times r^{x'} = r^{x+x'}$, but the log. of $r^{x+x'}$ is $x+x'$ $\therefore$ the log. of $ab = x+x' = \log. a + \log. b$. In like manner it may be shown that $\log. abc = \log. a + \log. b + \log. c$. Hence the logarithm of the product of any number of quantities is equal to the sum of their logarithms.

2. Again, $\frac{a}{b} = \frac{r^x}{r^{x'}} = r^{x-x'}$; but the log. of $r^{x-x'} = x - x'$ $\therefore$ the log. of $\frac{a}{b} = x - x' = \log. a - \log. b$; hence the logarithm of the quotient of any two numbers is equal to the difference of the logarithms of these numbers, or the log. of a fraction $\frac{a}{b}$ is equal to the log. of the numerator minus that of its denominator. If a is less than b, then $\log. a - \log. b$ is negative; consequently the logarithms of all proper fractions are negative.

* When the index of the logarithm is negative, the sign — is generally put above it in order to distinguish it from the decimal part, which must always be considered as + or affirmative.

3. Let $a = r^x$ be raised to the nth power, then $a^n = r^{xn}$; but the log. of r^{xn} is xn ∴ the log. of $a^n = xn = n$ times the log. of a. In like manner, taking the nth root of $a = r^x$, we have $a^{\frac{1}{n}} = r^{\frac{x}{n}}$; but the log. of $r^{\frac{x}{n}}$ is $\frac{x}{n}$ ∴ the log. of $a^{\frac{1}{n}} = \frac{x}{n}$ $= \frac{\log. a}{n}$; hence the logarithm of the nth power of any number is equal to its logarithm multiplied by n, and the logarithm of the nth root of any number is equal to its logarithm divided by n.

4. Let a, na, n^2a, n^3a, &c. be a series of numbers in geometrical progression, such that x is the log. of a, and y that of n; then $r^x = a$, and $r^y = n$; and the logarithms of the numbers in the geometrical progression will be r^x, r^{x+y}, r^{x+2y}, r^{x+3y}, &c., which evidently form an arithmetical progression. Hence, if a series of quantities are in Geometrical Progression, their logarithms are in Arithmetical Progression.

These principles being of the most extensive use in algebraical calculations, the following examples are given as exercises to the student:—

1. Log. $(a.b.c.d. \ldots) = \log. a + \log. b + \log. c + \log. d. \ldots$

2. Log. $\left(\frac{abc}{de}\right) = \log. a + \log. b + \log. c - \log. d - \log. e.$

3. Log. $(a^m.b^n.c^v) = m. \log. a + n \log. b + v \log. c.$

4. Log. $\left(\frac{a^m b^n}{c^v d^s}\right) = m \log. a + n \log. b - v \log. c - s \log. d.$

5. Log. $(a^2 - b^2) = \log. (a+b)(a-b) = \log. (a+b) + \log. (a-b).$

6. Log. $(a^2 - b^2)^{\frac{1}{2}} = \frac{1}{2} \log. (a+b) + \frac{1}{2} \log. (a-b).$

7. Log. $(a^3.a^{\frac{3}{4}}) = \log. a^3 + \frac{3}{4} \log. a = 3 \log. a + \frac{3}{4} \log. a = \frac{15}{4}$ log. a.

8. Log. $(a^3 - b^3)^{\frac{m}{n}} = \frac{m}{n} \log. (a-b) + \frac{m}{n} \log. (a^2 + ab + b^2)$, or making $(z^2 = ab) = \frac{m}{n}\{\log. (a-b) + \log. (a+b+z) + \log. (a+b-z)\}.$

9. Log. $(a^2 + b^2)^{\frac{1}{2}}$, making $2ab = z^2$, it becomes $\log. \{(a+b)^2 - z^2\}^{\frac{1}{2}} = \frac{1}{2}\{\log. (a+b+z) + \log. (a+b-z)\}.$

10. Log. $\frac{(a^2-b^2)^{\frac{1}{2}}}{(a+b)^2} = \frac{1}{2}\{\log. (a-b) - 3 \log. (a+b)\}.$

11. To insert m geometric means between a and y. In the equation $y = ar^{n-1}$, page 75, Prop. I., let $n = m+2$; then the ratio $r = \left(\frac{y}{a}\right)^{\frac{1}{m+1}}$, and $\log. r = \frac{\log. y - \log. a}{m+1}$; hence the several means are ar, ar^2, ar^3, &c. . . ar^m, and their logs. are $\log. a + \log. r$, $\log. a + 2 \log. r$, $\log. a + 3 \log. r$, &c. $\log. a + m \log. r$.

Let it be required to insert 10 means between 1 and 2; here $\log. a = 0$, and $\log. r = \frac{1}{11} \log. 2 = 0{\cdot}02736636$; hence $r = 1{\cdot}065041$, and the logarithms of the consecutive terms are $2 \log. r$, $3 \log. r$, $4 \log. r$, &c. The progression is therefore 1, 1·065041, 1·134312, 1·208089, 1·286665, 1·370351, 1·459480, 1·554406, 1·655506, 1·763182, 1·877862, 2.

PROB. I. To find the logarithm of any given number.

Let $r^x = N$, then if x be found in terms of r and N, it will be the logarithm of N to the *base* r.

Put $N = 1 + n$, and $r = 1 + a$; then $(1+a)^x = 1 + n$; and, raising both to the mth power, we have $(1+a)^{xm} = (1+n)^m$, whatever be the value of m. Expand both sides of this equation and it becomes

$$1 + xma + \frac{xm(xm-1)}{2}a^2 + \frac{xm(xm-1)(xm-2)}{2.3}a^3 + \text{\&c.} =$$
$$1 + mn + \frac{m(m-1)}{2}n^2 + \frac{m(m-1)(m-2)}{2.3}n^3 + \text{\&c.}$$

Now, expunging 1 from both sides of the equation, and dividing by m, we obtain

$$x(a + \frac{xm-1}{2}a^2 + \frac{(xm-1)(xm-2)}{2.3}a^3 + \text{\&c.}) = n + \frac{m-1}{2}n^2$$
$$+ \frac{(m-1)(m-2)}{2.3}n^3 + \text{\&c.};$$ and since m may be of any value, let us suppose $m = 0$, and the equation becomes

$$x(a - \tfrac{1}{2}a^2 + \tfrac{1}{3}a^3 - \tfrac{1}{4}a^4 + \text{\&c.}) = n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \text{\&c.}$$

$$\therefore \text{ the log. of } (1+n) = x = \frac{n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \text{\&c.}}{a - \frac{1}{2}a^2 + \frac{1}{3}a^3 - \frac{1}{4}a^4 + \text{\&c.}}$$

Substituting in this equation for n and a their values $N-1$ and $r-1$, we obtain

$$\text{Log. } N = \frac{(N-1) - \frac{1}{2}(N-1)^2 + \frac{1}{3}(N-1)^3 - \frac{1}{4}(N-1)^4 + \text{\&c.}}{(r-1) - \frac{1}{2}(r-1)^2 + \frac{1}{3}(r-1)^3 - \frac{1}{4}(r-1)^4 + \text{\&c.}}$$

$$= M\{(N-1) - \tfrac{1}{2}(N-1)^2 + \tfrac{1}{3}(N-1)^3 - \tfrac{1}{4}(N-1)^4 + \text{\&c.}\}$$

where M is the *modulus* of the system

$$= \frac{1}{(r-1) - \frac{1}{2}(r-1)^2 + \frac{1}{3}(r-1)^3 - \frac{1}{4}(r-1)^4 + \text{\&c.}}$$

This series, therefore, gives us the value of x in terms of r and N; but if N be any number greater than unity, it is evidently a diverging series, and of little use in the construction of logarithms.

In order to obtain a converging series, let us suppose $N = n - 1$; and, resuming the equation,

$\text{Log.}\ (1 + n = \frac{n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \&c.}{a - \frac{1}{2}a^2 + \frac{1}{3}a^3 - \frac{1}{4}a^4 + \&c.} = M(n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \&c.)$, proceeding as before, we get $\log.\ (1 - n) = M(-n - \frac{1}{2}n^2 - \frac{1}{3}n^3 - \frac{1}{4}n^4 - \&c.)$; and, subtracting this from the former, we obtain $\log.\ (1 + n) - \log.\ (1 - n) = \log.\ \frac{1+n}{1-n} = 2M(n + \frac{1}{3}n^3 + \frac{1}{5}n^5 + \frac{1}{7}n^7 + \&c.)$; and as this equation is true for every value of n,

Let $n = \frac{1}{N-1}$, then $\frac{1+n}{1-n} = \frac{N}{N-2}$, and consequently $\log.\ \frac{N}{N-2} = 2M\left\{\frac{1}{N-1} + \frac{1}{3(N-1)^3} + \frac{1}{5(N-1)^5} + \frac{1}{7(N-1)^7} + \&c.\right\}$; hence $\log.\ N = 2M\left\{\frac{1}{N-1} + \frac{1}{3(N-1)^3} + \frac{1}{5(N-1)^5} + \frac{1}{7(N-1)^7} + \&c.\right\} + \log.\ (N-2)$, which is a series rapidly convergent, and therefore very convenient for the construction of logarithms.

Before proceeding farther, however, it is necessary to assign some particular value to M; and since its value is arbitrary, let it be $= 1$, or the value first assumed by Lord Napier. Then $\log.\ N = 2\left\{\frac{1}{N-1} + \frac{1}{3(N-1)^3} + \frac{1}{5(N-1)^5} + \frac{1}{7(N-1)^7} + \&c.\right\} + \log.\ (N-2)$; but it is obvious that N must be some number greater than 2, and we must therefore first find the log. of 2, which may be done, by supposing $N = 4$. Hence $\log.\ 4 = \log.\ 2^2 = 2\ \log.\ 2 = 2\left(\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \&c.\right) + \log.\ 2$; and, expunging log. 2 from each side of the equation, it becomes $\log.\ 2 = 2\left(\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \frac{1}{9.3^9} + \frac{1}{11.3^{11}} + \frac{1}{13.3^{13}}\right) = 0{\cdot}6931472$. Having thus found log. 2 and, availing ourselves of the properties of logarithms, we readily obtain the logarithms of all numbers, by substituting in the formula 1, 2, 3, &c. for N: Thus,

$$\begin{aligned}
\text{Log. } 1 &= \quad \ldots\ldots\ldots &&= 0{\cdot}0000000\\
2 &= \quad \ldots\ldots\ldots &&= 0{\cdot}6931472\\
3 &= 2\left(\frac{1}{2}+\frac{1}{3.2^3}+\frac{1}{5.2^5}+\frac{1}{7.2^7}+\frac{1}{9.2^9}+\text{\&c.}\right) &&= 1{\cdot}0986123\\
4 &= \log. 2^2 = 2\log. 2 \quad \ldots &&= 1{\cdot}3862944\\
5 &= 2\left(\frac{1}{4}+\frac{1}{3.4^3}+\frac{1}{5.4^5}+\frac{1}{7.4^7}+\frac{1}{9.4^9}+\text{\&c.}\right)\\
&\quad + \log. 3 \quad \ldots &&= 1{\cdot}6094379\\
6 &= \log. 3 + \log. 2 \quad \ldots &&= 1{\cdot}7917595\\
7 &= 2\left(\frac{1}{6}+\frac{1}{3.6^3}+\frac{1}{5.6^5}+\frac{1}{7.6^7}\right)+\log. 5 &&= 1{\cdot}9459101\\
8 &= \log. 2^3 = 3\log. 2 \quad \ldots &&= 2{\cdot}0794415\\
9 &= \log. 3^2 = 2\log. 3 \quad \ldots &&= 2{\cdot}1972246\\
10 &= \log. 2 + \log. 5 \quad \ldots &&= 2{\cdot}3025851\\
\text{\&c.} &= \quad \text{\&c.} \quad \ldots &&= \quad \text{\&c.}
\end{aligned}$$

These are called Napierean Logarithms, from the name of their ingenious inventor; but they are likewise commonly known by the name of Hyperbolic Logarithms, from their connexion with the quadrature of the hyperbola.

Prob. II. To find the value of the *base* r in this system, whose *modulus* is 1.

Let log. N or log. $(n+1) = l$, and since $M = 1$, $l = n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \text{\&c.}$; reverting this series, we have $1+n$ or $N = 1 + l + \frac{l^2}{2} + \frac{l^3}{2.3} + \frac{l^4}{2.3.4} + \text{\&c.}$; and since the base of any system of logarithms is that number whose log. is 1, let $l = 1$, and we obtain the base $r = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \text{\&c.} = 2{\cdot}7182818$.

Since log. $2{\cdot}7182818 = 1$

$\therefore (2{\cdot}7182818)^{{\cdot}6931472}$ or log. $(2{\cdot}7182818)^2 = 2$

$(2{\cdot}7182818)^{1{\cdot}0986123}$ or log. $(2{\cdot}7182818)^3 = 3$

&c. &c.

Hence the numbers whose hyperbolic logarithms are 1, 2, 3, 4, &c. are *decimal* numbers, and therefore inconvenient for ordinary arithmetical computations.

Prob. III. Having given the logarithm of any number to the *base* r, to find its logarithm to any other base s.

Let $s^x = a$, or $x = \log. a$ to the base s; then, using the given logs. to the base r, we have $x = \frac{\log. a}{\log. s} = \log. a \times \frac{1}{\log. s}$; hence the log. of a to the base s is found by multiplying the log. of a to the base r by the reciprocal of the log. of s to the base r. But this reciprocal is the *modulus* of the system to which it belongs, and is therefore constant in that system; consequently if we suppose $s = 10$, the base of our common logarithms, we will obtain the modulus of these logarithms, or $M = \frac{1}{\text{hyp. log. } 10} = \frac{1}{2{\cdot}3025851} = {\cdot}4342944819$, and $2M = {\cdot}8685889638$. To construct a table of common logarithms, therefore, we have only to multiply the hyperbolic logarithms already found by this value of M, or, which is the same thing, substitute the value of 2M in the formula whence we derived the hyperbolic logarithms. Thus the common log. of N is

$${\cdot}86858896 \left\{ \frac{1}{N-1} + \frac{1}{3(N-1)^3} + \frac{1}{5(N-1)^5} + \&c. \right\} + \log. (N-2).$$

Now, making N successively 1, 2, 3, 4, &c. and availing ourselves of the properties of logarithms as before, we have

Log. $1 = \ldots \ldots = 0{\cdot}0000000$

$2 = {\cdot}86858896\left(\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \&c. \text{ to 7 terms}\right) \ldots = 0{\cdot}3010300$

$3 = {\cdot}86858896\left(\frac{1}{2} + \frac{1}{3.2^3} + \frac{1}{5.2^5} + \&c. \text{ to 10 terms}\right) \ldots = 0{\cdot}4771213$

$4 = \log. 2^2 = 2 \log. 2 \ldots = 0{\cdot}6020600$

$5 = \log. \frac{10}{2} = \log. 10 - \log. 2 \ldots = 0{\cdot}6989700$

$6 = \log. 3 + \log. 2 \ldots = 0{\cdot}7781513$

$7 = {\cdot}86858896\left(\frac{1}{6} + \frac{1}{3.6^3} + \frac{1}{5.6^5} + \frac{1}{7.6^7}\right) + \log. 5 \ldots = 0{\cdot}8450980$

$8 = \log. 2^3 = 3 \log. 2 \ldots = 0{\cdot}9030900$

$9 = \log. 3^2 = 2 \log. 3 \ldots = 0{\cdot}9542425$

$10 = \ldots \ldots = 1{\cdot}0000000$

&c. = &c. . . . . . = &c.

As exercises, the student may calculate the hyperbolic and also the common logarithms of the numbers from 10 to 20.

NOTE. To obtain the logarithms true to 7 places of decimals, *three* terms of the series will be sufficient for numbers between 10 and 29 inclusive; *two* terms for numbers between 29 and 400; and *one* term for all numbers above 400.

APPLICATION OF LOGARITHMS.

THE index or integral part of the logarithm of any whole or mixed number, as has already been shown, is always *one less than the number of integral figures* of which that number consists; and, in decimal fractions, the index, which is negative, is that number which points out *the distance of the first significant figure from the place of units*. Instead of negative indices, their *arithmetical complements to* 10 are often used. Thus if there is no cipher between the decimal point and the first significant figure of the decimal, the index is $\bar{1}$ or 9; if there is one cipher between them, the index is $\bar{2}$ or 8; if two ciphers are between them, it is $\bar{3}$ or 7, and so on.

The indices being thus readily found are omitted in the common logarithmic tables, and the decimal part only of the logarithms inserted.

TO FIND THE LOGARITHM OF A NUMBER FROM THE TABLES.

Look for the three highest figures in the margin on the left hand, and running along that line to the column which has the fourth figure at the top, you will find the logarithm for these four figures. If the number consists of more than four figures, take the difference between the logarithm thus found and the next greater, and multiply it by the remaining figures, and from the product cut off as many figures as are in the multiplier; the rest added to the logarithm for the first four figures gives the logarithm required.

NOTE. The mean differences given under D in the right-hand column may be used, except in the first three pages of the table, where they vary rapidly.

1. Required the logarithm of 73284.

Look in the margin for 732, and on that line in the column which has 8 at the top you will find ·864985, the logarithm of 7328, and the difference between it and the next logarithm is 60, which, multiplied by 4, gives 240: therefore, adding 24 to ·864985, we have 4·865009 for the logarithm of 73284, with 4 for an index, because the number has five places. If the number had been 732·84, the logarithm would have been the same, but the index would have been 2.

2. Required the log. of 6·1953. . . Ans. 0·792062.
3. . . . . of 47·5384. . . . 1·677044.
4. . . . . of ·003825. . . $\bar{3}$·582631.

TO FIND THE NUMBER CORRESPONDING TO A GIVEN LOGARITHM.

If the given logarithm be found in the table, the first three figures of the number will be found on the same line in the margin, and the fourth at the top of the column. But if the logarithm be not found exactly in the table, take the number answering the next less, and subtract this logarithm from the given one, and also from the next greater in the table; and, annexing ciphers to the first remainder, divide it by the other, to get the fifth, sixth, &c. figures. The integer places must be one more than the units in the index, and the rest are decimals.

5. Required the number corresponding to the logarithm 4·597179.

The next less logarithm is ·597146, and the number answering to it is 3955; the difference between it and the given logarithm is 33, and between it and the next greater in the table is 110. Divide 330 by 110, and the quotient 3, annexed to 3955, gives 39553 for the number sought.

6. Required the number of log.	3·774240.	Ans. 5946·2.
7.	2·147522.	140·45.
8.	$\bar{2}$·862489.	0·07286.

TO FIND THE ARITHMETICAL COMPLEMENT.

Subtract the logarithm from 10, an integer, or subtract the right-hand figure from 10, and all the rest from 9.

9. Thus the arithmetical complement of 3·642754 is 6·357246.

10. Required the ar. co. of 2·749367.	Ans. 7·250633.
11. . . . . of 1·360797.	8·639203.

TO PERFORM MULTIPLICATION BY LOGARITHMS.

Add the logarithms of the factors; the sum is the logarithm of the product.

NOTE. A negative index must be subtracted when the logarithm is added, and added when the logarithm is subtracted.

12. Multiply	37·68	log. 1·576111
by	9·25	log. 0·966142
Product	348·54	log. 2·542253

13. Multiply 5·735, 0·023, and 56·25 together.

	5·735	log. 0·758533
	0·023	log. $\bar{2}$·361728
	56·25	log. 1·750123
Product	7·41966	log. 0·870384

14. Required the product of 7·542 by ·963.	Ans. 7·26295.
15. . . . . ·00352 by ·864.	0·0030413.
16. . . . . ·0925 by 73·5.	6·79875.

TO PERFORM DIVISION BY LOGARITHMS.

Subtract the logarithm of the divisor from that of the dividend: the remainder is the logarithm of the quotient.

Or add the arithmetical complement of the divisor to the logarithm of the dividend: the sum, with its index lessened by 10, is the logarithm of the quotient.

17. Divide	9·7128	log. 0·987344	log. 0·987344
by	0·456	log. 9·658965	ar. co. 0·341035
Quotient	21·3	log. 1·328379	log. 1·328379

18. Required the quotient of 9 by 75. . Ans. 0·12.
19. . . . . 8964 by 3·84. . 2334·376.
20. . . . . 62·78 by 71·6. . ·876814.

TO WORK PROPORTION BY LOGARITHMS.

Add the logarithms of the second and third terms together, and from their sum subtract the logarithm of the first: the remainder is the logarithm of the fourth term, or answer.

Or add together the arithmetical complement of the first term, and the logarithms of the other two: the sum, with its index lessened by 10, is the logarithm of the answer.

21. First	36	log. 1·556303	ar. co. 8·443697
Second	144	log. 2·158362	log. 2·158362
Third	28	log. 1·447158	log. 1·447158
		3·605520	
Fourth	112	log. 2·049217	log. 2·049217

22. If 17 men do a piece of work in 28 days, in what time will 12 do it? Ans. 39·66667 or $39\frac{2}{3}$ days.

23. If $13\frac{1}{4}$ cwt. be carried 57 miles for £2·568, how far should $34\frac{1}{2}$ cwt. be carried for £8·56? Ans. 72·97116 miles.

TO INVOLVE A NUMBER BY LOGARITHMS.

Multiply the logarithm by the name of the power: the product is the logarithm of the power.

24. Numb. 32	log. 1·505150	Numb. ·009	log. $\bar{3}$·954243
	3		3
3d power 32768	log. 4·515450	·000000729	log. $\bar{7}$·862729

NOTE. After multiplying the negative index, the carriage to it from the logarithm must be subtracted from the product. If the positive index be used, 10 times the name of the power lessened by 1 must be taken from the index of the power.

25. Number ·0437	log. $\bar{2}$·640481, or	8·640481
	4	4
4th power ·000003649	log. $\bar{6}$·561924	4·561924

TO EXTRACT THE ROOT OF A NUMBER BY LOGARITHMS.

Divide its logarithm by the name of the root: the quotient is the logarithm of the root.

NOTE. If the given number be a decimal, and its index positive, prefix the name of the root lessened by 1 to the index, before dividing. If the index be negative, add to it the least number that will make the sum divisible by the name of the root: the quotient is the index of the root; but in dividing the logarithm, the number added only is to be considered as the index.

26. Number ·00130321 log. 4)$\bar{3}$·115014 log. 37·115014
$\bar{1}$

Fourth root ·19 log. $\bar{1}$·278754 log. 9·278754

27. Number 9261 log. 3)3·966658
Cube root 21 log. 1·322219

28. Required the square root of ·5329. . Ans. ·73.
29. . . . . cube root of ·041063625. . ·345.
30. . . . . fourth root of 7. . . 1·626567.

EXERCISES.

1. Reqd. the seventh power of 7·142. Ans. 947850.
2. . . . sixth root of 2. 1·1224625.
3. . . . ninth power of ·0375. ·0000000000001466.
4. . . . eighth root of ·02405. ·627536.
5. . . . compound interest of £67·495 for $5\frac{1}{3}$ years, at 4 per cent. Ans. £15·7033 = £15, 14s. $0\frac{3}{4}$d.
6. . . . rate of comp. int. at which £136·782 will, in $5\frac{1}{4}$ years, amount to £173·564. Ans. 4·64.
7. . . . time in which £53·5 will amount to £76·36, at $3\frac{1}{2}$ per cent. comp. int. Ans. 10·342 years.

OF CUBIC AND HIGHER EQUATIONS.

A CUBIC EQUATION, or one of the third degree, is an equation which contains the third power of the unknown quantity, as $x^3 - ax^2 + bx = c$.

A Biquadratic Equation, or one of the fourth degree, contains the fourth power of the unknown quantity, as $x^4 - ax^3 + bx^2 - cx = d$.

And, in general, an Equation of the nth degree is one which contains the nth power of the unknown quantity, as $x^n - ax^{n-1} + bx^{n-2} - cx^{n-3} + \&c. = \lambda$.

NOTE 1. Every equation has as many roots as there are units in the exponent of its highest power; that is, a simple equation has only one value of the root, a quadratic equation has two values or roots, a cubic equation has three roots, and so on.

NOTE 2. The methods usually given for the solution of equations of the third and fourth degree are too complicated to be of much practical use, and no general method has yet been discovered for resolving equations of higher degrees; but the roots of equations of all dimensions may be readily obtained, sufficiently near the truth, by either of the two following methods of approximation:

I. Find, by trials, the nearest integral value r of the root x, and substitute for x its equal $r \pm y$ in the given equation, and a new equation will arise involving only y and known quantities; then since y is a fraction, its square and higher powers are small when compared with itself, and may therefore be expunged from the equation, which will leave a simple equation, whence an approximate value of y may be easily obtained, and consequently a nearer value of the root. By substituting this value of r in the simple equation another value of y will be found, which will give a still nearer value of the root, and so on, to any degree of accuracy that may be required.

Required the value of x in the equation $x^3 - 15x^2 + 63x - 50 = 0$.

By a few trials we find that x lies between 1 and 2, but nearer to 1. Let therefore $1 = r$, and $x = r + y$.

$$\text{Then} \left\{ \begin{aligned} x^3 &= r^3 + 3r^2y + 3ry^2 + y^3 \\ -15x^2 &= -15r^2 - 30ry - 15y^2 \\ 63x &= 63r + 63y \\ -50 &= -50 \end{aligned} \right\} = 0.$$

And by expunging the terms y^3, $3ry^2$, $15y^2$, we have

$$r^3 - 15r^2 + 63r + 3r^2y - 30ry + 63y - 50 = 0$$

$$\therefore y = \frac{50 - r^3 + 15r^2 - 63r}{3r^2 - 30r + 63} = \frac{50 - 1 + 15 - 63}{3 - 30 + 63} = \frac{1}{36}$$

$= {\cdot}027$, and $x = 1{\cdot}027$ nearly.

Now, substituting $1{\cdot}027$ for r in the last equation, we obtain $y = \dfrac{50 - 1{\cdot}0832 + 15{\cdot}8209 - 64{\cdot}701}{3{\cdot}1642 - 30{\cdot}81 + 63} = \dfrac{{\cdot}0367}{35{\cdot}3542} = {\cdot}00103$,

and $\therefore 1{\cdot}027 + {\cdot}00103 = 1{\cdot}02803$, a still nearer value of x.

Again, substituting $1{\cdot}028$ for r, we have

$$y = \frac{50 - 1{\cdot}086373952 + 15{\cdot}85176 - 64{\cdot}764}{3{\cdot}170352 - 30{\cdot}84 + 63} = \frac{{\cdot}001386048}{35{\cdot}330352}$$

$= {\cdot}000039231$; consequently $x = 1{\cdot}028039231$, which is true to the ninth place of decimals.

II. Assume two numbers, differing only by unity in the last figure, as near the root as possible, and substitute them separately in the given equation instead of the unknown quantity; then collect the terms according to their signs, and mark the errors when in excess +, and when in defect —. Multiply the less error by the difference between the assumed numbers, and divide the product by the sum of the errors when they are unlike, but by their difference when they are alike. Add the quotient to the assumed number, whose error was multiplied when the assumed number is too small, otherwise subtract it, and the result will give the true root nearly.

To obtain the root still nearer, assume that last found, and another number differing from it only by unity in the last figure, and proceed with them in the same manner as before to get another correction, and so on, as far as is necessary.

Required the value of x in the equation $x^3 - 15x^2 + 63x = 50$.

Assume 1 and 1·1 as the trial numbers; then

1st Sup.		2d Sup.
1	x	1·1
63	$63x$	69·3
—15	$-15x$	—18·15
1	x^3	1·331
49	sums	52·481
50		50·
—1	errors	+ 2·481
	·1	
2·481	1	
3·481)·10000(	·03 cor.	
	1·00	

Hence x nearly $=$ 1·03

Again, assume 1·03 and 1·02 as the trial numbers; then

1st Sup.		2d Sup.
1·03	x	1·02
64·89	$63x$	64·26
—15·9135	$-15x^2$	—15·6060
1·092727	x^3	1·061208
50·069227	sums	49·715208
50		50
+ ·069227	errors	— ·284792
	·069227	
·284792	·01	
·354019)·00069227(·00196 correct.		

Hence x = 1·03 — ·00196 = 1·02804, still more nearly.

Lastly, assume 1·02804 and 1·02803 as the trial numbers; then

1st Sup.		2d Sup.
1·02804	x	1·02803
64·76652	$63x$	64·76589
—15·8529936240	$-15x^2$	—15·8526852135
1·0865007710	x^3	1·0864690653
50·0000271470	sums	49·9996738518
50·		50·
+ ·0000271470	errors	— ·0003261482
	·0000271470	
·0003261482	·00001	1·02804
·000353295)	·000000000271470(	·0000007684796
Hence x very nearly	=	1·0280392315204

When one of the roots of an equation has been found by either of these two methods, the rest may be found as follows:—

Divide the given equation by x *minus* the root found, and the quotient will be an equation depressed a degree lower; then find a root of this new equation by approximation, and the number thus obtained will be a second root of the given equation.

Depress the second equation a degree lower by dividing it by x *minus* the root last found, and then find a third root, and so on, till the equation is reduced to a quadratic, the two roots of which, with those before found, will be all the roots of the original equation. Thus, in the equation $x^3 - 15x^2 + 63x - 50 = 0$, we found, by the second operation, one of the roots $= 1\cdot02804$.

Hence $x - 1\cdot02804)x^3 - 15x^2 + 63x - 50(x^2 - 13\cdot97169x + 48\cdot63627 = 0$, and the two roots of this quadratic, when resolved in the usual way, are found to be $6\cdot57653$ and $7\cdot39543$, which are also roots of the given equation.

When the coefficient of the highest term is 1, the sum of all the roots of an equation is equal to the coefficient of the second term, and we therefore find that $1\cdot02804 + 6\cdot57653 + 7\cdot39543 = 15$, the coefficient of the second term.*

1. Given $x^3 - 2x - 5 = 0$ to find an approximate value of x. Ans. $x = 2\cdot09455148$.

2. Given $x^3 - 7x + 7 = 0$ to find an approximate value of x. Ans. $x = 1\cdot35689655$.

3. Given $x^4 - 16x^3 + 40x^2 - 30x + 1 = 0$ to find an approximate value of x. Ans. $x = 13\cdot12488$.

4. Given $x^3 - 17x^2 + 54x - 350 = 0$ to find an approximate value of x. Ans. $x = 14\cdot954067$.

5. Given $x^4 - 3x^2 + 75x - 10000 = 0$ to find an approximate value of x. Ans. $x = 9\cdot8860027$.

6. Given $\sqrt{144x^2 - (x^2 + 20)^2} + \sqrt{196x^2 - (x^2 + 24)^2} - 114 = 0$ to find an approximate value of x.

Ans. $x = 7\cdot123883$.

OF EXPONENTIAL EQUATIONS.

QUATIONS which contain quantities with unknown indices r exponents, as $a^x = b$, $a^{b^x} = c$, $x^x = a$, &c., are called Exonential or Transcendental Equations.

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* We may farther remark, that the coefficient of the third term is the sum all the products of the roots taken two by two, the coefficient of the fourth rm is the sum of the products taken three by three, &c., and the absolute term the continued product of all the roots: Thus $7\cdot39543 \times 6\cdot57653 + 7\cdot39543 \times 1\cdot02804 + 6\cdot57653 \times 1\cdot02804 = 63$, and $7\cdot39543 \times 6\cdot57653 \times 1\cdot02804 = 50$.

1. Given $a^x = b$ to find the value of x.

Since $a^x = b$, we have log. $(a^x) =$ log. b ∴ x log. $a =$ log. b, or $x = \frac{\text{log. } b}{\text{log. } a}$: Thus, let $a = 8$, and $b = 100$; then $8^x = 100$, and $x = \frac{\text{log. } 100}{\text{log. } 8} = \frac{2}{\cdot 90309} = 2{\cdot}2146187$.

2. Given $a^{b^x} = c$ to find the value of x.

NOTE. An exponential of this form means a to the power of b^x, and not a^b to the power x, which would then be expressed by a^{bx}.

Assume $z = b^x$, then $a^z = c$, and z log. $a =$ log. c ∴ $z = \frac{\text{log. } c}{\text{log. } a} = b^x$. Again, assume $y = \frac{\text{log. } c}{\text{log. } a}$, then $b^x = y$, and x log. $b =$ log. y ∴ $x = \frac{\text{log. } y}{\text{log. } b}$.

Thus let $a = 8$, $b = 2$, and $c = 2000$, then $8^{2^x} = 2000$; $\frac{\text{log. } c}{\text{log. } a} = \frac{\text{log. } 2000}{\text{log. } 8} = 2{\cdot}548 = y$, and $x = \frac{\text{log. } y}{\text{log. } 2} = \frac{\text{log. } 2{\cdot}548}{\text{log. } 2} = \frac{\cdot 406199}{\cdot 30103} = 1{\cdot}349$.

3. Exponentials of the form $x^x = a$ may be readily solved by the method of approximation employed, page 104. The assumed numbers being substituted for x in the equation x log. $x =$ log. a, and the operation repeated a sufficient number of times, x may be had to any degree of exactness.

Thus let $x^x = 100$ to find an approximate value of x.

Here we have x log. $x =$ log. $100 = 2$, and it is obvious that x lies between 3 and 4, but nearer to 4. Assume, therefore, $x = 3{\cdot}5$ and $3{\cdot}6$;

Then $3{\cdot}5$ log. $3{\cdot}5 = 3{\cdot}5 \times {\cdot}544068 = 1{\cdot}904233 =$ 1st result,
and $3{\cdot}6$ log. $3{\cdot}6 = 3{\cdot}6 \times 556303 = 2{\cdot}002691 =$ 2d result;
whence $1{\cdot}904238 - 2 = -{\cdot}095762 =$ first error,
and $2{\cdot}002691 - 2 = +\ 002691 =$ second error.
Sum of the errors $= \ {\cdot}098453$.

∴ $\cdot 0002691 \div {\cdot}098453 = {\cdot}00273 =$ first correction, and $3{\cdot}6 - {\cdot}00273 = 3{\cdot}59727 = x$ nearly.

Again, assuming $x = 3{\cdot}59727$ and $3{\cdot}59728$, and, using a table of logarithms to seven places, we have

$3{\cdot}59727$ log. $3{\cdot}59727 = 3{\cdot}59727 \times {\cdot}5559731 = 1{\cdot}9999854$
$3{\cdot}59728$ log. $3{\cdot}59728 = 3{\cdot}59728 \times {\cdot}5559743 = 1{\cdot}9999953$
whence $1{\cdot}9999854 - 2 = -{\cdot}0000146$ first error,
and $1{\cdot}9999953 - 2 = -{\cdot}0000047$ second error.
Difference of errors $\cdot 0000099$.

$\therefore$ $\cdot 000000000047 \div \cdot 0000099 = \cdot 00000474747$ second correction, which, added to $3{\cdot}59728$, gives $x = 3{\cdot}59728474747$ very nearly.

1. Given $16^x = 200$ to find an approximate value of x.
Ans. $x = 1{\cdot}91096$.

2. Given $6^x = 1500$ to find an approximate value of x.
Ans. $x = 4{\cdot}081587$.

3. Given $6^{3^x} = 3000$ to find an approximate value of x.
Ans. $x = 1{\cdot}22707$.

4. Given $12^{4^x} = 6500$ to find an approximate value of x.
Ans. $x = {\cdot}910447$.

5. Given $x^x = 2000$ to find an approximate value of x.
Ans. $x = 4{\cdot}8278226$.

6. Given $x^x = 50$ to find an approximate value of x.
Ans. $x = 3{\cdot}28726192$.

7. Given $(5x)^x = 80$ to find an approximate value of x.
Ans. $x = 1{\cdot}9320805$.

OF INDETERMINATE PROBLEMS.

When more quantities are sought than the number of conditions given, the problem is indeterminate or unlimited; but the answers in integers are commonly limited to a certain number.

I. When the equation is simple, after it is properly reduced, it will assume one of the following forms, viz. $ax = by + c$, $bxy = c - ax$, $ax + cxy = d + by$, where x and y are unknown, and the rest known quantities.

1. Let $ax = by + c$, then $x = \frac{by + c}{a}$, an integer; therefore $by + c$ is divisible by a. Take such multiples of $by + c$, and of ay, that their difference shall be of the form $y \pm d$, where the coefficient of y is 1, then making $y \pm d = a$, the least value of y will be found, and the rest are found by adding a continually. In like manner the values of x increase continually by b.

2. Let $bxy = c - ax$, then $x = \frac{c}{a + by}$; here $\frac{c}{b}$ is divisible by $\frac{a}{b} + y$. Take therefore $y = \frac{c - a}{b}$ or $\frac{c - ar}{br}$, r being any number which will make y an integer.

3. Let $ax + cxy = d + by$, then $x = \frac{d + by}{a + cy}$; or $cx = \frac{cd + cby}{a + cy}$

$= b + \frac{cd - ab}{a + cy}$; and making $e = cd - ab$, then $cx = b + \frac{e}{a + cy}$, which agrees with the second form. Take therefore $y = \frac{e - ar}{cr}$.

1. To find a number which, being divided by 17, shall leave a remainder of 7, and, being divided by 26, shall leave 13. Take x and y, the quotients, then the number is $17x + 7 = 26y + 13$; whence $x = \frac{26y + 6}{17}$, an integer; therefore $\frac{52y + 12 - 51y}{17} = \frac{y + 12}{17} = 1$, or $y + 12 = 17$; whence $y = 5$, 22, 39, 56, &c. and $x = 8$, 34, 60, 86, &c.

2. To compound 100 gallons of spirits, worth 72d., by mixing some at 56d., some at 60d., and some at 80d.

Here $x + y + z = 100$, and $56x + 60y + 80z = 7200$; whence $6x + 5y = 200$, or $x = 33 - \frac{5y - 2}{6}$, an integer; therefore $\frac{6y + 2 - 5y}{6} = \frac{y + 2}{6} = 1$, and $y = 4$; $\therefore x = 30$ and $z = 66$.

II. If the equation is a quadratic, as $x^2 = a + by + cy^2$, then x may be taken $= a^{\frac{1}{2}} + my$, or $= m + c^{\frac{1}{2}}y$; and if either a or c is a square the irrationality will disappear, for, in the first case, $a + 2a^{\frac{1}{2}}my + m^2y^2 = a + by + cy^2$, or $2a^{\frac{1}{2}}m + m^2y = b + cy$. And in the other $m^2 + 2mc^{\frac{1}{2}}y + cy^2 = a + by + cy^2$, or $m^2 + 2mc^{\frac{1}{2}}y = a + by$.

If neither a nor c is a square, assume $a + by + cy^2 = (d + ey) \times (f + gy)$, and suppose m such, that $dm^2 + em^2y = f + gy$, then $y = \frac{dm^2 - f}{g - em^2}$, which value of y being substituted for it will make the irrationality disappear. Again, if $a + by + cy^2$ can be divided into two parts, so that one of them is a square, and the other the product of two simple factors, it will then be of this form $s^2 + nr$ and $x = \sqrt{s^2 + rn}$. Take $\sqrt{s^2 + nr} = s + pn$, and the given equation will be reduced to a simple one.

1. Let $x^2 = 4 + 5y + 28y^2$, then $y = \frac{4m - 5}{28 - m^2}$; if $m = 5$, then $y = 5$, and $x = 27$.

2. Let $x^2 = 8 + 32y + 4y^2$, then $y = \frac{m^2 - 8}{32 - 4m}$; if $m = 6$, then $y = 3\frac{1}{2}$, and $x = 13$.

3. Let $x^2 = 15 + 19y + 6y^2$, then $y = \frac{3m^2 - 5}{3 - 2m^2}$; if $m^2 = \frac{20}{13}$, then $y = 5$.

Here $15 + 19y + 6y^2$ is taken $= (3 + 2y) \times (5 + 3y)$.

4. To divide a^2 into two squares. Here $x^2 = a^2 - y^2$; therefore $x = a - my$; whence $y = \frac{2am}{m^2 + 1}$.

5. To divide a into two squares. Let $\frac{r+1}{\sqrt{2}}x$ and $\frac{r-1}{\sqrt{2}}x$ be the roots, then the numbers are $\frac{a}{2} \times \frac{(r+1)^2}{r^2+1}$, and $\frac{a}{2} \times \frac{(r-1)^2}{r^2+1}$.

6. To find a square, which, added to a given square, a^2, shall make a square.

Assume $na - y$ for the root, then $n^2a - 2nay + y^2 = a^2 + y^2$, and $y = \frac{n^2 - 1}{2n} a$.

PROBLEMS.

1. The duties on certain goods amounted to £2460, out of which a discount was allowed of $2\frac{1}{2}$ per cent. upon the sum actually paid for prompt payment. What did the discount amount to? Ans. £60.

2. A merchant discounted two bills at the bank, one of them for £550, payable in 7 months, and the other for £720, payable in 4 months; and he received for the whole £1200. At what rate per cent. per annum was the interest charged? Ans. £13·267 per cent. per annum.

3. The common difference of four numbers in arithmetical progression is 4, and their continual product is 21945. What are the numbers? Ans. 7, 11, 15, 19.

4. The sum of ten numbers in arithmetical progression is 120, and the sum of their cubes is 29160. What are the numbers? Ans. 3, 5, 7, 9, &c.

5. Given the sum of the numbers 0, 1, 2, 3, &c. = 1225; to find the sum of their squares. Ans. 40425.

6. Two persons set out at the same time from two places 462 miles distant, to meet one another. The first goes 1 mile the first day, 2 the second, and so on. The second travels each day the cubes of the number of miles that the first travelled on that day. In what time will they meet? Ans. 6 days.

7. A gentleman sold an estate for the value of the trees

upon it above 7 feet circumference, at one pound for the first, two for the second, four for the third, and so on, doubling the price of each successive tree. The value of the estate came to £65535. How many trees of the above description were upon it? Ans. 16 trees.

8. A gentleman had seven children, whose ages differed successively by one year. In giving them new clothes, he determined to bestow as many yards of lace on the trimming of the youngest as he was years old, on the second as many as the sum of the ages of the two youngest, on the third as many as the sum of the ages of the three youngest, and so on; and he agreed with the tailor to pay for making each suit the product in pence of the child's age by the number of yards of lace on his suit. The bill amounted to £7, 10s. 6d. What were the ages of the children? Ans. The youngest 5 years.

9. A merchant discounted two bills; the first had 6 months to run, and the other 8 months. The value of both came to £308, 6s. 8d., and the discount to £8, 6s. 8d. Had interest been charged upon the bills, it would have come to 4s. $8\frac{2}{3}$d. more than the discount. Required the value of the bills.

Ans. The first was £205, and the other £$103\frac{1}{3}$.

10. If £400 be the present value of an annuity to continue 23 years after the expiration of 8 years, what would be its value for 21 years after the expiration of 10 years, interest at 5 per cent.? Ans. £344·8624.

11. A gentleman had 10 different annuities of £100 each; their continuance differed by one year each, and the longest was for 60 years. He sold them all at 5 per cent. compound interest. What money did he receive for them?

Ans. £18653·2142.

12. A bookseller purchases a work for £40, and pays for printing 1000 copies of it £15, for paper £20, and for incidents £10. He sells the edition in 10 years at 3s. each copy. How much does he gain per cent. per annum?

Ans. £11, 19s. per cent. per annum.

13. A person who owes his creditor £320 just now, and £96 more at the end of five years, wishes to pay the whole in one payment. What is the proper time for doing this, according to the true principle of equation of payments, viz. that the simple interest shall be equal to the discount?

Ans. At the end of one year.

14. A usurer lent £186 for a certain time, and gained £31; and by lending £360 at the same rate for another time, he gained £90. The sum of the times they were lent amounted to 20 months. How long time was each sum lent?

Ans. The first 8 months, the other 12 months.

ELEMENTS OF GEOMETRY.

DEFINITIONS.

1. **Geometry** treats of magnitude or continued quantity, and of its relation to number.

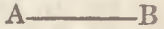
2. A **Solid** is that which has three dimensions, length, breadth, and thickness.

3. A **Surface**, or **Superficies**, is the boundary of a solid, and has only length and breadth.

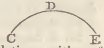
4. A **Line** is the boundary of a surface, and has only length.

5. A **Point** is the extremity of a line. It has position, but not magnitude, as A.

6. A **Straight Line** is one every part of which points in the same direction, as AB.

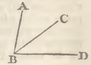


7. A **Curve** changes continually its direction, or it has unlike sides, a concave and a convex, as CDE.



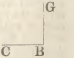
8. An **Angle** is the measure of the relative position of two straight lines which meet, or it is their inclination to one another.

Note. An angle is denoted by three letters, of which the second is at the point where the lines meet, and the other two are upon the containing lines, one on each. Thus the uppermost angle is named ABC, the other CBD, and the whole angle ABD.

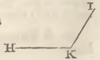


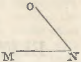
9. A straight line is said to be **Perpendicular** to another, when it does not incline towards one end more than towards the other. Thus AB is perpendicular to CD.

10. A **Right Angle** is that made by a perpendicular, as CBG.



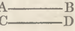
11. An **Obtuse Angle** is greater than a right angle, as HKI.





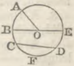
12. An ACUTE ANGLE is less than a right angle, as MNO.

13. A PLANE is a surface with which a straight line will coincide, when drawn between any two points in it.



14. PARALLELS are straight lines in a plane, which never meet, though extended ever so far both ways, as AB and CD.

15. A CIRCLE is a figure contained by a curve ABD, called the *circumference*, which is equally distant from a point O within it, called the *centre*.



16. The RADIUS AO is a straight line, drawn from the centre to the circumference.

17. The DIAMETER BE is a straight line, drawn through the centre O, and terminated both ways at the circumference.

18. A CHORD CD is a straight line joining any two points of the circumference.

19. An ARC BCD is any part of the circumference.

20. A SEMICIRCLE is a portion of the circle cut off by a diameter, as BAE.

21. A SEGMENT is a portion CFD, cut off by a chord CD.

22. A SECTOR is a part cut off by two radii, as AOB.

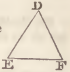
NOTE 1. If the radii contain a right angle, the sector is called a *Quadrant;* and, if half a right angle, it is called an *Octant.*

NOTE 2. The circumference of every circle is supposed to be divided into 360 equal parts, called *degrees*, and a degree into 60 equal parts, called *minutes*, and a minute into 60 *seconds*, and so on. Degrees are marked by a small circle at the top of the right-hand figure, minutes with one accent, seconds with two accents, &c.; thus 29° 12′ 45″ denote 29 degrees, 12 minutes, and 45 seconds.

NOTE 3. If two diameters AC, BE, are perpendicular to one another, they divide both the circle and the circumference into four equal parts, and form four right angles at the centre; and if the arc CB of one of these parts be divided into 90 degrees, and radii drawn to the points of division, they will divide the right angle BOC into 90 equal angles, each of which is said to be an angle of one degree, and any angle AOD at the centre is said to consist of as many degrees as the arc AD upon which it stands. The arc AD is called the *measure* of the angle AOD. Hence a right angle AOB contains 90°, an obtuse angle AOD more, and an acute angle COD less than 90 degrees.

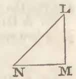
23. A TRIANGLE is a figure contained by three straight lines

24. An Equilateral Triangle has its three sides equal, as DEF.

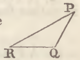


25. An Isosceles Triangle has two of its sides equal, as GHK.

26. A Right-angled Triangle has one right angle, as LMN. The side LN opposite the right angle is called the *Hypotenuse*, the sides NM and LM about the right angle are called the *Base* and *Perpendicular*, or the *Legs* of the Triangle.



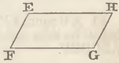
27. An Obtuse-angled Triangle has one obtuse angle, as PQR.



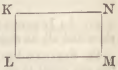
28. All others are called Acute-angled Triangles.

29. A Quadrilateral is a figure bounded by four straight lines.

30. A Parallelogram is a quadrilateral, of which the opposite sides are parallel, as EFGH.

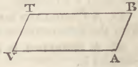


31. A Rectangle is a parallelogram which has right angles, as KLMN.

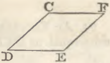


32. A Square is a rectangle which has all its sides equal, as PQRS.

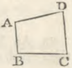
33. A Rhomboid is a parallelogram which has no right angles, as TVAB.



34. A **Rhombus** is a rhomboid which has all its sides equal, as CDEF.

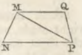


35. A **Trapeze**, or **Trapezium**, is a quadrilateral which has not its opposite sides equal, as ABCD.

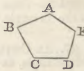


36. A **Trapezoid** has two sides parallel, but not the other two, as MNPQ.

37. A **Diagonal** is a straight line, which joins two opposite angles of a figure, as MP.

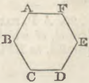


38. A **Polygon** is a figure contained by more than four straight lines, as ABCDE.



39. A **Polygon** of five sides is called a *Pentagon;* one of six sides, a *Hexagon;* of seven sides, a *Heptagon;* of eight sides, an *Octagon;* of nine sides, a *Nonagon;* of ten sides, a *Decagon,* &c.

40. A **Regular Polygon** is that which has all its sides and all its angles equal, as ABCDEF.



41. An **Irregular Polygon** is that which has not all its sides and all its angles equal.

42. **Similar Figures** are such as have all the angles of the one equal to all the angles of the other, and the corresponding sides about the angles of each proportional.

43. The **Perimeter** of a figure is the sum of all its sides.

44. A **Proposition** is a general term, either implying something to be demonstrated, or some operation to be performed. In the former case it is called a *Theorem,* and in the latter a *Problem.*

45. A **Corollary** is some property which obviously results from the demonstration of a proposition.

46. A **Scholium** is a remark or observation upon what precedes it.

47. An Axiom is a self-evident truth.

48. A Postulate is a request to admit the possibility of performing some operation.

AXIOMS.

1. Things which are equal to the same thing are equal to each other.

2. When equals are added to equals the sums are equal.

3. When equals are taken from equals the remainders are equal.

4. When equals are added to unequals the sums are unequal.

5. When equals are taken from unequals the remainders are unequal.

6. Things which are like multiples of the same or of equal things are equal to each other.

7. Things which are like submultiples of the same or of equal things are equal to each other.

8. The whole of any thing is equal to the sum of all its parts.

9. The whole of any thing is greater than a part of it.

10. Magnitudes which coincide with each other, or fill the same space, are equal to each other in every respect.

11. All right-angles are equal to each other.

12. Two straight lines which intersect each other cannot both be parallel to the same straight line.

POSTULATES.

1. Let it be granted that a circle may be described round any point, as a centre, and with any radius.

2. That a terminated straight line may be produced to any length in the same direction.

3. That a straight line may be drawn from any point to any other point.

Note. *A straight line* may be drawn between two points, by laying a *ruler* or another straight line upon these points, and tracing a line along the side of it.

But the only original method of producing a straight line is, by stretching a hair or thread through the two points; and as the thread assumes invariably the same position as often as it is stretched through the same points, and a less portion of it lies between the points when it is stretched, than when it lies loosely between them, it follows,

That a straight line between two points has only one position.

That both sides of a straight line are exactly alike.*

* If a hair stretched between the points A and B coincide with the trace AB, and if then the part of it at A be brought to

A——B

That a part of a straight line is in every respect similar to another part of it, or to another straight line of the same length.

That a straight line is the shortest distance from one point to another.

From these properties of a straight line it is inferred,

That two straight lines will coincide when they are applied to one another, in what way soever the application is made.

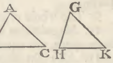
That one straight line cannot cut another in more points than one.

And consequently that two straight lines which intersect can neither have a common segment nor enclose a space.

That a straight line is less than a curve, or than the sum of any number of straight lines joined together, which terminate at the same points with it.

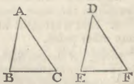
That straight lines which have the same position, in respect of the same straight line, must either coincide or be parallel to one another.*

THEOREM I. Two triangles ABC, GHK are equal in every respect, when an angle BAC and the two sides AB, AC, which contain it in one of them, are respectively equal to an angle HGK, and the sides GH, GK containing it in the other.



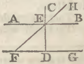
For, if the triangle ABC be laid on GHK, so that A be on G and AB on GH, then AC will lie along GK, for the angle $A = G$, and B will be on H, and C on K; therefore BC will coincide with HK, the triangle ABC with GHK, the angle B with H, and C with K. They are all therefore equal.

THEOREM II. If a side AB, and the two adjacent angles at A and B of one triangle ABC, be equal to a side DE, and the adjacent angles at D and E of another, the triangles are in all respects equal.



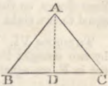
For, if the triangle ABC be laid on DEF, A on D, and AB on DE, then B will be on E, AC on DF and BC on EF, because the angles at A and B are equal to those at D and E; therefore the angle C shall be on F, and the triangle ABC will coincide altogether with DEF, and be equal to it.

B, and that at B to A, so that the upper side of it may now be the lower one, the stretched hair will again coincide with the trace AB.



* If the straight lines AB and CD intersect in E, the angle CEB shows their relative situations; and these situations would remain though they should intersect in any other point of CD, as at D; in which case AB would become FG, and EC would coincide with DE. Of course, if the angle EDG be equal to CEB, the lines AB and FG would have the same direction, and if they have the same direction,

THEOREM III. In an isosceles triangle ABC, the angles at B and C, opposite to the equal sides AC and AB, are equal to one another.



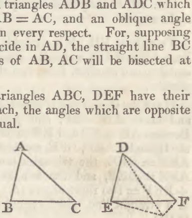
Bisect the angle BAC by AD, then the triangles ABD, ACD, have AB = AC, AD common, and the angle BAD = CAD; therefore, they are equal in every respect (1.), and have the angle ABC = ACB.

Cor. 1. An equilateral triangle is also equiangular.

Cor. 2. The straight line AD which bisects the angle BAC, bisects also BC at right angles, and conversely.

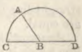
Cor. 3. Two right-angled triangles ADB and ADC which have equal hypotenuses AB = AC, and an oblique angle DAB = DAC, are equal in every respect. For, supposing their perpendiculars to coincide in AD, the straight line BC which joins the extremities of AB, AC will be bisected at right angles by AD.

THEOREM IV. If two triangles ABC, DEF have their three sides equal, each to each, the angles which are opposite to the equal sides will be equal.



Let DE be the least side, and if possible let the angle BAC be less than EDF; and if AB be laid on DE, A on D, and B on E, then AC will fall within the angle EDF as on DG, and BC on EG. Join FG, then the angle EGF = EFG (3.), because EF = EG; but DGF, a part of the first, is equal to DFG, for DG = DF, which is greater than the other.* As this cannot be, the angle BAC must be equal to EDF, ABC = DEF, and ACB = DFE.

THEOREM V. The adjacent angles ABC and ABD on the same side of the straight line CD, make together two right angles, or 180°.



For their measuring arcs AC and AD make $\frac{1}{2}$ of the circumference or 180°.

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the angle EDG would be equal to CEB; and for the same reason the angle HEB would be equal to EFD.

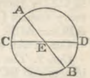
These things seem to follow immediately from the definitions of a straight line and of an angle, and, if admitted as principles, they would render several parts of geometry easy, which are at present difficult.

* The point G cannot fall within the triangle DEF, for then DFE being

Cor. 1. On the contrary, if the angles ABC, ABD make together 180°, CB and BD are in a straight line.

Cor. 2. All the angles that can be made at the point B by lines drawn on the same side of the line CD are together equal to two right angles.

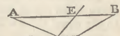
THEOREM VI. The vertical angles AEC and BED, made by two straight lines AB and CD, which cut in E, are equal to one another.



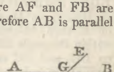
For the arcs CAD and ADB being each $\frac{1}{2}$ of the circumference are equal; therefore, AC = BD, and the angle AEC = BED.

Cor. All the angles about a point are together equal to four right angles.

THEOREM VII. If a straight line EF meet two straight lines AB and CD, and make the alternate angles AEF, EFD equal to one another, these two straight lines AB and CD are parallel.



If not, let them meet if possible in B, and make AE = BF, and join AF. Because AE = FB and EF common to the triangles AEF, BFE, and the angle AEF = BFE, the triangles are equal (1.), and the angle AFE = BEF, and the two angles AFE + EFB = AEF + BEF = two right angles (5.), therefore AF and FB are in a straight line, which cannot be; therefore AB is parallel to CD.



Cor. 1. If the *exterior* angle EGB be = the *interior* and opposite angle EHD, or the two *interior* angles BGF, EHD equal together to two right angles, the lines AB and CD are parallel, for in each of these cases the angle AGF = EHD.

Cor. 2. Straight lines AB, CD perpendicular to the same straight line EF are parallel, for the right angles AGF, EHD are equal.

Assumption. If two straight lines be parallel, a straight line which is perpendicular to one of them is also perpendicular to the other.

THEOREM VIII. If a straight line EF cut two parallels AB and CD, it will make the alternate angles AGH and GHD

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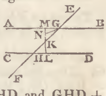
equal to DFG + EFG, would be equal to DGF + EGF, which is greater than two right angles.

equal to one another, the exterior angle EGB = the interior and opposite GHD, and the two interior angles BGH, GHD, on the same side of it, equal to two right angles.



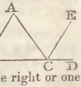
Bisect GH in K, and draw KL perpendicular to CD, it is also perpendicular to AB. And because the angle HKL = GKM (6.) HLK, KMG right angles, and HK = KG; therefore the angle AGH = GHD (3. Cor. 3.) Also AGH = EGB (6.); therefore EGB = GHD and GHD + BGH = EGB + BGH = two right angles.

Cor. 1. If the two interior angles be less than two right angles, the straight lines will meet if produced far enough.

Cor. 2. A straight line which meets one of two parallels will, if produced, meet the other also.

Scholium. When a straight line meets two parallels, the angles are equal, which are either on the same side of it, and also of the parallels, or on different sides both of it and of the parallels. And the two angles are together equal to two right angles, which are either on the same side of the cutting line, and on different sides of the parallels, or on different sides of it, and on the same side of the parallels.

Theorem IX. The exterior angle ACD of a triangle is equal to both the interior and opposite angles ABC + BAC, and the three angles ABC + BAC + ACB, are together equal to two right angles.



Draw CE parallel to AB; it will make (8.) the angle ACE = BAC, and ECD = ABC; therefore ACD = ABC + BAC, and ABC + BAC + ACB = ACD + ACB = (5.) to two right angles.

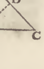
Cor. 1. In any triangle, there can be only one right or one obtuse angle.

Cor. 2. In a right-angled triangle, the two acute angles are together equal to a right angle.

Cor. 3. An angle of an equilateral triangle is two-thirds of a right angle, or it is 60°.

Cor. 4. When two angles of a triangle are known, the third angle is got by subtracting their sum from 180°.

Theorem X. The greater side AC of a triangle ABC has the greater angle opposite to it.

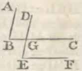


In AC take CD = BC and join BD. The angle DBC = BDC (3.), but BDC is greater than BAD or BAC (9.); much more then is CBA greater than BAC.

Cor. 1. If the angle ABC be greater than ACB, the side AC will be greater than the side AB.

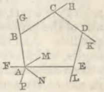
Cor. 2. An equiangular triangle is also equilateral.

THEOREM XI. If two angles ABC, DEF have their sides parallel, and in the same direction, they are equal.



Let DE, produced if necessary, meet BC in G. Then the angle ABC = DGC, and DGC = DEF (8.); therefore the angle ABC = DEF.

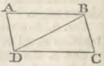
THEOREM XII. All the exterior angles FAB, GBC, &c. of any rectilineal figure, are together equal to four right angles, or 360°.



Draw AM parallel to BC, AN parallel to CD, AP to DE. Then the angle GAM = GBC, MAN = HCD, &c. (8.); therefore all the exterior angles are equal to the angles about the point A, that is, to four right angles (6. Cor.)

Cor. Since each interior angle, with its adjacent exterior, makes two right angles (5.), all the interior angles, together with four right angles, make twice as many right angles as the figure has sides. Thus the interior angles of a quadrilateral make four right angles, of a pentagon six right angles, of a hexagon eight, of a heptagon ten, &c.

THEOREM XIII. The opposite sides and the opposite angles of a parallelogram ABCD are equal to one another, and the diagonal BD bisects it.

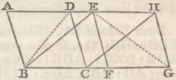


Since BD meets the parallels, it makes (8.) the angle BDC = ABD and DBC = ADB, and the side DB is common to the triangles ADB and DBC; they are therefore in all respects equal (2.)

THEOREM XIV. The lines joining the corresponding extremities of equal and parallel lines are themselves equal and parallel (fig. to Theorem XIII.)

Draw the diagonal BD, because AB, DC are parallel, the angle ABD = BDC (8.); and since AB = DC, and DB common to both triangles, they are therefore equal in every respect (1.); wherefore the angle ADB = CBD, the side AD = BC, and parallel to it (7.)

THEOREM XV. Parallelograms ABCD, EBCH, EFGH upon the same or on equal bases BC = FG, and between the same parallels AH and BG, are equal to one another.

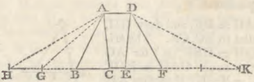


Draw BE, CH. Since AD=BC=FG=EH (13.) AE = DH, and AB = DC, and the angle HDC = EAB (8.); therefore the triangle EAB = HDC (1.); take these equals from ABCH, and the remainder ABCD = EBCH. For the same reason EFGH = EBCH; therefore ABCD = EFGH.

Cor. Triangles upon the same or on equal bases, and between the same parallels, are equal; for (13.) they are the halves of the parallelograms.

Scholium. If ABCD be a rectangle, it is $= BC \times AB$; therefore if EFGH be any parallelogram, it will be $= FG \times$ perpendicular between EH and FG. And the triangle $EFG = \frac{1}{2}FG \times$ perpendicular on it.

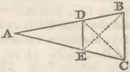
THEOREM XVI. Triangles ABC, DEF between the same parallels AD and BF, are to one another as their bases. BC : EF : : ABC : DEF.



Let CB, BG, GH be all equal, and n their number, so that $CH = n \times CB$, and draw AG, AH, the triangles ABC, AGB, AHG are equal (15. Cor.), and therefore $AHC = n \times ABC$. Take EK the least number of times EF, which is greater than CH, and let $FK = m \times EF$, and draw DK, then the triangle $DFK = m \times DEF$. And because CH or $n \times BC$ is not less than FK or $m \times EF$, but less than EK or $(m+1) \times EF$, m is the quotient by which $n \times BC$ contains EF. And the triangle AHC or $n \times ABC$ is not less than DFK, or $m \times DEF$, but less than DEK, or $(m \times 1) \times DEF$, therefore m is also the quotient by which $n \times ABC$ contains DEF, so that $n \times BC$ divided by EF, and $n \times ABC$ divided by DEF, give the same quotient. Wherefore BC : EF : : ABC : DEF.

Cor. Triangles and parallelograms of equal altitudes are to one another as their bases; and, *conversely*, triangles and parallelograms upon the same or on equal bases are to one another as their altitudes.

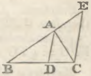
THEOREM XVII. Parallels BC, DE divide diverging straight lines proportionally. $AD : DB :: AE : EC$.



Draw BE and DC. The triangle $BED = DEC$ (15. Cor.); therefore $ADE : DEB :: ADE : DEC$. But (16.) $AD : DB :: ADE : DEB$ and $AE : EC :: ADE : DEC$; therefore $AD : DB :: AE : EC$.

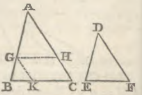
Cor. Straight lines which meet three parallels are cut proportionally by them.

THEOREM XVIII. If the angle BAC of a triangle ABC be bisected by AD, the segments of BC have the same ratio with the sides. $BD : DC :: BA : AC$.



Draw CE parallel to AD. The angle $BEC = BAD$ or $= DAC$; that is, $= ACE$ (8.); therefore $AE = AC$, and $BD : DC :: BA : AE$ or AC (17.)

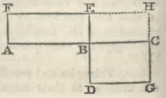
THEOREM XIX. Triangles ABC, DEF which have two angles equal, each to each, $A = D$, and $B = E$, have their sides proportional.



Make $AG = DE$, and draw GH, GK parallel to BC, CA. The triangle $AGH = DEF$ (2.) for $AG = DE$, the angle $GAH = EDF$ and $AGH = B = E$; therefore $AH = DF$ and $GH = EF$. But $AB : AC :: AG : AH$ (17.) $:: DE : DF$. And $AB : BC :: AG : KC = GH :: DE : EF$.

Cor. If the sides be proportional, or if the sides about two equal angles be proportional, the triangles are equiangular.

THEOREM XX. If four straight lines be proportional $AB : BC :: DB : BE$, the rectangle contained by AB and BE, the extremes, is equal to that contained by DB and BC, the means.



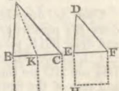
Let DE be perpendicular to AC, and complete the rectangles AE, EC, and CD. Then $AE : EC :: AB : BC$ (16. Cor.) $:: DB : BE :: DC : CE$; therefore $AE = CD$.

Cor. 1. If three straight lines be proportional, the rectangle contained by the extremes is equal to the square of the mean.

Cor. 2. If the rectangle AE, contained by AB and BE, the extremes, be equal to the rectangle DC, contained by DB and BC, the means, then AB : BC : : DB : BE.

Cor. 3. Any parallelogram or triangle contained by the extremes is equal to a parallelogram, or a triangle which has an equal angle contained by the means.

THEOREM XXI. Similar triangles, viz. such as have equal angles, are to one another as the squares of their like sides. ABC : DEF : : CG : FH.



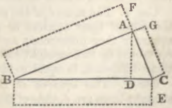
Find BK a third proportional to BC and EF the like sides, so that BC : EF : : EF : BK, then join AK, and draw KL parallel to BG. Because AB : DE : : BC : EF (19.); that is, : : EF : BK, the triangle ABK = DEF, and GK = FH (20. Cor. 3.) But GC : GK or HF : : BC : BK : : ABC : ABK = DEF.

Cor. 1. Any similar figures, viz. those composed of the same number of similar triangles similarly placed, are to one another as the squares of their like sides.

Cor. 2. If three straight lines BC, EF, BK be proportional, the first BC is to the third BK, as any figure upon the first BC to a similar figure upon the second EF.

Cor. 3. If the area of any polygon, of which the side is 1, be multiplied by the square of any straight line, it will give the area of a similar polygon described on that line.

THEOREM XXII. The figure BE described upon the hypotenuse BC of a right-angled triangle ABC, is equal to the figures BF and CG, similarly described upon the other two sides BA and AC.



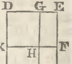
Draw AD perpendicular to BC. Because the angle B is common to the triangles BAC, BDA, and BAC, BDA are right angles, BD : BA : : BA : BC (19.); therefore BD : BC : : BF : BE (21. Cor. 3.) For the same reason, DC : CB : : CG : BE. Wherefore BD + DC : BC : : BF + CG : BE. Consequently, since BD + DC = BC, BF + CG = BE.

Cor. 1. If the greatest of three similar figures be equal to the sum of the other two, a right-angled triangle can be made of their like sides.

Cor. 2. The square of BC is equal to the squares of BA and AC; and, therefore, if any two of the sides be given, the third side may be found from them.

Cor. 3. If a, b, c be three straight lines, and $a^2 = b^2 + c^2$, or $b^2 = a^2 - c^2$, these lines will form a right-angled triangle, of which a will be the hypotenuse.

THEOREM XXIII. The squares of two straight lines AB, BC, together with twice the rectangle $AB \times BC$ contained by them, is equal to the square of their sum AC.



Upon AC describe the square ADEC, and draw BG parallel to CE. Make $CF = CB$, and draw FHK parallel to AC. Because $CF = CB$, FE or $DK = AB$ or DG; therefore DH and HC are the squares of AB and BC, and each of the figures AH and HE is the rectangle contained by AB and BC. But these four make the whole figure CD, which is the square of AC; therefore $AC^2 = AB^2 + BC^2 + 2\ (AB \times BC.)$

Cor. If $AB = BC$, the four figures CH, HD, AH, HE will be squares, and equal to one another; therefore 4 times the square of AB is equal to the square of 2 AB.

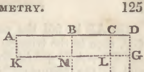
THEOREM XXIV. The squares of two straight lines, AC and CB, lessened by twice the rectangle $AC \times CB$, contained by them, are equal to the square of AB, their difference (fig. to Theorem 23.)

For CD and CH are the squares of AC and CB, and each of the figures AF and CG is the rectangle contained by AC and CB, and these two make AF, FG, and CH, which taken from CD + CH, leave DH the square of AB; therefore $AB^2 = AC^2 + CB^2 - 2\ (AC \times CB.)$

Cor. 1. The square of the sum of two straight lines exceeds the sum of their squares as much as this sum exceeds the square of their difference; and therefore 4 times the rectangle contained by two straight lines, together with the square of their difference, is equal to the square of their sum.

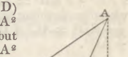
Cor. 2. The squares of the sum and difference of two straight lines are double of the squares of the lines.

THEOREM XXV. The rectangle contained by the sum AC, and difference DC of two straight lines, AB and BC is equal to the difference of their squares.



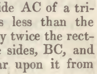
Make BD = BA, and upon DB make the square DBEF, draw CH parallel to DF, and make DG = DC, and complete the rectangle AKGD. AC is the sum of AB and BC, and DC or DG their difference, and because DC = DG or BM and DF = AB, the figure ABMK = CDFH, and CAKL = BL + CF; that is, to the difference of DE and EL, which are the squares of AB and BC. Therefore $AB^2 - BC^2 = (AB + BC) \times (AB - BC.)$

Theorem XXVI. The square of the side AB of a triangle opposite to an obtuse angle ACB, is greater than the squares of AC and CB, the other two sides, by twice the rectangle $BC \times CD$, contained by either side BC, and the part of it intercepted between the perpendicular AD, from the opposite angle and the obtuse angle.



For $BD^2 = BC^2 + CD^2 + 2\,(BC \times CD)$ (23.); add AD^2 to each, and $BD^2 + DA^2 = BC^2 + CD^2 + DA^2 + 2\,(BC \times CD)$, but $BD^2 + DA^2 = BA^2$, and $CD^2 + DA^2 = CA^2$ (22. Cor. 2.); therefore $BA^2 = BC^2 + CA^2 + 2\,(BC \times CD.)$

Theorem XXVII. The square of the side AC of a triangle opposite to an acute angle ABC, is less than the squares of the other two sides AB and BC, by twice the rectangle $CB \times BD$ contained by either of these sides, BC, and the part of it BD, between the perpendicular upon it from the opposite angle and the acute angle.

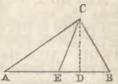


For $BC^2 + BD^2 = 2\,(BC \times BD) + DC^2$ (24.); add AD^2 to each, and $CB^2 + BD^2 + DA^2 = 2\,(BC \times BD) + DC^2 + DA^2$, but $BD^2 + DA^2 = BA^2$ and $CD^2 + DA^2 = CA^2$ (22. Cor. 2.); therefore $CB^2 + BA^2 = 2\,(CB \times BD) + CA^2$.

Cor. Hence the angle ABC is obtuse or acute, according as the square of AC is greater or less than the sum of the squares of AB and BC, and the difference in each case is $2\,(CB \times BD.)$

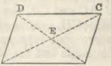
Theorem XXVIII. The sum of the squares of the sides AC, CB of any triangle ABC is equal to twice the square of the line CE drawn from the vertex to the middle point of the base, together with twice the square of AE, the half of the base.

On AB let fall the perpendicular CD, then in the triangle CEA, $AC^2 = CE^2 + AE^2 + 2\,(AE \times ED)$ (26.), and in the triangle CEB, $CB^2 = CE^2 + EB^2 - 2\,(EB \times ED)$ (27.); therefore, since $AE = EB$, by adding the corresponding sides together, we have $AC^2 + CB^2 = 2\,CE^2 + 2\,AE^2$.



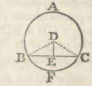
Theorem XXIX. The two diagonals AEC, BED of a parallelogram ABCD bisect each other, and the sum of their squares is equal to the sum of the squares of its four sides.

For since the triangles AEB, DEC are equiangular (6. and 8.), and the side $AB = CD$, they are equal in every respect; $AE = EC$, and $DE = EB$. Now, in the triangle ADC, $2\,AE^2 + 2\,ED^2 = AD^2 + DC^2$ (28.), wherefore, doubling that, we have $4\,AE^2 + 4\,ED^2 = AC^2 + DB^2$ (23. Cor.) $= AD^2 + DC^2 + CB^2 + AB^2$.



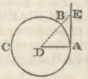
Theorem XXX. A straight line, DE, drawn from the centre D, of a circle ABC, perpendicular to a chord BC, bisects the chord and the arc BFC subtended by it.

Draw DB, DC, they are equal, the angle $DBE = DCE$ (3.) and BED, DEC are right angles; therefore $BE = EC$ (2.) and the angle $BDE = CDE$; consequently, if they be laid on one another, DB will coincide with DC, and the arc BF with FC.



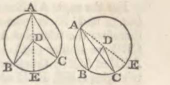
Theorem XXXI. A perpendicular AE, to the diameter of a circle AC at its extremity A touches the circle.

From any point E in AE, draw ED to the centre, then DE $>$ DA or DB (10. Cor. 1.), for the angle DAE $>$ DEA (9.); therefore E, that is, every point of AE, except A, is without the circle, and consequently AE touches it.



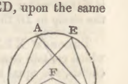
Theorem XXXII. An angle BDC, at the centre D of a circle, is double of the angle BAC at the circumference, when they stand upon the same arc BC.

Draw ADE, then the angle BDE = DAB + DBA (9.), it is therefore = 2 BAD (3.); and for the same reason, EDC = 2 DAC; therefore, by adding or subtracting, BDC = 2 BAC.



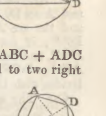
Cor. If BDE + EDC be greater than two right angles, still the two, BDE, EDC together, are double of BAC.

THEOREM XXXIII. Angles BAD, BED, upon the same arc BCD, or in the same segment of a circle BAED, are equal.



Join B and D with F, the centre of the circle. Then (32.) the angles BAD and BED are each of them = half the angle BFD, and consequently equal to one another.

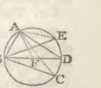
THEOREM XXXIV. The opposite angles ABC + ADC of a quadrilateral ABCD in a circle, are equal to two right angles.



Join AC, BD. The angle ADC = ADB + BDC = ACB + BAC (33.); therefore ADC + ABC = ACB + BAC + ABC = two right angles (9.)

Cor. The exterior angle EBC is = the interior, and opposite angle ADC (9.)

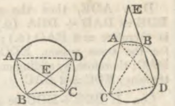
THEOREM XXXV. The angle BAD in a semicircle BAED is a right angle, the angle BAC in a greater segment is acute, and the angle BAE in a segment less than a semicircle is obtuse.



Let F be the centre, join AF. The angle FBA = FAB, and FDA = FAD (3.); therefore BAD = ABD + ADB, and is therefore a right angle (9.) But BAC is $<$ BAD, and BAE $>$ BAD.

THEOREM XXXVI. If through any point E, two straight lines AC, BD be drawn, to cut the circle ABCD, and the points of their intersection with the circle be joined, the triangles thus formed are similar.

For the angle ADB=ACB (33.), and E is common; therefore the triangles ADE, CEB are similar (19.) Also ABE=ACD; therefore the triangles ABE, ECD are similar.

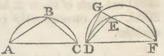


Cor. 1. The rectangle $CE \times EA = BE \times ED$ (20.)

Cor. 2. If BE be equal to, or the same with, ED, that is, if BD be either perpendicular to the diameter AC, or touch the circle in D, then $AE \times EC = ED^2$ (20. Cor. 1.)

Theorem XXXVII. Segments of circles ABC, DEF, which contain equal angles ABC, DEF, and stand upon equal chords, are equal to one another.

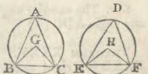
If AC be applied to DF, and A to D, C will be on F, and the arc ABC will be on DEF; if not, let it fall on DGF, and meet FE in G, join DG; and the angle DGF = ABC = DEF, which is impossible (9.); therefore ABC coincides with DEF, and is equal to it.



Cor. The arc ABC is equal to the arc DEF.

Theorem XXXVIII. If two equal angles, BGC, EHF, be at the centres of equal circles, ABC, DEF, the arcs BKC, ELF upon which they stand, are equal to one another.

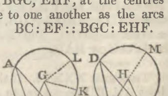
Draw BC, EF. Because BG, GC are = EH, HF, and the angle BGC = EHF, the base BC = EF (1.); and because the angle BAC = EDF, the segment BAC = EDF (37.); therefore the remaining segment BKC = ELF, and the arc BKC = the arc ELF.



Cor. The greater angle stands upon the greater arc.

Theorem XXXIX. Angles BGC, EHF, at the centres of equal circles ABC, DEF, are to one another as the arcs BC, EF upon which they stand. BC : EF :: BGC : EHF.

Take any number n, of arcs CK, KL, each equal to BC, so that $BL = n \times BC$ be greater than EF, and draw GK, GL, the angles BGC, CGK, KGL (38.) are equal, and the angle



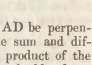
BGL $= n \times$ BGC. Take m such a number, that when FM $= m \times$ EF, then EM is the least multiple of EF, which is greater than BL; therefore FHM $= m \times$ EHF.

And since $n \times$ BC or BL is not less than FM or $m \times$ EF, but less than EM or $(m+1) \times$ EF; therefore $n \times$ BGC or BGL is not less than FHM or $m \times$ EHF, but less than EHM or $(m+1) \times$ EHF. Wherefore m is the quotient by which $n \times$ BC contains EF, and also the quotient by which $n \times$ BGC contains EHF. Therefore BC : EF : : BGC : EHF.

Cor. 1. The sector BGC = sector CGK = sector KGL; therefore BC : EF : : sector BGC : sector EHF.

Cor. 2. An angle BGC at the centre, is to four right angles as the arc BC to the whole circumference.

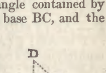
Theorem XL. In any triangle ABC, if AD be perpendicular to BC, the rectangle or product of the sum and difference of the sides AC, AB is equal to the product of the base BC, by the difference between it and the double of one of its segments.



From A, with the greater side AC for a radius, describe a circle meeting AB produced in E and F, and CB in G; then BE = CA + AB, and BF = CA — AB, and because CG = 2 CD (30.), GB = 2 CD — CB, but (36. Cor. 1.) CB $\times$ BG = EB $\times$ BF.

Cor. If EB $\times$ BF $\div$ CB = radius, then CD $= \frac{1}{2}$ (BC + radius), and BD $= \frac{1}{2}$ (BC — radius).

Theorem XLI. Any triangle ABC, is a mean proportional between the rectangle contained by half the perimeter and its excess above the base, and the rectangle contained by half the sum and half the difference of the base BC, and the difference of the sides AC and AB.

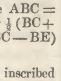


Make AD = AB, and AE = AC, join DB and CE, meeting one another in F, and parallel to them draw AG and AH. The triangles ADB, ACE being isosceles, the angle ACE = AEC = BEF, and CDB = ABD = EBF (3.); therefore DFC = BFE, or each is a right angle (5.) And since AH is parallel to FC, and GA parallel to FH, the angles at H and G are also right angles (8.) And DH = HB; CG = GE; HF $= \frac{1}{2}$ (DF + FB), and FG $= \frac{1}{2}$ (CF — FE.) The rectangle $\frac{1}{2}$ (DC + CB) $\times \frac{1}{2}$ (DC — CB) $= \frac{1}{2}$ (DF + FB) $\times \frac{1}{2}$ (DF — FB) (40.) = FH $\times$ HD = AG $\times$ HD. And for the same reason,

$\frac{1}{2}(CB+BE)\times\frac{1}{2}(CB-BE)=CG\times GF$. And because the triangle $ACE=CG\times GA$, or $CG\times FH$, and the triangle $CBE=CG\times BF$; therefore the triangle $ABC=CG\times BH$, or $CG\times DH$. And the triangles AGC, DHA are similar; therefore $AG:GC::DH:HA=FG$, and multiplying the two first by DH, and the two last by GC, the rectangle $AG\times DH$, or $DH\times HF:GC\times DH::DH\times GC:FG\times GC$; that is, $\frac{1}{2}(DC+CB)\times\frac{1}{2}(DC-CB)$: the triangle ABC : : as the triangle $ABC:\frac{1}{2}(CB+BE)\times\frac{1}{2}(CB-BE)$.

Cor. If P be $\frac{1}{2}$ the perimeter, then the triangle $ABC=\sqrt{\{P\times(P-BC)\times(P-AC)\times(P-AB)\}}$, for $\frac{1}{2}(BC+BE)=\frac{1}{2}(BC+CA-AB)=P-AB$, and $\frac{1}{2}(BC-BE)=\frac{1}{2}(BC+AB-AC)=P-AC$.

Theorem XLII. If a quadrilateral ABCD be inscribed in a circle, the area of the figure is a mean proportional between the excess of the square of half the sum of two adjacent sides AD, DC above the square of half the difference of the other two, AB, BC, and the excess of the square of half the sum of the latter AB, BC above the square of half the difference of the former AD, DC.



Let $AF=\frac{1}{2}(AD+DC)$, and $AG=\frac{1}{2}(AB+BC)$, then $DF=\frac{1}{2}(AD-DC)$, and $GB=\frac{1}{2}(AB-BC)$; and AF^2-BG^2: area : : area : AG^2-DF^2. Produce AD, BC to E. Because the triangles ABE, DCE, are similar, $AB:DC::AE:EC::BE:ED$; therefore (putting $P=\frac{1}{2}$ sum of AB, BE, and AE), $AB:DC::\frac{1}{2}(AB+AE+EB)=P:\frac{1}{2}(DC+DE+EC)$, and (*by conv.*) $AB:BA-DC::P:\frac{1}{2}(AB+AD+BC-CD)=AG+DF$. Again, $AB:CD::\frac{1}{2}(AB+BE-AE)=P-AE:\frac{1}{2}(CD+DE-EC)$, and (*comp.*) $AB:AB+CD::P-AE:\frac{1}{2}(AB+BC+CD-AD)=AG-DF$, and multiplying the corresponding terms of these proportions $AB^2:AB^2-CD^2::P\times(P-AE):AG^2-DF^2$. In like manner it may be proved that $AB^2:AB^2-CD^2::(P-BE)\times(P-AB):AF^2-BG^2$. But because the triangles ABE, DCE are similar $AB^2:DC^2::ABE:DCE$ and $AB^2:AB^2-DC^2::ABE:ABCD$. Therefore $P\times(P-AE):AG^2-DF^2::ABE:ABCD$, and (*altern.*) $AG^2-BF^2:ABCD::P\times(P-AE):ABE$; that is, (41.) $::ABE:(P-BE)\times(P-AB)$, or $::ABCD:AF^2-BG^2$. Therefore ABCD is a mean proportional between AG^2-BF^2, and AF^2-BG^2.

Cor. Hence the quadrilateral $ABCD=\sqrt{\{(AF^2-BG^2)\times(AG^2-BF^2)\}}$.

THEOREM XLIII. A quadrilateral ABCD, inscribed in a circle, is a mean proportional between the rectangle under the excesses of half the perimeter above two of its sides, and the rectangle under its excesses above the other two sides. Fig. to Theorem XLII.

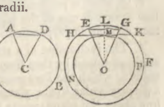
Let P be half the perimeter, then $AF^2 - BG^2 = (AF + BG) \times (AF - BG)$ (42.) $= \frac{1}{2}(AD + DC + AB - BC) \times \frac{1}{2}(AD + DC - AB + BC) = (P - BC) \times (P - AB)$, and $AG^2 - DF^2 = \frac{1}{2}(AB + BC + AD - DC) \times \frac{1}{2}(AB + BC - AD + DC) = (P - DC) \times (P - AD)$; therefore ABCD is a mean proportional between $(P - BC) \times (P - AB)$, and $(P - DC) \times (P - AD)$.

THEOREM XLIV. The area of any circle ABD is equal to the rectangle contained by the radius AC, and a straight line equal to half the circumference ABD.

If not, let the rectangle be less than the circle ABD, or equal to the circle EGM. Draw FD, touching this circle in E, and meeting the circumference ABD in F and D, and join CD, meeting the arc EG in H. Let EG be a fourth part of the circumference EGM. From EG take away its half, and from the remainder its half, and so on, till the arc EK is found less than EH. Draw CKL, and make $EN = EL$. Then LN is the side of a regular polygon, described about the circle EGML; and it is plain that this polygon is less than the circle ABD. Because the triangle $NLC = \frac{1}{2} NL \times CE$, the polygon is $= \frac{1}{2}$ the perimeter $\times$ CE. But the perimeter is less than the circumference ABD, and CE is less than CA; therefore the polygon is less than $\frac{1}{2}$ the circumference ABD $\times$ CA; that is, less than the circle EGM, which it contains; therefore the rectangle is not less than the circle ABD. And it may be shown, by a similar construction about ABD, that it is not greater. Therefore the circle is equal to the rectangle contained by the radius and the half of the circumference.

Cor. Any sector of a circle is equal to the rectangle or product of the radius, and half the arc of the sector.

THEOREM XLV. The circumferences of the circles ABD, EFG, are to one another as their radii.



If possible, let the radius AC, be to the radius EO, as the circumference ABD to a circumference MNP, less than EFG. Draw the radius OML, and HMK touching the circle MNP in M; and let LF be a fourth part of the circumference EFG. Take away its half, and the half of the remainder, and so on, till an arc LG is found less than LK, and draw GE parallel to HK, it will be the side of a regular polygon in the circle EFG; and this polygon is greater than MNP. Let AD be the side of a similar polygon inscribed in the circle ADB, and join EO, OG, AC, CD. The triangles ACD, EOG being similar AC : EO : : AD : EG; that is, as the perimeter of the polygon in ADB to the perimeter of the polygon in EFG; but AC : EO : : circumference ADB : circumference MNP; the perimeters, therefore, are as these circumferences; but this is impossible, for the perimeter of the polygon in ADB is less than the circumference; and, on the contrary, the perimeter of the polygon in EFG is greater than the circumference MNP. Therefore AC is not to EO as the circumference ADB to a circumference less than EFG; and in the same manner it may be shown that EO is not to AC as the circumference EFG, to a circumference less than ADB. Therefore AC : EO : : the circumference ABD : the circumference EFG.

Cor. 1. Hence circles are to one another as the squares of their radii, or of their diameters.

Cor. 2. If p be the circumference of a circle, of which the diameter is 1, or $\frac{1}{2}$ the circumference, of which the radius is 1, then $1 : p : : \mathrm{CA} : \frac{1}{2}$ the circumference $\mathrm{ADB} = p \times \mathrm{CA}$, and therefore $p \times \mathrm{CA} \times \mathrm{CA} = p \times \mathrm{CA}^2 =$ area of the circle ADB.

OF THE INTERSECTIONS OF PLANES.

DEFINITIONS.

1. A STRAIGHT line is perpendicular, or at right angles to a plane, when it makes right angles with every straight line meeting at its *foot* in that plane.

The *foot* of a perpendicular is the point at which it meets the plane.

2. A plane is perpendicular to a plane, when every straight line drawn in one of the planes, perpendicularly to their common section, is perpendicular to the other plane.

3. The inclination of a straight line to a plane is the acute angle contained by that line, and another line drawn from the point in which the first meets the plane, to the point in which a perpendicular to the plane drawn from any point of the first line above the plane, meets the plane.

4. The inclination of a plane to a plane is the angle contained by two straight lines drawn from any point of their common section at right angles to it, one upon the one plane, and the other upon the other plane.

5. Two planes, or a straight line and a plane, are parallel when they do not meet though produced indefinitely.

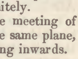
6. A solid angle is that which is made by the meeting of more than two plane angles, which are not in the same plane, in one point, the inclination of all the planes being inwards.

THEOREM XLVI. A straight line cannot be partly in a plane and partly out of it.

For when a straight line is drawn between any two points in a plane it coincides wholly with it (Def. 13. page 112.)

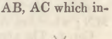
THEOREM XLVII. Two straight lines AB, AC which intersect each other lie in the same plane.

Let any plane pass through the straight line AB, and let the plane be turned about AB until it pass through the point C, then the line AC which has its two points A, C in the same plane, lies wholly in it (46.)



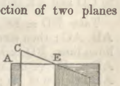
Cor. 1. Any three lines which meet one another not in the same point are in the same plane.

Cor. 2. Two parallels AB, CD are in the same plane, for, join them by the line GH, it is obvious that the plane of the two straight lines AG, GH is also the plane of the line CH or CD.

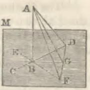


THEOREM XLVIII. The common section of two planes AB, CD is a straight line.

Let E and F, any two points in their common section, be joined by the straight line EF; then that line being wholly in the plane AB, and also wholly in the plane CD (46.), is therefore their common section.



THEOREM XLIX. If a straight line AB is at right angles to two other straight lines BF, BD, which intersect at its foot in the plane MN, it will also be at right angles to the plane MN.



Draw any straight line BG through the point B in the plane in which BD, BF are situated; and through any point G in BG draw a straight line FD, such that $FG = GD$, then join AF, AG, and AD.

Since the base of the triangle BDF is bisected in G, $BD^2 + BF^2 = 2(BG^2 + DG^2)$ (28.), and in the triangle AFD we have likewise $AF^2 + AD^2 = 2(AG^2 + DG^2)$, but the angles ABD, ABF are right angles; therefore $AD^2 - BD^2 = AB^2$, and $AF^2 - BF^2 = AB^2$ (22. Cor. 2.); whence $AB^2 + AB^2 = 2(AG^2 - DG^2)$, and by taking the halves of these, we get $AB^2 = AG^2 - DG^2$; therefore the triangle ABG is right angled at B, and in the same manner it may be shown that AB is perpendicular to every straight line drawn through the point B in the plane MN; wherefore AB is at right angles to the plane MN.

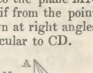
Cor. 1. The perpendicular AB is shorter than any oblique line AG, and it measures the distance of any point A from the plane.

Cor. 2. At a given point B in a plane only one perpendicular can be erected, and also from any point out of a plane only one perpendicular can be let fall upon the plane.

Cor. 3. Oblique lines equally distant from the perpendicular are equal, and of two oblique lines, that is the longer which is the more remote from the perpendicular.

Scholium. This Theorem proves the accuracy of the first definition, page 132.

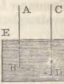
Theorem L. If AB is a perpendicular to the plane MN and CD, a line situated in the same plane; if from the point B the foot of the perpendicular BE be drawn at right angles to CD and AE joined, AE will be perpendicular to CD.



Take $ED = EC$, and join BD, BC, AD, AC; then since $ED = EC$, the oblique lines, $BD = BC$, and consequently $AD = AC$ (49. Cor. 3.), and the line AE has two of its points equally distant from the extremities D and C, wherefore AE is a perpendicular at the middle of DC.

Cor. Hence DC is perpendicular to the plane ABE, since DC is at once perpendicular to the two straight lines AE, BE.

Theorem LI. If two straight lines AB, CD are perpendicular to the same plane, they are parallel to each other.

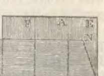


Draw the straight line BD in the plane EF; then since AB, CD are perpendicular to the plane EF, they are each perpendicular to the line BD in that plane, therefore they are parallel to each other (7. Cor. 2.)

Cor. 1. If one of two parallel straight lines is perpendicular to a plane, the other is also perpendicular to that plane.

Cor. 2. If two straight lines are each parallel to a third, though not in the same plane with it, they are parallel to each other; for, conceive a plane perpendicular to any one of them, then the other two being each parallel to it, they must also be perpendicular to the same plane, and therefore parallel to each other.

Theorem LII. If a straight line AB is at right angles to a plane MN, any plane CD passing through AB is at right angles to the plane MN.



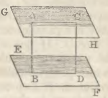
Let MD be the common section of the planes CD, MN. From any point G in MD draw GF in the plane CD at right angles to MD; then, as AB is perpendicular to the plane MN, it is perpendicular to MD (Def. 1. page 132), hence ABG is a right angle = FGB, and GF is parallel to AB (7. Cor. 2.); but since AB is at right angles to the plane MN, FG is also at right angles to that plane (51.); therefore the plane CD is at right angles to the plane MN. In like manner it may be shown that any other plane passing through AB is at right angles to the plane MN.

Cor. Hence a line standing at right angles to one of two perpendicular planes, at any point B in their common section MD, must also be in the other plane.

Theorem LIII. If a straight line AB, without a given plane EF, is parallel to a straight line CD in that plane, AB is also parallel to the plane EF.

For if the line AB which lies in the plane AD could meet the plane EF, it would meet it in some point of the line CD, the common intersection of the two planes; but AB cannot meet CD since they are parallel (Def. 14. page 112), and therefore it cannot meet the plane EF, hence it is parallel to that plane (Def. 5. page 133).

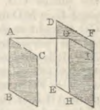
Theorem LIV. Two planes GH, EF perpendicular to the same straight line AB are parallel to each other.



From any point C, in the plane GH, draw CD parallel to AB, and it will also be perpendicular to both of the planes (51. Cor. 1.) Join AC, BD, and the angles at A, B, C, D are all right angles, hence the figure ABCD is a rectangle (Def. 31. page 113), and $CD = AB$; consequently the plane GH is parallel to the plane EF.

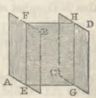
Cor. All straight lines perpendicular to one of two parallel planes are also perpendicular to the other.

Theorem LV. If two straight lines AB, AC meeting one another are respectively parallel to two other straight lines DE, DF which meet one another, but are not in the same plane with the first two, the plane which passes through AB, AC is parallel to that which passes through DE, DF.



Let AG be perpendicular to the plane BC, and let it meet the plane EF in G. In the plane EF draw GH, GI parallel to ED, DF, they will also be parallel to AB, AC (51. Cor. 2.); whence the angles GAB, GAC are both right angles, and so are also the angles AGH, AGI (Assumption, page 118); and since AG is perpendicular to both the planes BC and EF, they are therefore parallel to each other (54.)

Theorem LVI. The sections EF, GH of two parallel planes AB, CD with a third plane EFGH are parallel.

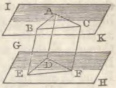


Since the planes AB, CD are parallel, the straight lines EF, GH, which are wholly in these planes, do not meet, though produced indefinitely (Def. 5. page 133); but these straight lines are in the same plane EFGH, whence EF is parallel to GH (Def. 14. page 112).

Cor. Hence parallel lines included between two parallel planes are equal, and parallel planes are every where equidistant.

Theorem LVII. If two angles BAC, EDF, not situated in the same plane, have their sides parallel, and lying in the same direction, those angles will be equal and their planes parallel.

Make BA = ED, AC = DF, and join BE, AD, CF, BC, EF; then since BA and ED, AC and DF, are respectively equal and parallel, BE and CF will be each equal and parallel to AD (14.), and therefore equal and parallel to each other, whence BC is also equal and parallel to EF;

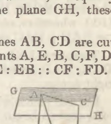


consequently the triangles BAC, EDF, having their corresponding sides equal, are equal in every respect (4.), and the angle BAC = EDF.

And since the points A, B, C, in the plane IK, are equally distant from the points D, E, F, in the plane GH, these planes are parallel (56. Cor.)

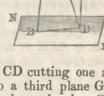
Theorem LVIII. If two straight lines AB, CD are cut by parallel planes GH, LK, NM in the points A, E, B, C, F, D, they will be cut in the same ratio, or AE : EB : : CF : FD.

Join AC, BD, AD, and let AD meet the plane LK in X, and join EX, XF; then since the intersections EX, BD of the parallel planes LK, NM are parallel (56.) AE : EB : AX : XD (17.), and, in like manner, AC and XF being parallel AX : XD : : CF : FD; now the ratio AX : XD being the same in both, therefore AE : EB : : CF : FD.



Theorem LIX. If two planes AB, CD cutting one another are each of them perpendicular to a third plane GH, their common section EF is also perpendicular to the plane GH.

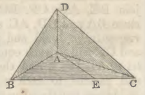
From the point F erect a perpendicular to the plane GH; and since EF is in both planes AB, CD (52. Cor.), it must therefore be their common section; consequently the common section of the two planes AB, CD is perpendicular to the plane GH.



Theorem LX. If a solid angle at A is contained by three plane angles BAC, CAD, DAB, the sum of any two of these is greater than the third.

It is only the case, in which the third angle is greater than either of the other two with which it is compared, that requires to be demonstrated.

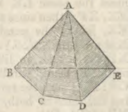
Let BAC be the greatest, and in the plane BAC draw the straight line AE, making the angle BAE = BAD. Make AE = AD, and through E draw any straight line BEC, cutting AB, AC in the points B and C, and join BD, CD.



Since AE = AD, the angle BAE = BAD, and BA common to the two triangles BAE, BAD; therefore the other sides BE, BD are equal (1.) But BD + DC $>$ BE + EC,* hence DC $>$ EC; again, since AD = AE, AC common to the two triangles EAC, DAC and the base EC $<$ DC; therefore the angle EAC $<$ DAC (10.), wherefore, adding BAD = BAE, we have BAD + DAC $>$ BAE + EAC or BAC.

Theorem LXI. The sum of the plane angles BAC, CAD, DAE, EAB, which contain a solid angle A, is less than four right angles.

Let the planes which contain the solid angle A be cut by another plane, and let their common sections with it be BC, CD, DE, EB; then the solid angle at B is contained by the three plane angles CBA, ABE, EBC, any two of which is greater than the third (60.); therefore the angles CBA + ABE $>$ EBC. For the same reason the angles BCA + ACD $>$ BCD, the angles CDA + ADE $>$ CDE, and the angles DEA + AEB $>$ DEB; whence all the angles at the bases of the plane triangles, whose common vertex is A, are together greater than the sum of all the interior angles of the rectilineal figure BCDE. But the angles of these triangles are equal to twice as many right angles as there are triangles (9.), or as there are sides in the figure BCDE; and the interior angles of the figure BCDE, together with four right angles, are also equal to twice as many right angles as the figure has sides (12. Cor.); therefore all the angles of the triangles are equal to all the interior angles of the figure with four right angles, and as all the angles at the bases of the triangles are greater than all the interior angles of the figure, the remaining angles of the triangles, or those which contain the solid angle A, are less than four right angles.



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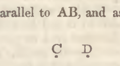
* This is evident from the properties of a straight line, for if the sum of any two sides of a triangle was not greater than the third side, a straight line would not be the shortest distance between two points.

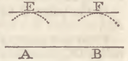
PROBLEMS.

Prob. I. To draw a straight line parallel to AB, and as far from it as the point C is from D.

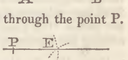


With the distance CD for a radius, describe arcs E and F from the centres A and B, and draw the straight line EF to touch these arcs without cutting them.

Prob. II. To draw a parallel to AB through the point P.

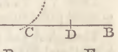


From P, with any sufficient radius, describe an arc cutting AB in C. Lay the radius on AB from C to D, and from D cut the arc again in E, and draw PE.



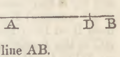
Or, with the nearest distance of P from AB for a radius, describe an arc E, from D, taken as far as possible from P, and draw a line from P to touch the arc E.

Prob. III. To bisect a given straight line AB.

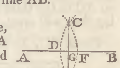


With a radius greater than half the line, describe from B the arc CDE, and from A the arc CFE, cutting the former in C and E. Draw CE cutting AB in G.

Prob. IV. To raise a perpendicular to AB at a given point in it, as C.



With any radius, from C, cut AB in D and E; and with a greater radius describe arcs from D and E, cutting one another in F, and draw CF.



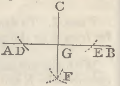
If the perpendicular is to be raised at B, the end of AB,

Place one foot at G, above AB, and extending the other to B, describe a circle cutting AB in H; then lay the radius on the circumference, from H to K, from K to I, and from I to M, and draw BM.

Or a straight line through H and G will give M.

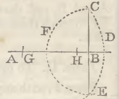
PROB. V. To let fall a perpendicular upon AB from the point C above it.

With a sufficient radius from C cut AB in D and E, and from these points describe arcs on the other side of AB, cutting one another in F, and draw CF, cutting AB in G.



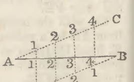
If the point C be above the end of AB,

From any point G in AB, with the radius GC, describe the arc CDE; and from any other point H, in AB, with the radius HC, describe the arc CFE, cutting the former in E, and draw CE.

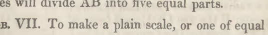


PROB. VI. To divide a straight line AB into any number of equal parts, suppose five.

Through A and B draw any parallels AC and BD, on different sides of AB. Take any convenient distance, and lay it four times (one less than the given number) from A on AC, and from B on BD; then join the first on AC to the fourth on BD, the second on AC to the third on BD, and so on in order, and the joining lines will divide AB into five equal parts.



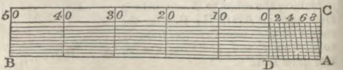
PROB. VII. To make a plain scale, or one of equal parts.



Draw any straight line AB, and take any convenient distance, and lay it eleven times from A to B, and divide the last one BD into 10 equal parts; then each of the large divisions will be 10, and each of the small divisions 1.

For a scale of feet and inches, divide BD into 12 equal parts; then each of the large divisions will be a foot, and each of the small ones an inch.

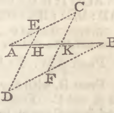
PROB. VIII. To make a diagonal scale.



Having drawn AB, and divided it as in the plane scale, draw AC perpendicular to AB, and on it lay any small distance 10 times, and through the points of division draw parallels to AB, and through the great divisions of AB draw parallels to AC; divide AD and C0 each into 10 equal parts, and draw a line from 0 to the first division of AD, and from the first division of 0C to the second of AD, and so on.

To take from this scale a number consisting of three figures, as 546, call one of the large divisions 100, or take 5 of them, call one of the divisions on 0C 10, or take 4 of them, and for the units reckon one parallel on the diagonal for each unit; or count 6 parallels on the diagonal through 4, and bring the foot on the large 5, along that division to the sixth parallel.

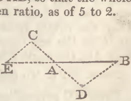
Prob. IX. To divide a straight line AB in any proportions, as of 3, 5, 7.



Draw any parallels AC and BD, through A and B on different sides of AB. From any scale of equal parts take the extent from 0 to 3, and lay it on AC, from A to E. Take 7 from the same scale, and lay it on BD, from B to F; then take 5, and lay it from E to C, and from F to D; and join ED, CF, cutting AB in H and K. AH : HK : : 3 : 5, and HK : KB : : 5 : 7.

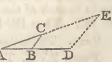
Note. In the same way, AB may be divided similarly to a given divided line.

Prob. X. To produce a straight line AB, so that the whole shall be to the produced part in a given ratio, as of 5 to 2.



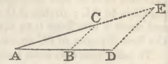
Through A draw any straight line DAC, lay 2 from A to C, and 5 from C to D towards A. Join BD, and parallel to it draw CE. Then BE : EA : : 5 : 2.

Prob. XI. To find a third line proportional to two given straight lines, as 4 and 6.



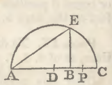
Make any angle BAC, and lay the first term 4 from A to B, and the second term 6 both from A to C and from B to D. Join BC, and draw DE parallel to it. Then CE = 9 is the third proportional.

Prob. XII. To find a fourth line proportional to three given ones, as 8, 6, and 12.



Make any angle BAC. Lay the first 8 from A to B, the second 6 from B to D, and the third 12 from A to C. Join BC, and draw DE parallel to it. Then CE is the fourth proportional.

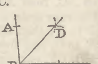
Prob. XIII. To find a mean proportional between two straight lines, as 9 and 4.



On the same straight line lay AB 9 and BC 4, and bisect AC in D; then with the radius DA describe the semicircle AEC, and draw BE perpendicular to AC. It is the mean proportional, for AB : BE : : BE : BC.

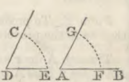
Note. Make AP = AE, then AP or AE is a mean proportional between AC and AB; therefore AC : AB : : $AC^2 : AP^2$.

Prob. XIV. To bisect a given angle ABC.



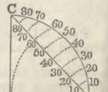
From B, with any radius, cut the sides in A and C. From A describe the arc D, and from C cut it in D, and join BD, the angle ABD = CBD.

Prob. XV. To make, at A in AB, an angle equal to the angle CDE.



From D, with any radius, cut DC, DE, in C, E; and from A, with the same radius, describe the arc FG. Take the extent from C to E, and lay it on the arc from F to G, and draw AG, the angle FAG = CDE.

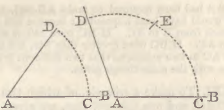
Prob. XVI. To make a scale of chords.



Draw AC perpendicular to AB. From A, with any radius, describe the arc BC, and let it be divided into 90 equal parts, (it is here divided into 9,) and draw BC; and, with one foot in B, transfer the extents to each of the divisions, from the arc to BC. Then BC is a line of chords.

Note. The radius AB is equal to the chord of 60°.

Prob. XVII. To make an angle of any number of degrees, at A in AB.

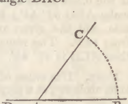


Take 60° from the line of chords, and from A describe an arc, cutting AB in C.

If the given angle do not exceed 90°, as 54°, take it from the line of chords, and lay it on the arc from C to D; draw AD, then BAD is the angle required.

If the given angle be greater than 90°, as 112°, take a less number from the chords; lay it from C to E, lay the rest from E to D, and draw AD; then BAD is the angle required.

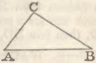
Prob. XVIII. To measure a given angle BAC.



With the chord of 60°, from A describe the arc BC. Lay BC on the line of chords, and it will show the number of degrees in the angle BAC.

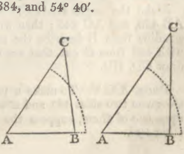
If the extent from B to C be greater than the line of chords, measure part of the arc, and then the rest, and add them. Or produce BA to D, and measure CAD, which, subtracted from 180°, leaves BAC.

Prob. XIX. To make a triangle, of which the three sides are given, viz. 186, 257, and 324 feet.



Draw a straight line AB. Take 324 from the diagonal scale, and lay that extent from A to B. Take 186 from the scale, and from A describe an arc; then with 257 for a radius, from B cut that arc in C, and join AC, CB.

Prob. XX. To make a triangle, of which two sides and an angle are given, viz. 256, 384, and 54° 40′.

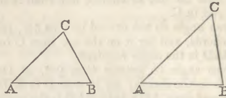


Make the angle BAC 54° 40′, and make AB 256; then, if the given angle be between the given sides, make AC 384, and join BC.

But if one of the sides be opposite to the given angle, with 384 for a radius, from B cut AC in C, and join BC.

NOTE. If it had been required to make AB 384, and BC 256, the problem would have been impossible; because 256 for a radius would not reach from B to AC. If BC were 340, it would be perpendicular to AC. If BC were greater than 340, but less than 384, it would cut AC in two points, so that two different triangles could then be made with the same things given.

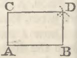
PROB. XXI. To make a triangle, of which two angles 43° 36′, and 57° 44′, and one side 297 feet, are given.



Make the angle BAC 43° 36′, and make AB 297. Then, if the other given angle is to be adjacent to the given side, make ABC 57° 44′; but if it is to be opposite to the given side, add the given angles, and subtract the sum 101° 20′ from 180°. The remainder 78° 40′ is the angle ABC, and then ACB is 57° 44′.

NOTE. If in either of these problems a right angle is given, it is to be made 90°, or a perpendicular is to be drawn.

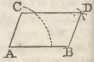
PROB. XXII. To make a rectangle, of which the sides are given; suppose 428 and 246 feet.



Draw AC perpendicular to AB; make AB 428, and AC 246 feet; then with 246 for a radius, from B describe the arc D; and with 428 for a radius, from C cut that arc in D, then join BD, CD.

NOTE. If AC be made equal to AB, the figure will be a square.

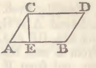
PROB. XXIII. To make a parallelogram, of which two sides, 436 and 243 feet, and an angle, 67° 30′, are given.



Make the angle BAC 67° 30′; make AB 436, and AC 243; then with 243 for a radius from B describe the arc D, and with 436 from C cut that arc in D, then draw CD, BD.

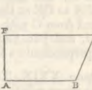
PROB. XXIV. To make a parallelogram, of which there are given two sides 421 and 234 feet, and the perpendicular upon one of them, suppose the longest, from the end of the other 196.

Draw CD parallel to AB, at the distance of 196 feet from it; and with 234 for a radius from A cut CD in C, and make AB, CD each 421; then join AC, BD, and let fall the perpendicular CE.



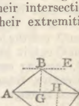
Prob. XXV. To make a quadrilateral, of which all the sides, 256, 348, 436, and 297 feet, and an angle contained by the two first, 87° 44′, are given.

Make the angle BAF 87° 44′; make AB 256, and AF 348; then from F, with 436 for a radius, describe an arc, nd with 297 from B cut that arc in C, nd draw FC, CB.



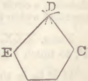
Prob. XXVI. To make a quadrilateral, of which are given two sides 268 and 394, the diagonal from their intersection 73, and the perpendiculars upon it from their extremities 88 and 234 feet.

Make AC 473, and draw parallels to it on different sides at the distances of 188 nd 234, as BE, DF. With 268 for a radius from A cut BE in B, and with 394 ut DF in D. Join AB, BC, CD, DA, nd let fall the perpendiculars BG, DH, n AC.



Prob. XXVII. To make a pentagon of which all the sides re given, 236, 194, 253, 318, and 372 feet; and two angles, uppose those at the extremities of the second side, 112° and 24°.

Make AB 194 feet; at A make the ngle BAE 112°, and at B the angle ABC 24°; make AE 236, and BC 253; then ith 318 for a radius from C describe the c D, and from E with 372 cut it in D, nd draw CD, ED.

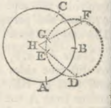


Note. In like manner may any polygon be made, of which all e sides are given, and all the angles except three.

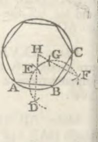
Prob. XXVIII. Given two sides of a figure 234 and 348, e diagonals 438, 385, 452, and 537, and the perpendiculars pon the diagonals from the angles 183, 248, 315, 212, and 4; to construct the figure.

First, by Prob. XXVI., make the quadrilateral ABFG, of which AB is 234, BG 438, BF 385, AH 183, and FK 248. From B with the radius 315 describe an arc, and from F draw FC to touch it; make FC 452, and join BC. From F with 212 describe an arc, draw CE to touch it, and make CE 537. Draw a parallel to CE at the distance of 274 from it, and from C with 348 cut the parallel in D, and join CD, DE, EF, and draw the perpendiculars BL, FM, DN.

Prob. XXIX. To describe a circle that shall pass through three given points, A, B, C, not in a straight line.



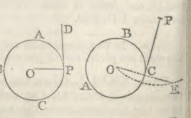
With a radius greater than half the distance of B from A or C describe a circle about B; with the same radius from A cut the circle in D and E, then from C cut it in F and G. Join DE, FG, meeting one another in H; it is the centre, from which the circle described through A shall pass through B and C.



Note 1. If ABC be a triangle, a circle may be described about it by this problem. And in the same way, by taking three points in the circumference, or in any arc of a circle, the centre of that circle may be found.

Note 2. The circumference which passes through three of the angular points of a regular polygon passes through all the rest; and therefore a circle may be described about it, or inscribed within it, by this problem.

Prob. XXX. To draw a straight line from a given point P, to touch a given circle ABC.



If P be in the circumference, draw PO to the centre, and PD perpendicular to it.

If P be without the circle; from P describe the arc OE through the centre O, and from O, with the diameter of ABC for a radius, cut the arc in E;

then draw EO, meeting the circumference in C. Join PC, and it will touch the circle.

PROB. XXXI. To make a regular polygon of a given number of sides in a given circle ABC.

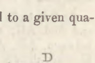
Divide 360° by the number of sides; the quotient is the angle at the centre subtended by one of them. Draw a radius AO, and make the angle AOB equal to the quotient. Join AB, and place straight lines all around the circle equal to AB, and they will form the polygon required.

PROB. XXXII. To make a regular polygon of a given number of sides, upon a given straight line, as AB 365 feet.

Divide 360° by twice the number of sides, and subtract the quotient from 90°. At A and B make the angles BAO, ABO, each equal to the remainder, and the point O in which the sides meet is the centre of the circle containing the polygon. From O describe a circle through A, and place lines equal to AB all round in it.

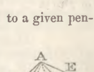
PROB. XXXIII. To make a triangle equal to a given quadrilateral ABCD.

Draw the diagonal AC, and parallel to it, through D, draw DE, meeting BC produced, if necessary, in E, and join AE; then the triangle ABE is equal to the quadrilateral ABCD. For the triangle ACE = ACD.



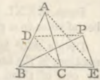
PROB. XXXIV. To make a triangle equal to a given pentagon ABCDE.

Join AC, and draw BF parallel to it, meeting CD in F, then join AF, and the triangle AFC = ABC; and thus the pentagon is reduced to the quadrilateral AFDE. Let this be reduced as before to the triangle AFG, then AFG = ABCDE.



NOTE. In the same way may any polygon be reduced to a triangle, only the number of operations will increase with the number of the sides of the figure.

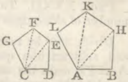
PROB. XXXV. To reduce a triangle ABC to another, which shall have its base in the same straight line with that of the given triangle, and its vertex at a given point P.



Draw PD parallel to BC, meeting AB in D. Join DC, and through A draw AE parallel to DC, and join PB, PE. If DE were joined, the triangle ADC = EDC, and ABC = DBE = PBE.

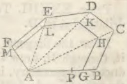
NOTE. By this and the preceding problem, any polygon may be reduced to a triangle, which shall have its vertex at a given point.

PROB. XXXVI. To construct a figure upon a given straight line AB, which shall be similar to a given figure CDEFG.



Join CE, CF, to reduce the given figure to triangles. At A make the angle BAH = DCE, HAK = ECF, and KAL = FCG. Also at B make the angle ABH = CDE; at H make AHK = CEF; and at K make AKL = CFG. Then ABHKL is similar to CDEFG.

PROB. XXXVII. To construct a figure which shall be similar to a given figure ABCDEF, and have a given ratio to it, as that of 7 to 9.

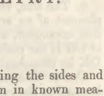


As 9 is to 7, so make AB to AP, and find AG, a mean proportional between AB, AP, by Prob. XIII. Having drawn the diagonals AC, AD, AE, draw GH parallel to BC, meeting AC in H, draw HK parallel to CD, KL parallel to DE, and LM to EF; then the figure AGHKLM is similar to ABCDEF, and has to it the ratio of 7 to 9.

PLANE TRIGONOMETRY.

TRIGONOMETRY is the method of determining the sides and angles of triangles, and of expressing them in known measures. This is done by means of the ratios which certain straight lines in and about the circle have to its radius.

DEFINITIONS.



1. The SINE BG of an arc AB, is a straight line drawn from B, one of its extremities, perpendicular to the diameter AE, which passes through the other.

2. The VERSED SINE AG of an arc AB, is that portion of the diameter AE upon which the sine is perpendicular, intercepted between the sine and the arc.

3. The TANGENT AF of an arc AB is a perpendicular to the radius CA at one extremity of the arc, and meets at F the diameter MB, which passes through the other extremity B.

4. The SECANT CF of an arc AB, is a straight line drawn from C the centre, to F the farthest extremity of the tangent.

5. The sine, versed sine, tangent, and secant of an arc AB, are called the sine, versed sine, tangent, and secant of the angle ACB measured by that arc to the radius AC.

6. The COMPLEMENT of an arc AB, or angle ACB, is what it wants of a quadrant or 90°. Thus BD or BCD is the complement of AB or ACB.

7. The SUPPLEMENT of an arc AB, or of an angle ACB, is what it wants of 180°. Thus BE or AM is the supplement of AB, and BCE or ACM the supplement of ACB.

Cor. 1. An arc or angle, and its supplement, have the same sine, tangent, and secant; for BG is the sine of BE or BCE, AF the tangent of AM or ACM, and CF the secant of AM or ACM.

Cor. 2. The versed sine EG of BCE (or the supplemental versed sine of ACB), together with AG the versed sine of ACB, is equal to the diameter AE.

8. What the arc wants of the whole circumference, or the angle of four right angles, is sometimes called the *explement:* Thus BDEMLA is the explement of AB, or of ACB.

9. The sine, versed sine, tangent, and secant of the complement of an arc or angle, are called the cosine, coversed sine, cotangent, and cosecant of the arc or angle. Thus BH or CG is the cosine of AB or ACB, DH is its coversed sine, DK its cotangent, and CK its cosecant.

Cor. 1. The cosine CG, together with the versed sine AG, is equal to the radius AC.

Cor. 2. The radius is equal to the sine or versed sine of 90°, and to the tangent or cotangent of 45°.

Note 1. We generally write sin. for sine, cos. for cosine, tan. for tangent, sec. for secant, ver. for versed sine, cov. for coversed sine, suv. for supplemental versed sine, cot. for cotangent, cosec. for cosecant, cho. for chord, R. or rad. for radius, and D. or dia. for diameter.

From these definitions the equations which express the values of the trigonometrical lines in terms of each other are easily derived.

1. Since the diameter which bisects an arc, bisects also the chord at right angles, it follows that half the chord of any arc is equal to the sine of half that arc: Thus $BG = \frac{1}{2}BL$.

2. In the right-angled triangle CGB, $CB^2 = CG^2 + GB^2$, or the square of the radius is equal to the sum of the squares of the sine and cosine of any arc; hence $\sin. = \sqrt{(R^2 - \cos.^2)}$, $\cos. = \sqrt{(R^2 - \sin.^2)}$, or if radius $= 1$, then $\sin. = \sqrt{(1 - \cos.^2)}$, and $\cos. = \sqrt{(1 - \sin.^2)}$.

The triangles CGB, CAF, CDK, being evidently similar, we have

3. CG : GB : : CA : AF, or the cosine of an arc is to its sine as the radius to the tangent; therefore $\tan. = \frac{R \times \sin.}{\cos.}$ $= \frac{\sin.}{\cos.}$, if radius $= 1$.

4. GB : CG : : CD : DK, or the sine of an arc is to its cosine as the radius is to the cotangent; hence $\cot. = \frac{R \times \cos.}{\sin.}$ $= \frac{\cos.}{\sin.}$, if radius $= 1$.

5. CG : CB or CA : : CA : CF, or the radius is a mean

proportional between the cosine of an arc and its secant; whence sec. $= \frac{R^2}{\cos.} = \frac{1}{\cos.}$, if radius $= 1$.

6. GB : CB or CD : CD : CK, or the radius is a mean proportional between the sine of an arc and its cosecant; therefore cosec. $= \frac{R^2}{\sin.} = \frac{1}{\sin.}$, if radius $= 1$.

7. AF : CA or CD : : CD : DK, or the radius is a mean proportional between the tangent of an arc and its cotangent; hence tan. $= \frac{R^2}{\cot.} = \frac{1}{\cot.}$, if radius $= 1$, and cot. $= \frac{R^2}{\tan.} = \frac{1}{\tan.}$, if radius $= 1$.

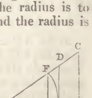
8. The triangle CAF being right angled, $CF^2 = CA^2 + AF^2$, or the square of the secant is equal to the sum of the squares of the radius and the tangent; hence sec. $= \surd(R^2 + \tan.^2) = \surd(1 + \tan.^2)$, if radius $= 1$, and tan. $= \surd(\sec.^2 - R^2) = \surd(\sec.^2 - 1)$, if radius $= 1$.

9. In the right-angled triangle CDK, $CK^2 = CD^2 + DK^2$, or the square of the cosecant is equal to the sum of the squares of the radius and the cotangent; therefore cosec. $= \surd(R^2 + \cot.^2) = \surd(1 + \cot.^2)$, if radius $= 1$, and cot. $= \surd(\text{cosec.}^2 - R^2) = \surd(\text{cosec.}^2 - 1)$, if radius $= 1$.

10. From the similar triangles EGB, BGA, EG : GB : : GB : GA, or the sine of an arc is a mean proportional between the versed sine and its supplemental versed sine; that is, between the versed sine and the sum of the radius and cosine; therefore vers. $= \frac{\sin.^2}{R + \cos.} = \frac{\sin.^2}{1 + \cos.}$, if radius $= 1$.

11. Since $CG^2 = BH^2 = DH \times HI$ or $DH \times (CD + CH)$, it is obvious that the cosine of an arc is a mean proportional between the sum and the difference of the radius and the sine, or between the coversed sine and the sum of the radius and sine; hence cov. $= \frac{\cos.^2}{R + \sin.} = \frac{\cos.^2}{1 + \sin.}$, if radius $= 1$.

THEOREM I. In any right-angled plane triangle ABC, the hypotenuse AC is to either of the sides as the radius is to the sine of the angle opposite to that side; and the radius is to the tangent of an acute angle as the adjacent side to the opposite side.



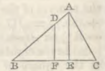
From A, as a centre with any radius CE, describe the arc EF, and draw the perpendiculars DE, FG; then FG is the sine, and DE the tangent of EF, or of the angle A.

The triangles AGF, AED, and ABC, having the angle A common, and the angles AGF, AED, ABC right angles, are therefore similar; hence AC : CB : : AF : FG, or AC : CB : : rad. : sin. A. And AE : ED : : AB : BC, or rad. : tan. A : : AB : BC.

Cor. 1. Hence the radius is to the cosine of an angle as the hypotenuse to the adjacent side. For AG is the cosine of the arc EF, or of the angle A; and AF : AG : : AC : AB, or rad. : cos. A : : AC : AB.

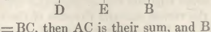
Cor. 2. Hence also the radius is to the secant of an angle as the adjacent side to the hypotenuse. For AD is the secant of the arc EF, or of the angle A; and AE : AD : : AB : AC, or rad. : sec. A : : AB : AC.

THEOREM II. In any triangle ABC, the sides are to one another as the sines of their opposite angles. AB : AC : : sin. C : sin. B.



Make BD = AC, and draw AE, DF, perpendicular to BC. Making AC or BD the radius, AE is the sine of C, and DF the sine of B, and (18. El. Geo.) AB : BD = AC : : AE : DF : : sin. C : sin. B.

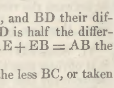
THEOREM III. Half the difference of two unequal quantities AB and BC, added to half their sum, gives the greater, and half the difference taken from half the sum, gives the less.



Make AD = BC, then AC is their sum, and BD their difference; bisect BD in E, then BE or ED is half the difference, and AE = EC half the sum, but AE + EB = AB the greater, and EC — EB = BC the less.

Cor. Half the difference BE, added to the less BC, or taken from the greater AB, gives half the sum.

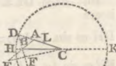
THEOREM IV. In any triangle ABC, of which the sides are unequal, the sum of the sides AC + AB is to their difference as the tangent of half the sum of the opposite angles B and C, to the tangent of half their difference. CA + AB : CA — AB : : tan. $\frac{1}{2}$ (B + C) : tan. $\frac{1}{2}$ (B — C).



Make AD = AB, and AE = AC, and join DB, CE, meeting one another in F. The angle DFC = BFE (41. El. Geo.) or each is a right angle, and the triangles CDF, EBF, are similar; therefore DC : EB : : DF : FB (18. El. Geo.); and DC = CA + AB, and BE = CA — AB; and because ABC + ACB = ACE + AEC, therefore $ACF = \frac{1}{2}(B+C)$, and $BCF = \frac{1}{2}(B-C)$; therefore $AC+AB : AC-AB :: DF = \tan. \frac{1}{2}(B+C) : BF = \tan. \frac{1}{2}(B-C)$, the radius being CF.

Cor. Hence (Theor. 2. Trig.) sin. BCE : sin. BEC : BE : BC; that is, $\sin. \frac{1}{2}(B-C) : \sin. \frac{1}{2}(B+C) :: AC-AB : BC$. Also sin. DBC or CBF : sin. BDC :: DC : CB; that is, $\cos. \frac{1}{2}(B-C) : \cos. \frac{1}{2}(B+C) :: AC+AB : BC$.

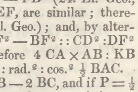
Theorem V. In any triangle ABC, four times the product of the two sides AC, AB, is to the product of the perimeter, by the excess of the sides above the base, as the square of the radius to the square of the cosine of half the angle BAC, opposite to the base.



Make AD = AB, and AE = AC, and join DB, CE, meeting in F, the angles at F are right angles. From C, with the radius CD, describe a circle meeting DF in G, and BC in H and K. Then DF = FG (30. El. Geo.), BK = BC + CD, is the perimeter, and HB the excess of DC above CB. Make AL = AB, then LC = BE. Now $4\,CA \times AB = 4\,CA \times AL = CD^2 - CL^2 = CD^2 - BE^2$ (24. El. Geo., Cor. 1.), and $HB \times BK = DB \times BG$ (36. El. Geo., Cor. 1.) $= DF^2 - FB^2$ (24. El. Geo., Cor. 1.) But the triangles CDF, BEF, are similar; therefore $CD^2 : DF^2 :: EB^2 : BF^2$ (19. El. Geo.); and, by alternation and division, $CD^2 - BE^2 : DF^2 - BF^2 :: CD^2 : DF^2$ $:$ rad.^{2} : cos.^{2} $CDF = \frac{1}{2}BAC$; therefore $4\,CA \times AB : KB \times BH = (DC+CB) \times (DC-CB) ::$ rad.^{2} : cos.^{2} $\frac{1}{2}$ BAC.

Cor. Since $HB = KC - CB = KB - 2\,BC$, and if $P = \frac{1}{2}BK$, then $CA \times AB : P \times (P - BC) ::$ rad.^{2} : cos.^{2} $\frac{1}{2}$ A.

Prob. The sines and cosines of two arcs being given, to find the sines and cosines of their sum and of their difference.



Let C be the centre of the circle, AB the greater arc $= a$, and BD or BE the less arc $= b$. Join DE, BC, then BC bisects DE at right angles in I. Draw EF, BG, IL, DH perpendicular to CA, and IK, EM perpendicular to DH; then BG = sin. a, GC = cosine a, DI or IE = sin. b, CI = cos. b, DH = sin. $(a+b)$, CH = cos. $(a+b)$, EF = sin. $(a-b)$, and CF = cos. $(a-b)$.

Because the angle C is common to the two triangles CBG, CIL, and the angles CLI, CGB are, by construction, right angles, therefore these triangles are similar; whence

CB : CI :: BG : IL, or rad. : cos. b :: sin. a : IL $= \frac{\text{sin. } a \text{ cos. } b}{\text{rad.}}$.

CB : CI :: CG : CL, or rad. : cos. b :: cos. a : CL $= \frac{\text{cos. } b \text{ cos. } a}{\text{rad.}}$.

The triangles DIK, CBG having their three sides perpendicular, each to each, are also similar; whence

CB : DI :: CG : DK, or rad. : sin. b :: cos. a :: DK $= \frac{\text{cos. } a \text{ sin. } b}{\text{rad.}}$.

CB : DI :: BG : IK, or rad. : sin. b :: sin. a :: IK $= \frac{\text{sin. } a \text{ sin. } b}{\text{rad.}}$.

But since DI = IE, then EN = IK, and IN = DK; hence

DH or sin. $(a+b) = \text{IL} + \text{DK} = \frac{\text{sin. } a \text{ cos. } b + \text{cos. } a \text{ sin. } b}{\text{rad.}}$.

EF or sin. $(a-b) = \text{IL} - \text{DK} = \frac{\text{sin. } a \text{ cos. } b - \text{cos. } a \text{ sin. } b}{\text{rad.}}$.

CH or cos. $(a+b) = \text{CL} - \text{IK} = \frac{\text{cos. } a \text{ cos. } b - \text{sin. } a \text{ sin. } b}{\text{rad.}}$.

CF or cos. $(a-b) = \text{CL} + \text{IK} = \frac{\text{cos. } a \text{ cos. } b + \text{sin. } a \text{ sin. } b}{\text{rad.}}$.

And if radius be made unity, these expressions become,

I. Sin. $(a+b) =$ sin. a cos. $b +$ sin. b cos. a.

II. Sin. $(a-b) =$ sin. a cos. $b -$ sin. b cos. a.

III. Cos. $(a+b) =$ cos. a cos. $b -$ sin. a sin. b.

IV. Cos. $(a-b) =$ cos. a cos. $b +$ sin. a sin. b.

Scholium. Many important formulæ may be deduced from these four expressions, of which the following are the most useful.

Taking the sum and difference of I. and II., and also of III. and IV., we derive,

V. Sin. $(a+b) +$ sin. $(a-b) = 2$ sin. a cos. b.

VI. Sin. $(a+b) -$ sin. $(a-b) = 2$ sin. b cos. a.

VII. Cos. $(a+b) +$ cos. $(a-b) = 2$ cos. a cos. b.

VIII. Cos. $(a+b) -$ cos. $(a-b) = 2$ sin. a sin. b.

Substituting in these formulæ c for $(a+b)$, and d for $(a-b)$, they become,

IX. Sin. $c +$ sin. $d = 2$ sin. $\frac{1}{2}(c+d)$ cos. $\frac{1}{2}(c-d)$.

X. Sin. $c -$ sin. $d = 2$ sin. $\frac{1}{2}(c-d)$ cos. $\frac{1}{2}(c+d)$.

XI. Cos. $c +$ cos. $d = 2$ cos. $\frac{1}{2}(c+d)$ cos. $\frac{1}{2}(c-d)$.

XII. Cos. $c -$ cos. $d = 2$ sin. $\frac{1}{2}(c+d)$ sin. $\frac{1}{2}(c-d)$.

These expressions are frequently used in calculation, for reducing two terms to a single one.

Assuming $a = b$ in formulæ I. and III., they become,

XIII. $\text{Sin. } 2a = 2 \sin. a \cos. a.$

XIV. $\text{Cos. } 2a = \cos.^2 a - \sin.^2 a.$

Which give the sine and cosine of the double arc when the sine and cosine of the simple arc are known.

And substituting in XIV. for $\cos.^2 a$ and $\sin.^2 a$ their values $1 - \sin.^2 a$ and $1 - \cos.^2 a$, we obtain the following expressions: $\cos. 2a = 1 - 2 \sin.^2 a$ and $\cos. 2a = 2 \cos.^2 a - 1$, whence, by transposition,

XV. $\text{Sin.}^2 a = \frac{1 - \cos. 2a}{2}$, and $\cos.^2 a = \frac{1 + \cos. 2a}{2}$, which are useful in transforming the squares of the sine or cosine of any arc into the cosine of double that arc.

By extracting the square root in XV. we get

XVI. $\text{Sin. } a = \sqrt{\left(\frac{1 - \cos. 2a}{2}\right)}$, and $\cos. a = \sqrt{\left(\frac{1 + \cos. 2a}{2}\right)}$.

And if in these expressions we take $a = \frac{1}{2}e$, we have

XVII. $\text{Sin. } \frac{1}{2}e = \sqrt{\left(\frac{1 - \cos. e}{2}\right)}$, and $\cos. \frac{1}{2}e = \sqrt{\left(\frac{1 + \cos. e}{2}\right)}$,

from which we may obtain the sine and cosine of half an arc in terms of the cosine of that arc.

Dividing IX., X., XI., XII. by each other, and observing that $\frac{\sin. a}{\cos. a} = \tan. a = \frac{1}{\cot. a}$, we derive the following:

XVIII. $\frac{\text{Sin. } c + \sin. d}{\sin. c - \sin. d} = \frac{\sin. \frac{1}{2}(c+d) \cos. \frac{1}{2}(c-d)}{\cos. \frac{1}{2}(c+d) \sin. \frac{1}{2}(c-d)} = \frac{\tan. \frac{1}{2}(c+d)}{\tan. \frac{1}{2}(c-d)}.$

XIX. $\frac{\text{Sin. } c + \sin. d}{\cos. c + \cos. d} = \frac{\sin. \frac{1}{2}(c+d)}{\cos. \frac{1}{2}(c+d)} = \tan. \frac{1}{2}(c+d).$

XX. $\frac{\text{Sin. } c + \sin. d}{\cos. c - \cos. d} = \frac{\cos. \frac{1}{2}(c-d)}{\sin. \frac{1}{2}(c-d)} = \cot. \frac{1}{2}(c-d).$

XXI. $\frac{\text{Sin. } c - \sin. d}{\cos. c + \cos. d} = \frac{\sin. \frac{1}{2}(c-d)}{\cos. \frac{1}{2}(c-d)} = \tan. \frac{1}{2}(c-d).$

XXII. $\frac{\text{Sin. } c - \sin. d}{\cos. c - \cos. d} = \frac{\cos. \frac{1}{2}(c+d)}{\sin. \frac{1}{2}(c+d)} = \cot. \frac{1}{2}(c+d).$

XXIII. $\frac{\text{Cos. } c + \cos. d}{\cos. c - \cos. d} = \frac{\cos. \frac{1}{2}(c+d) \cos. \frac{1}{2}(c-d)}{\sin. \frac{1}{2}(c+d) \sin. \frac{1}{2}(c-d)} = \frac{\cot. \frac{1}{2}(c+d)}{\tan. \frac{1}{2}(c-d)}.$

XXIV. $\frac{\text{Sin. } (c+d)}{\sin. c + \sin. d} = \frac{2 \sin. \frac{1}{2}(c+d) \cos. \frac{1}{2}(c+d)}{2 \sin. \frac{1}{2}(c+d) \cos. \frac{1}{2}(c-d)} = \frac{\cos. \frac{1}{2}(c+d)}{\cos. \frac{1}{2}(c-d)}.$

XXV. $\frac{\text{Sin. } (c+d)}{\sin. c - \sin. d} = \frac{2 \sin. \frac{1}{2}(c+d) \cos. \frac{1}{2}(c+d)}{2 \sin. \frac{1}{2}(c-d) \cos. \frac{1}{2}(c+d)} = \frac{\sin. \frac{1}{2}(c+d)}{\sin. \frac{1}{2}(c-d)}.$

Resuming formulæ I. and III., and substituting the values of sin. $(a+b)$ and cos. $(a+b)$ in the equation tan. $(a+b)$ $= \frac{\sin. (a+b)}{\cos. (a+b)}$, we obtain tan. $(a+b) = \frac{\sin. a \cos. b + \sin. b \cos. a}{\cos. a \cos. b - \sin. b \sin. a}$, and since sin. $a = \cos. a$ tan. a, and sin. $b = \cos. b$ tan. b; by again substituting these values in the right-hand member of the equation, and dividing both its terms by cos. a cos. b, it becomes,

XXVI. Tan. $(a+b) = \frac{\tan. a + \tan. b}{1 - \tan. a \tan. b}$, which gives the value of the tangent of the sum of two arcs in terms of the simple arcs.

In like manner we derive the expression for the tangent of the difference of two arcs, or

XXVII. Tan. $(a-b) = \frac{\tan. a - \tan. b}{1 + \tan. a \tan. b}$.

And if we take $a = b$, we obtain for the duplicate of the arc,

XXVIII. Tan. $2a = \frac{2 \tan. a}{1 - \tan.^2 a}$; whence also

XXIX. Cot. $2a = \frac{1}{\tan. 2a} = \frac{\frac{1}{2} \tan. a}{2 \tan. a} = \frac{1}{2} \cot. a - \frac{1}{2} \tan. a$.

From the preceding formulæ the following are easily derived, and will form a useful exercise to the student.

XXX. Cot. $(a+b) = \frac{\cot. a \cot. b - 1}{\cot. b + \cot. a}$.

XXXI. Cot. $(a-b) = \frac{\cot. a \cot. b + 1}{\cot. b - \cot. a}$.

XXXII. Sec. $(a+b) = \frac{\sec. a \sec. b \operatorname{cosec}. a \operatorname{cosec}. b}{\operatorname{cosec}. a \operatorname{cosec}. b - \sec. a \sec. b}$.

XXXIII. Sec. $(a-b) = \frac{\sec. a \sec. b \operatorname{cosec}. a \operatorname{cosec}. b}{\operatorname{cosec}. a \operatorname{cosec}. b + \sec. a \sec. b}$.

XXXIV. Cosec. $(a+b) = \frac{\sec. a \sec. b \operatorname{cosec}. a \operatorname{cosec}. b}{\sec. a \operatorname{cosec}. b + \sec. b \operatorname{cosec}. a}$.

XXXV. Cosec. $(a-b) = \frac{\sec. a \sec. b \operatorname{cosec}. a \operatorname{cosec}. b}{\sec. a \operatorname{cosec}. b - \sec. b \operatorname{cosec}. a}$.

If in formulæ I. and III. we successively take $b = a$, $2a$, $3a$, &c. and substitute s for sin. a; c and $(1 - s^2)^{\frac{1}{2}}$ for cos. a, we readily obtain the following multiple arcs:

Sin. $a=s$.	Cos. $a=c$.
Sin. $2a=2s\,(1-s^2)^{\frac{1}{2}}$.	Cos. $2a=2c^2-1$.
Sin. $3a=3s-4s^3$.	Cos. $3a=4c^3-3c$.
Sin. $4a=(4s-8s^3)\,(1-s^2)^{\frac{1}{2}}$.	Cos. $4a=8c^4-8c^2+1$.
Sin. $5a=5s-20s^3+16s^5$.	Cos. $5a=16c^5-20c^3+5c$.
Sin. $6a=(6s-32s^3+32s^5)\,(1-s^2)^{\frac{1}{2}}$.	Cos. $6a=32c^6-48c^4+18c^2-1$.
&c. &c. &c.	&c. &c. &c.

In like manner from formulæ XXVI. and XXX. we obtain

Tan. $a=t$.	Cot. $a=\text{cot.}$
Tan. $2a=\dfrac{2t}{1-t^2}$.	Cot. $2a=\dfrac{\text{cot.}^2-1}{2\,\text{cot.}}$.
Tan. $3a=\dfrac{3t-t^3}{1-3t^2}$.	Cot. $3a=\dfrac{\text{cot.}^3-3\,\text{cot.}}{3\,\text{cot.}^2-1}$.
Tan. $4a=\dfrac{4t-4t^3}{1-6t^2+t^4}$.	Cot. $4a=\dfrac{\text{cot.}^4-6\,\text{cot.}^2+1}{4\,\text{cot.}^3-4\,\text{cot.}}$.
Tan. $5a=\dfrac{5t-10t^3+t^5}{1-10t^2+5t^4}$.	Cot. $5a=\dfrac{\text{cot. }5-10\,\text{cot.}^3+5\,\text{cot.}}{5\,\text{cot.}^4-10\,\text{cot.}^2+1}$.
&c. &c. &c.	&c. &c. &c.

The powers of the sines and cosines of arcs in terms of the sum and difference of certain multiples of these arcs may be deduced from Formulæ V., VI., VII., VIII.: Thus,

Sin. $a=\sin.\ a$.	Cos. $a=\cos.\ a$.
2 Sin.$^2a=1-\cos.\ 2a$.	2 Cos.$^2a=\cos.\ 2a+1$.
4 Sin.$^3a=3\sin.\ a-\sin.\ 3a$.	4 Cos.$^3a=\cos.\ 3a+3\cos.\ a$.
8 Sin.$^4a=\cos.\ 4a-4\cos.\ 2a+3$.	8 Cos.$^4a=\cos.\ 4a+4\cos.\ 2a+3$.
16 Sin.$^5a=\sin.\ 5a-5\sin.\ 3a+10\sin.\ a$.	16 Cos.$^5a=\cos.\ 5a+5\cos.\ 3a+10\cos.\ a$.
&c. &c. &c.	&c. &c. &c.

OF THE SIGNS OF THE TRIGONOMETRICAL LINES.

In Analytical Trigonometry, and its application to Astronomy, it is necessary to attend to the changes which the several quantities undergo in the different quadrants of the circle.

Geometrical quantities, when expressed analytically, are estimated from some given point or line, and are considered as + or —, according as they lie on the one or on the other side of that point or line.

The sines are estimated from the diameter EA, and the cosines from the centre C; and if we consider the sines as *positive* when they lie above the diameter, and the cosines when they lie on the right-hand side of the centre, it is obvious, that, in the first quadrant AD, the sines and cosines are both positive. In the second quadrant DE, the sine lying still above the diameter is *positive*, but the cosine having changed its position in regard to the centre is now *negative*. The sine changing its position in the third quadrant EI, is now set off below the diameter, and the cosine remaining as in the second quadrant, they are therefore both *negative*. And, in the fourth quadrant, the sine still lying below the diameter is *negative*, while the cosine having resumed its original position in regard to the centre is *positive*.

The signs of the other quantities may be easily determined from the preceding equations, for since $\tan. = \frac{\sin.}{\cos.}$, it follows, that when the signs of the sine and cosine are *alike*, that of the tangent is positive, and when they are *unlike*, the sign of the tangent is negative.

The following table exhibits the mutations of the signs of the different quantities for each quadrant of the circle:—

Quadrants.	Sin.	Cos.	Tan.	Cot.	Sec.	Cosec.	Vers.	Cov.
1.	+	+	+	+	+	+	+	+
2.	+	—	—	—	—	+	+	+
3.	—	—	+	+	—	—	+	+
4.	—	+	—	—	+	—	+	+

NOTE 1. The signs of the sine and cosecant, of the cosine and secant, and of the tangent and cotangent, are respectively *alike*; and the signs of the versed and coversed sines are always *positive*, the former being always set off from A in the same direction, and the latter from E in the contrary direction.

NOTE 2. The sines, cosines, &c. may be considered not only as belonging to arcs less than four quadrants, but also to those arcs increased by any number of complete circumferences.

OF THE CONSTRUCTION OF A TABLE OF SINES, COSINES, &c.

Various methods may be employed for computing the numerical values of the sines, cosines, &c., but we shall only exhibit two.

I. If x be any arc of a circle, whose radius is unity, it was shown by Newton that (See Appendix)

$$\text{Sin. } x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \text{\&c., and}$$

$$\text{Cos. } x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \text{\&c.}$$

Now by means of these series, and the ratio between the diameter and circumference of the circle, the sines and cosines of any arc may be found.

When the radius is unity, half the circumference is 3·141592653589793, &c., and as there are 180° or 10800′ in a semicircle, it is obvious that, if we divide the former by the latter, we will obtain the length of an arc of 1 minute $= \cdot 00029088821$; whence, if the arc is 1 minute,

$$x = \cdot 00029088821$$
$$-\tfrac{1}{6}x^3 = -\cdot 0000000000004$$
$$\therefore \text{Sin. } x = \cdot 0002908882 = \text{the sine of 1 minute.}$$

Again from $1\cdot 0000000000$
Take $\tfrac{1}{2}x^2 = 0\cdot 0000000423$

$$\therefore \text{Cos. } x = \cdot 9999999577 = \text{the cosine of 1 minute.}$$

Let the arc be 5°, to find its sine and cosine.

Here $\frac{5 \times 3\cdot 14159265}{180} = \cdot 08726646 = x =$ the length of an arc of 5°; hence $x = 0\cdot 08726646$

$$-\tfrac{1}{6}x^3 = -0\cdot 00011076$$
$$+\tfrac{1}{120}x^5 = +0\cdot 00000004$$
$$\therefore \text{Collecting, sin. } x = 0\cdot 08715574 = \text{the sine of 5 degrees.}$$

And for the cosine $1\cdot 00000000$

$$-\tfrac{1}{2}x^2 = -0\cdot 00380771$$
$$+\tfrac{1}{24}x^4 = +0\cdot 00000241$$
$$\therefore \text{Collecting, cos. } x = 0\cdot 99619470 = \text{the cosine of 5 degrees.}$$

This method may be employed for the sines and cosines at the beginning and end of the quadrant, for when the arc does not exceed 10′, the first two terms of the series give the sine and cosine true to 15 places; and when it does not exceed 1°, the first three terms give them true to the same number of places, but the nearer the arc is to 45°, the more slowly do these series converge; and therefore the greater are the number of the terms that must be employed in the calculation.

NOTE. It is necessary to compute the sines only, as the cosines are more easily found from the equation, $\cos. = \sqrt{(1 - \sin.^2)}$.

II. It was shown, XIII., that $\sin. 2a = 2 \sin. a \cos. a$; whence, after computing the sine and cosine of 1′ by the last method, and substituting 1′ for a, we obtain the sine of 2′: Thus,

$$\text{Sin. } 2' = 2 \sin. 1' \cos. 1';$$

And for the sine of 3′ and upwards we may employ formula

V. $\text{Sin. } (a+b) + \sin. (a-b) = 2 \sin. a \cos. b,$

Or $\sin. (a+b) = 2 \sin. a \cos. b - \sin. (a-b)$,

where, if a is taken successively $= 2', 3', 4'$, &c., and $b = 1'$, we have

$$\text{Sin. } 3' = 2 \text{ sin. } 2' \text{ cos. } 1' - \text{sin. } 1'.$$
$$\text{Sin. } 4' = 2 \text{ sin. } 3' \text{ cos. } 1' - \text{sin. } 2'.$$
$$\text{Sin. } 5' = 2 \text{ sin. } 4' \text{ cos. } 1' - \text{sin. } 3'.$$
&c. &c. &c.

In like manner, if a is taken successively $= 2', 3', 4'$, &c., and $b = 1'$, and these values substituted in Formula VII., we get

$$\text{Cos. } 3' = 2 \text{ cos. } 2' \text{ cos. } 1' - \text{cos. } 1'.$$
$$\text{Cos. } 4' = 2 \text{ cos. } 3' \text{ cos. } 1' - \text{cos. } 2'.$$
$$\text{Cos. } 5' = 2 \text{ cos. } 4' \text{ cos. } 1' - \text{cos. } 3'.$$
&c. &c. &c.

When the sines and cosines have been computed for every minute of the quadrant as far as 30°, the remainder of the table may be found by subtraction only.

For dividing Formulæ V. and VIII. by 2, we obtain

$$\text{Sin. } a \text{ cos. } b = \tfrac{1}{2} \text{ sin. } (a + b) + \tfrac{1}{2} \text{ sin. } (a - b).$$
$$\text{Sin. } a \text{ sin. } b = \tfrac{1}{2} \text{ cos. } (a - b) - \tfrac{1}{2} \text{ cos. } (a + b).$$

And if a is taken $= 30°$, then $\text{sin. } a = \text{sin. } 30° = \frac{1}{2}$; whence

$$\tfrac{1}{2} \text{ Cos. } b = \tfrac{1}{2} \text{ sin. } (30° + b) + \tfrac{1}{2} \text{ sin. } (30° - b).$$
$$\tfrac{1}{2} \text{ Sin. } b = \tfrac{1}{2} \text{ cos. } (30° - b) - \tfrac{1}{2} \text{ cos. } (30° + b).$$

Multiplying by 2, and transposing, these expressions become

$$\text{Sin. } (30° + b) = \text{cos. } b - \text{sin. } (30° - b).$$
$$\text{Cos. } (30° + b) = \text{cos. } (30° - b) - \text{sin. } b.$$

Now if b is taken successively $= 1', 2', 3'$, &c., we have for the sines

$$\text{Sin. } 30°\ 1' = \text{cos. } 1' - \text{sin. } 29°\ 59'.$$
$$\text{Sin. } 30°\ 2' = \text{cos. } 2' - \text{sin. } 29°\ 58'.$$
$$\text{Sin. } 30°\ 3' = \text{cos. } 3' - \text{sin. } 29°\ 57'.$$
&c. &c. &c.

And for the cosines

$$\text{Cos. } 30°\ 1' = \text{cos. } 29°\ 59' - \text{sin. } 1'.$$
$$\text{Cos. } 30°\ 2' = \text{cos. } 29°\ 58' - \text{sin. } 2'.$$
$$\text{Cos. } 30°\ 3' = \text{cos. } 29°\ 57' - \text{sin. } 3'.$$
&c. &c. &c.

By either of these methods the sines and cosines may be computed as far as 45°; and it is obvious, from the definitions, that the sines and cosines will also be found from 45° to 90°, for sine 50° = cosine 40°, and cosine 60° = sine 30°, &c.

The tangents and secants may be readily obtained by Formulæ III. and V., pages 150 and 151, when the cotangents and cosecants will also be known.

The versed sines are $= 1 \mp \cos.$, according as the arc is greater or less than 90°, and the coversed sines are the complements of the versed sines to 1.

The sines, cosines, &c. which we have been computing are called Natural Sines, Cosines, &c., and when these are arranged in a table from 1′ up to 90°, they form what is termed the *Trigonometrical Canon.*

If the logarithms of all the natural sines, cosines, &c. be taken from the common logarithmic tables, and 10 added to their indices, these will form the tables of logarithmic sines, cosines, &c.

The logarithmic sines, &c. are supposed to be computed to the radius 10,000,000,000, in order that the smallest arc, likely to be used in calculation, may not have a negative index; but the natural sines, &c. are computed to the radius 1, hence the reason of adding 10 to the indices.

OF THE TABLES OF SINES, TANGENTS, &C.

The common tables have the degrees at the top, and the minutes on the left-hand side, when the degrees are less than 45°; but if greater, the degrees are marked at the bottom, and the minutes on the right-hand side.

1. Required the logarithmic sine of 37° 23′ 12″.

Turn to the page which has 37° at the top, and come down the column titled *Sine* at the top, to the line that has 23′ on the left-hand side, and you will find 9·783292, the sine of 37° 23′; and the difference between it and the sine of 37° 24′ is 166. Then as 60″ is to 12″, so is 166 to 33, the proportional difference for 12″, which, added to 9·783292, gives 9·783325, the logarithmic sine of 37° 23′ 12″.

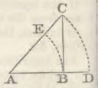
2. Required the degrees and parts of a degree of which 10·273846 is the logarithmic tangent.

Look for the nearest tangent 10·273716, and because it is titled *Tang.* at the bottom, take the degrees at the foot, and the minutes on the right-hand side, where are found 61° 58′. The difference between this tangent and the one above it is 305, and the difference between it and the given one is 130; therefore 305 : 130 : : 60″ : 26″, so that 10·273846 is the tangent of 61° 58′ 26″.

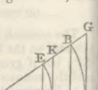
3. Required the nat. sine of 57° 26′ 20″.	Ans. ·842818.
4. . . . . log. cosine of 67° 31′ 40″.	9·582331.
5. . . . . log. secant of 73° 27′ 45″.	10·545700.
6. Nat. cosine is ·747682, what is the arc?	41° 36′ 36″.
7. Log. secant is 10·475546, what is the arc?	70° 27′ 19″.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

The first thing to be done in resolving right-angled triangles is to make one of the sides the radius of a circle, the centre of which is at an acute angle, and thus to determine what the other sides would be in that circle.



If from the centre A, with the radius AC, the arc CD be described, then BC will be the sine of CAB, and AB its cosine. But if the centre be at C, and the circle pass through A, then AB is the sine of C, and BC its cosine. Hence when the hypotenuse is radius, the other sides are the sines of their opposite angles, or the cosines of their adjacent angles. Again, if from the centre A, with the radius AB, the arc BE be described, then BC is the tangent of A, and AC is its secant.



Suppose ACB any angle, and AB an arc described with the radius of the circle, from which the sines, tangents, &c. in the tables were calculated; then BF is the sine in the tables, CF the cosine, AG the tangent, and CG the secant in the tables. Let CEH be a right-angled triangle. If CE be radius EH will be the sine of C, and CH its cosine. Hence CE : EH : : CB : BF (Theor. 1. Trig.); that is, CE is to EH as the radius of the tables is to the sine of C in the tables. In like manner CE is to CH as the radius is to the cosine of C in the tables (Theor. 1. Trig., Cor. 1.) In the same way if CDK is the triangle, and CD the radius, CD is to DK as the radius is to the tangent of C in the tables (Theor. 1. Trig.), and DC is to CK as the radius is to the secant of C in the tables (Theor. 1. Trig., Cor. 2.); so that after determining the names of the sides of the triangle, any two sides are to one another as their names in the tables.

The terms of the proportion, however, must be so arranged, that the thing required shall be the last term, thus:

To find EH, R : sin. C : : CE : EH

To find CE, sin. C : R : : HE : EC

To find C, CE : EH : : R : sin. C.

And these three are all the variations which are requisite. But the student should accustom himself to state them without hesitation. Before proceeding to the numerical solution, he should also construct the triangle geometrically, as directed in Problems XVII. to XXI. PRACTICAL GEOMETRY, distinguishing the given sides by a dash across them, and the given angles by one or two dots.

1. In the triangle ABC, right angled at B, are given the hypotenuse AC 324 feet, and the angle BAC 48° 17′; to find the base AB, and perpendicular BC.

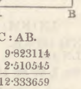
If AC be radius, and A the centre, CB is the sine of A, and AB its cosine. Wherefore,

R : sin. A :: AC : CB, and R : cos. A :: AC : AB.

Sin. A 48° 17′ log.	9·872998	cos. A log.	9·823114
AC 324 log.	2·510545	AC 324 log.	2·510545
Sum	12·383543	Sum	12·333659
Radius	10·000000	Rad.	10·000000
CB 241·85	log. 2·383543	AB 215·6 log.	2·333659

NOTE. Instead of subtracting the logarithm of the first term from he sum of the logarithms of the second and third, it is preferable o take the arithmetical complement of the first and add the three ogether.

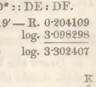
2. Given DE 1254 feet, and the angle D 51° 19′; to find the hypotenuse DF, and the perpendicular EF.



DE being radius, EF is the tangent and DF the secant of D.

R : tan. D :: DE : EF.		R : sec. D* :: DE : DF.	
Tan. D 51° 19′ — R.	0·096545	Sec. D 51° 19′ — R.	0·204109
DE 1254	log. 3·098298	DE 1254	log. 3·098298
EF 1566·18	log. 3·194843	DF 2006·35	log. 3·302407

3. Given the angle G 43° 38′, and the opposite side HK 186 feet; to find the hypotenuse GK, and the base GH.



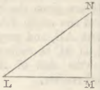
This may be wrought as the last by first finding GKH. Or, GK being radius, KH is the sin. of G; and GH being radius, HK is the tan. of G.

* The log. secant is readily found by subtracting the log. cosine from 20.

Sin. G : R : : HK : KG, and tan. G : R : : KH : HG.

HK 186 + R.	log. 12·269513	HK + R.	log. 12·269513
Sin. G 43° 38′	log. 9·838875	tan. G	log. 9·979274
GK 269·549	log. 2·430638	GH 195·09	log. 2·290239

4. Given the hypotenuse LN 415 inches, and the perpendicular MN 249; to find the angles, and LM.



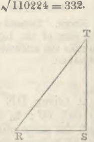
LN being radius, NM is the sine and LM the cosine of L; whence

LN : NM : : R : sin. L. R : cos. L : : NL : LM.

NM 249 + R.	log. 12·396199	Cos. L 36° 52′ 12″ — R.	$\bar{1}$·903089
LN 415	log. 2·618048	LN 415	log. 2·618048
Sin. L 36° 52′ 12″	log. 9·778151	LM 332	log. 2·521137

Note. LM is equal to the square root of the product of the sum and difference of LN and NM $= \sqrt{664 \times 166} = \sqrt{110224} = 332$.

5. Given the base RS 53 miles, and the perpendicular ST 67; to find the angles, and hypotenuse RT.



If RS is made the radius, then ST is the tangent and RT the secant of R; therefore

RS : ST : : R : tan. R. R : sec. R : : SR : RT

ST 67 + R.	log. 11·826075	Sec. R 51° 39′ 16″ — R.	0·207326
RS 53	log. 1·724276	RS 53	log. 1·724276
Tan. R 51° 39′ 16″	10·101799	RT 85·4284	log. 1·931602

Note. The square of RT is equal to the sum of the squares of RS and ST; therefore RT $= \sqrt{53^2 + 67^2} = \sqrt{7298} = 85·4284$.

6. Given the hypotenuse 893, and the base 586 chains.
Ans. Angle at base 48° 59′ 17″, perpendicular 673·833 ch.

7. Given the base 326 yards, and the vertical angle 64° 40′.
Ans. Hypotenuse 360·686, perpendicular 154·33 yards.

8. Given the perpendicular 286, and vertical angle 71° 24′.
Ans. Hypotenuse 896·666, base 849·8314.

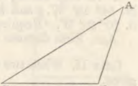
9. Given the hypotenuse 963 links, and vertical angle 41° 48′. Ans. Base 641·87, perpendicular 717·892 links.

SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

CASE I. When two sides and an angle opposite to one of them is given.

Any two sides of a triangle are to one another as the sines of the angles opposite to them. Thus BC : CA : : sin. A : sin. B, or sin. A : sin. B : : CB : CA (Theor. 2. Trig.)

The former order is to be used when an angle is required, and the latter when a side.



1. Given two sides AB 532, and BC 358 feet, and the angle at C 107° 40′; to find the angles at A and B, and the side AC.

AB : BC : : sin. C : sin. A.

Sin. C (107° 40′) 72° 20′*		9·979019
BC 358 feet	log.	2·553883
		12·532902
BA 532	log.	2·725912
Sin. A 39° 53′		9·806990

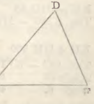
B=180°—(C+A), and sin. C : sin. B : : BA : AC.

Sin. B 32° 27′		9·729621
BA 532	log.	2·725912
		12·455533
Sin. C 107° 40′		9·979019
AC 299·58	log.	2·476514

2. Given AB 232, and BC 345 yards, and the angle at C 37° 20′; to find the angles at A and B, and the side AC.

By proceeding in the same way, the angle at A may be either 64° 24′ or 115° 36′, and therefore the angle at B may be either 78° 16′ or 27° 4′, and AC 374·56 or 174·07. For AB being less than BC, there are two triangles which have each the given things in them.

3. Two places are 560 feet from one another, and at a station 258 feet from the first place, their distance subtended an angle of 63° 28′. Required the distance of the station from the other place. Ans. 625·468 feet.



4. Given two angles D 63° 48′, and E 49° 25′, and the side EF opposite to D 275 yards; to find DE and DF. The angle at F is=180°—(D+E)=66° 47′.

* When the angle is greater than 90°, take the sine, tangent, &c. of its supplement.

Sin. D : sin. E : : EF : FD.

Sin. E 49° 25′	log.	9·880505
EF 275	log.	2·439333
		12·319838
Sin. D 63° 48′	log.	9·952918
FD 232·766	log.	2·366920

Sin. D : sin. F : : FE : ED.

Sin. F 66° 47′	log.	9·963325
EF 275	log.	2·439333
		12·402658
Sin. D 63° 48′	log.	9·952918
DE 281·67	log.	2·449740

5. Given the angles at E 49° 25′, and at F 63° 48′, and the side EF 275; to find ED and DF.

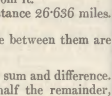
Ans. ED 268·488, and DF 227·255.

6. A ship sailing due north observes a cape bearing N. 54° 12′ W.; and after sailing 27 miles, the cape bore S. 70° 30′ W. Required her distances from it.

Ans. First distance 30·957, second distance 26·636 miles.

Case II. When two sides and the angle between them are given.

Add and subtract the sides to get their sum and difference. Subtract the angle from 180°, and take half the remainder, to get half the sum of the unknown angles. Then as the sum of the sides is to their difference, so is the tangent of half the sum of the unknown angles to the tangent of half their difference (Theor. 4. Trig.) Having thus found the half difference, add it to the half sum to get the angle opposite to the greater side, and subtract it to get the less angle; after which the third side is found by Case I.



7. Given the sides GH 133, and HK 176 yards, and the angle at H 73° 16′; to find the angles at G and K, and the side GK.

KH+HG : KH—HG : : tan. ½(G+K) : tan. ½(G—K).

KH — HG 43	log.	1·633468
Tan. ½(180° — H) 53° 22′		10·128679
		11·762147
KH + GH 309	log.	2·489958
Tan. ½(G — K) 10° 36′	log.	9·272189
Angle G	63° 58′	
Angle K	42° 46′	

Sin. G : sin. H : : HK : KG.

Sin. H 73° 16′		9·981209
HK 176	log.	2·245513
		12·226722
Sin. G 63° 58′		9·953537
GK 187·58	log.	2·273185

8. Given GH 237, and GK 482 feet, and the angle at G 77° 48′; to find the angles at H and K, and HK.

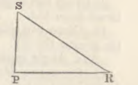
Ans. H 73° 59′ 39″, K 28° 12′ 21″, and HK 490·1135 feet.

9. Given HK 78, and KG 168, and the angle K 128° 26′.
Ans. H 35° 48′ 20″, G 15° 45′ 40″, HG 224·943.

CASE III. When the three sides are given.

Add the three sides, and from half the sum subtract the side opposite to the angle sought; then take the arithmetical complements of the logs. of the two sides containing the angle sought, and the logarithms of the half sum and of the remainder, and add these four together, and half the sum will be the log. cosine of half the angle sought. (Theor. 5. Trig. Cor.)*

10. Given the sides SP 230, PR 365, and SR 426 feet; to find the angles.



SP 230 ar. co. log.		7·638272
PR 365 ar. co. log.		7·437707
SR 426		
½)1021		
½ Sum 510·5	log.	2·707996
426		
Rem. 84·5	log.	1·926857
		½)19·710832
½P 44° 12′ 24″ cosine		9·855416
P 88° 24′ 48″		

In the same manner the angle S is found to be 58° 55′ 25″.

11. Given the sides SP 1248, PR 728, and RS 956 feet.
Ans. The angle R 94° 40′ 50″, P 49° 46′ 16″.

12. Given SP 375, PR 275, and RS 196.
Ans. The angle S 45° 17′ 26″, P 30° 25′ 58″.

PROMISCUOUS EXAMPLES.

1. Given the hypotenuse of a right-angled triangle 528 feet, and one of the acute angles 39° 27′.
Ans. The opposite side 335·493, adjacent side 407·7104 feet.

2. Given the base 256, and the adjacent angle 57° 28′.
Ans. Hypotenuse 476·022, perpendicular 401·324 feet.

3. Given the perpendicular 297 feet, and the angle at the base 36° 48′.
Ans. Hypotenuse 495·806, base 397·0073 feet.

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* Let $s =$ half the sum of the three sides, then $\mathrm{SP} \times \mathrm{PR} : s \times (s - \mathrm{SR}) :: \mathrm{rad.}^2 : \cos.^2 \frac{1}{2}\mathrm{P}$, or $\frac{\mathrm{rad.}^2}{\mathrm{SP} \times \mathrm{PR}} \times s \times (s - \mathrm{SR} = \cos.^2 \frac{1}{2}\mathrm{P} = 2 \log. \mathrm{R} - (\log. \mathrm{SP} + \log. \mathrm{PR}) + \log. s + \log. (s - \mathrm{SR}) = 2 \log. \cos. \frac{1}{2}\mathrm{P} =$ ar. co. log. SP + ar. co. log. PR + log. s + log. $(s - \mathrm{SR}) = 2 \log. \cos. \frac{1}{2}\mathrm{P}$, which is the rule.

4. Given the hypotenuse 1268, and perpendicular 428 yards.
Ans. The base 1193·583, adjacent angle 19° 43′ 37·3″.

5. Given the base 674, and the perpendicular 438 yards.
Ans. Hypotenuse 803·8166 yards, angle at base 33° 1′ 4·4″.

6. Given the hypotenuse 97, and the base 38 miles.
Ans. Perpendicular 89·247 miles, angle at base 66° 56′ 11″.

7. Given the base 326, and the vertical angle 67° 30′.
Ans. The hypotenuse 352·86, perpendicular 135·034.

8. In an oblique triangle, given two angles 46° 48′ and 114° 26′, and the side opposite the lesser 254 feet.
Ans. Other sides 317·2328, and 112·0974 feet.

9. Given two angles 56° 24′ and 74° 28′, and the side between them 354. Ans. Other sides 451·0104, and 389·898.

10. Given two sides 572 and 748, and the angle opposite to the greater 67° 30′.
Ans. Angle opposite less 44° 57′ 1·5″, third side 748·269.

11. Given two sides 356 and 294, and the angle opposite to the lesser 51° 27′.
Ans. Other angles 71° 15′ 39″ and 57° 17′ 21″, or 108° 44′ 21″ and 19° 48′ 39″; third side 316·309 or 127·4079.

12. Given two sides 1864 and 1235, and included angle 73° 38′.
Ans. Other angles 68° 21′ 15·48″ and 38° 0′ 44·52″, third side 1924·155.

13. Given two sides 436 and 219, and included angle 127°.
Ans. Other angles 35° 52′ 45·72″ and 17° 7′ 14·28″, third side 594·125.

14. Given the three sides 456, 327, and 184 yards.
Ans. Angles 123° 55′ 10·8″, 36° 31′ 5·72″, and 19° 33′ 43·48″.

15. Given the sides 2586, 1482, and 1234.
Ans. Angles 144° 14′ 52·6″, 19° 33′ 49″, and 16° 11′ 18·4″.

16. Given two angles 57° 12′ and 24° 45′, and the side between them 365 poles.
Ans. Other sides 154·33, and 309·86 poles.

17. Given two sides 120 and 112 feet, and the angle opposite the less 57° 27′.
Ans. Angle opposite the greater 64° 34′ 21″ or 115° 25′ 39″, and third side 112·65 or 16·47 feet.

MENSURATION OF SURFACES.

THE area or surface of a figure is the number of square inches, feet, yards, &c. which it contains.

A square constructed upon a straight line, of which the length is an inch, is called a *square inch;* and the same is to be understood of a square foot, &c. This is called the *measuring unit,* and the area of any figure is computed by the number of those squares which it contains.

TABLE OF LINEAL MEASURES.

Inches.	Feet.				
12	1	Yards.*			
36	3	1	Poles.		
198	$16\frac{1}{2}$	$5\frac{1}{2}$	1	Furlongs.	
7920	660	220	40	1	Mile.
63360	5280	1760	320	8	1

TABLE OF SQUARE MEASURES.

Square In.	Square Feet.				
144	1	Sqr. Yds.			
1296	9	1	Sqr. Pls.		
39204	$272\frac{1}{4}$	$30\frac{1}{4}$	1	Roods.	
1568160	10890	1210	40	1	Acre.
6272640	43560	4840	160	4	1

NOTE. The acre contains 10 square chains, each 16 perches, or 100,000 square links. The chain is 66 feet in length, and is divided into 100 links, each 7·92 inches.

* The imperial yard is the distance between the centres of the points in the old studs fixed in the brass rod belonging to the House of Commons, and titled "Standard Yard, 1760." When used, the brass must be at the temperature of 62 degrees of Fahrenheit's thermometer.

The length of a pendulum vibrating seconds of mean time, at the level of the sea, in the latitude of London, contains 39·1393 imperial inches.

SCOTCH LAND MEASURE.

Ells.	Falls.		
36	1	Roods.	
1440	40	1	Acre.
5760	160	4	1

NOTE. A Scotch ell = 37·0598 imperial inches. The Scotch chain is 74·1196 imperial feet, and consequently the Scotch acre is = 1·26118345 imperial acre.

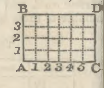
PARALLELOGRAMS.

PROB. I. To measure a right-angled parallelogram.

RULE. Multiply one of the sides by the other.

That is, $AC \times AB$ = the area (El. Geom. 15. Schol.)

1. Required the area of the rectangle ABDC, of which the sides are AB 4 yards, and AC 6.*



$$\begin{array}{r} 6 \\ 4 \\ \hline \end{array}$$

Area 24 square yards.

NOTE. If AC be divided into 6 equal parts or yards, and AB into 4, and lines be drawn parallel to the sides, the rectangle will be divided into 24 squares, each of them a square yard.

2. Required the area of a square, each side 37 feet.
Ans. 1369 square feet.

3. Required the area of a rectangle, the sides 326 and 153 feet. Ans. 49878 sq. feet = 1 acre 23 per. $6\frac{1}{4}$ yds.

4. Required the area of a square, each side 3525 links.
Ans. 124·25625 ac. = 124 ac. 1 ro. 1 per.

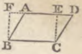
5. A rectangular space, 68 feet 3 inches long by 56 feet 8 inches broad, is to be paved with stones each 2 feet 3 inches by 10 inches. Required how many stones it will take, and what will be the expense at 2s. 3d. for a square yard.
Ans. $2062\frac{2}{3}$ stones, expense £48, 6s. $10\frac{1}{2}$d.

PROB. II. To measure any parallelogram.

RULE. Multiply one of the sides by the perpendicular let fall upon it from the opposite side.

That is, $BC \times FB$ = the area (El. Geom. 15. Schol.)

* The student should always construct the figures upon his slate before he begins his computations.



1. Required the area of the parallelogram ABCD, of which the sides are AB 214, and BC 354, and the perpendicular CE 192 feet.

354
192

9|67968 square feet.

4840)7552 square yards.

Ans. 1 acre 2 roods 9 perches $19\frac{2}{9}$ yards.

2. Required the area of a rhombus, the side 358, and the perpendicular on it 194 feet. Ans. 69452 feet.

3. Required the area of a rhombus, of which the diagonals are AC 436, and BD 623 yards.

NOTE. AC and BD bisect one another at right angles. For in the triangles AED, CED, the side AE = CE (El. Geom. 29.) AD = CD (El. Geom., Def. 34.), and ED common; whence these triangles are equal in every respect, and the angle AED = CED, or each is a right angle (El. Geom. 5.)

Ans. $623 \times 218 = 135814$ yards, $= 28$ ac. 9 per. $21\frac{2}{9}$ yds.

4. Required the area of a rhomboid, the sides 1234 and 762, and the perpendicular on the former 658 links.

Ans. 8·11972 ac. = 8 ac. 19 per. 4 yds. $6\frac{1}{4}$ feet.

5. Required the area of a parallelogram, the sides 56 feet 3 inches and 42 feet 10 inches, and the perpendicular on the latter 47 feet 3 inches. Ans. 2023 feet $10\frac{1}{2}$ inches.

6. Required the area of a rhomboid, the sides 24 and 18 poles, and the perpendicular upon the latter 96 yards.

Ans. 9504 sq. yds. = 1 acre 3 roods 34 per. $5\frac{1}{2}$ yards.

7. Required the area of a rhombus, the diagonals $6\frac{1}{2}$ feet and $3\frac{1}{4}$ feet. Ans. 10 feet 81 inches.

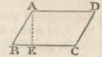
PROB. III. Given two sides and an angle of a parallelogram; to find the area.

RULE. Multiply the product of the two sides by the natural sine of the angle.

That is, $BC \times BA \times \sin. B =$ the area.*

Or add the logarithms of the sides and the logarithm sine of the angle: the sum, after taking 10 from the index, will be the logarithm of the area.

* The area, by Prob. 2., is $BC \times AE$, but rad. $1 : \sin. B :: BA : AE = BA \sin. B$; hence $BC \times BA \times \sin. B =$ the area.



1. Required the area of the rhomboid ABCD, of which the sides are AB 278, and BC 456 feet, and the angle B 58° 46′.

Sin. 58° 46′ = ·85506
456

389·90736
278

43560)108394·24608 square feet.

Ans. 2 acres 1 rood 38 perches $4\frac{1}{3}$ yards.

2. Required the area of a rhombus, the side 172 ells, and an angle 72° 30′. Ans. 28214·74 ells.

3. Required the area of a rhomboid, the sides 136 and 97 yards, and the angle 73° 16′.

Ans. 12633·4 sq. yds. = 2 ac. 2 ro. 17 per. 19 yds. 1·35 ft.

4. Required the area of a rhomboid, the sides 628 and 425 links, and the angle 126°.

Ans. 2·159267 ac. = 2 ac. 25 per. 14 yds. 5·4 feet.

5. Required the area of a rhombus, the side 57 poles, and the angle 67° 45′.

Ans. 3007·08 per. = 18 ac. 3 ro. 7·08 per.

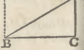
6. Required the area of a rhombus, the side 157 inches, and the angle 29° 12′. Ans. 12025·26 sq. in. = 83 ft. $73\frac{1}{4}$ in.

TRIANGLES.

Prob. IV. Given the base and the perpendicular of a triangle; to find the area.

Rule. Multiply the base by the perpendicular, and half the product will be the area.

That is, $\frac{1}{2}(BC \times AC)$ = the area (El. Geom. 15. Schol.)



1. Required the area of the right-angled triangle ABC, of which the sides about the right angle are BC 254, and AC 136 yards.

254
68

4840)17272 square yards.

Ans. 3 acres 2 roods 10 perches $29\frac{1}{4}$ yards.

2. Required the area of a triangle ABC, the base CB 396, the side AB 278, and the perpendicular AE 174 feet.

Ans. 396×87 = 34452 square feet, = 3 ro. 6 per. $16\frac{1}{2}$ yds.

3. Required the area of a triangle, one angle 43°, adjacent side 296, and perpendicular on it 176 yards.

Ans. 26048 sq. yards, = 5 ac. 1 ro. 21 per. $2\frac{3}{4}$ yds.

4. Required the area of a triangle, the sides 156 and 97 poles, and the perpendicular upon the latter 102 poles.

Ans. 4947 perches, = 30 acres 3 roods 27 perches.

5. Required the area of a triangle, the side 684 links, the angle adjacent 137°, and the perpendicular 928 links.

Ans. 3·17376 acres, = 3 acres 27 perches $24\frac{1}{4}$ yards.

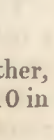
Prob. V. *Given two sides and the included angle of a triangle; to find the area.*

Rule. Multiply one side by half of the other, and by the natural sine of the included angle.

That is, $\frac{1}{2}AB \times BC \times \sin. B =$ the area.*

Or add the logarithms of one side and of half the other, and the logarithm sine of the angle: the sum, rejecting 10 in the index, is the logarithm of the area.

1. Required the area of the triangle ABC, of which the side AB is 534, and BC 872 links, and the angle B 63° 40′.



Sin. 63° 40′ = ·89623
872

781·51256
267

100000)208663·85352 square links.

2·0866385

Ans. 2 acres 13 perches 26 yards.

2. Required the area of a triangle, having given an angle 78° 30′, and the containing sides 933 and 471 Scotch links.

Ans. 215310·59 links, = 2 acres 24 falls 17·88 ells.

3. Required the area of a triangle, two sides 12 feet 9 inches, and 7 feet 3 inches, and the included angle 57° 38′.

Ans. 5621·5 inches, = 4 yards 3 feet $5\frac{1}{2}$ inches.

4. Required the area of a triangle, an angle 54° 30′, and the containing sides 328 and 157 yards.

Ans. 20961·96 sq. yds. = 4 ac. 1 ro. 12 per. 29 yds.

5. Required the area of a triangle, an angle 128°, and the sides about it 38 and 93 poles.

Ans. 1392·414 per. = 8 ac. 2 ro. 32 per. $12\frac{1}{2}$ yds.

6. Required the area of a triangle, an angle 17° 54′, and the adjacent sides 27 and 12 miles.

Ans. 49·791834 miles.

---

* This rule is obvious from Prob. 3., for a triangle is half a parallelogram of the same base and altitude.

7. Required the area of a triangle, an angle 93°, and the sides about it 137 and 428 ells.

Ans. 29277·834 sq. ells = 5 ac. 13 falls 9·8 ells.

Prob. VI. Given the three sides of a triangle; to find the area.

Rule. Add the three sides together, and from half the sum subtract each side separately. Then multiply the half sum and the three remainders successively, and the square root of the last product will be the area.

That is, if a, b, c, represent the sides of the triangle, and s half their sum, then $\sqrt{\{s \times (s-a) \times (s-b) \times (s-c)\}}$ = the area.*

Or add the logarithms of the half sum and of the three remainders, and half the sum will be the logarithm of the area.

1. Required the area of the triangle ABC, of which the sides are AB 221, BC 255, and AC 238 feet.

---

* Let AB $= a$, BC $= b$, and AC $= c$, then $b : a + c :: a - c : \frac{a^2 - c^2}{b} =$ BE — EC $\therefore \frac{1}{2}b + \frac{a^2 - c^2}{2b} = \frac{b^2 + a^2 - c^2}{2b} =$ BE; hence $\sqrt{\left\{a^2 - \left(\frac{b^2 + a^2 - c^2}{2b}\right)^2\right\}} = \sqrt{\frac{2a^2b^2 - a^4 + 2b^2c^2 - b^4 + 2a^2c^2 - c^4}{4b^2}} =$ AE, and by Prob. 4. the area is = AE × ½BC = AE × $\frac{1}{2}b$

$$= \sqrt{\frac{2a^2b^2 - a^4 + 2b^2c^2 - b^4 + 2a^2c^2 - c^4}{16}} = \sqrt{\left(\frac{-a^2 + b^2 + c^2 + 2bc}{4} \times \frac{a^2 - b^2 - c^2 + 2bc}{4}\right)} = \sqrt{\left(\frac{a+b+c}{2} \times \frac{-a+b+c}{2} \times \frac{a-b+c}{2} \times \frac{a+b-c}{2}\right)}$$

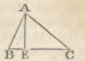
$$= \sqrt{\left\{\frac{a+b+c}{2} \times \left(\frac{a+b+c}{2} - a\right) \times \left(\frac{a+b+c}{2} - b\right) \times \left(\frac{a+b+c}{2} - c\right)\right\}}$$

$= \sqrt{\{s \times (s-a) \times (s-b) \times (s-c)\}}$ (where s is $= \frac{a+b+c}{2} =$ half the sum of the three sides of the triangle).—(See also El. Geom. 41. Cor.)

Cor. 1. The expression $\sqrt{\left(\frac{-a^2 + b^2 + c^2 + 2bc}{4} \times \frac{a^2 - b^2 - c^2 + 2bc}{4}\right)}$ becomes, by reduction, $= \frac{1}{4}\sqrt{\{((b+c)^2 - a^2) \times (a^2 - (b-c)^2\}}$, and by putting s, = the sum, and d = the difference of b and c, we obtain $\frac{1}{4}\sqrt{\{(s^2 - a^2) \times (a^2 - d^2)\}}$, which is another rule for the area.

Cor. 2. If the triangle is equilateral, and its side $= a$, the rule for the area becomes $\sqrt{(\frac{3}{2}a \times \frac{1}{2}a \times \frac{1}{2}a \times \frac{1}{2}a)} = \frac{1}{4}a^2\sqrt{3}$.

Cor. 3. If the triangle is isosceles, and each of the two equal sides be represented by a, and the other by b, the rule will be $\sqrt{\left\{\left(a + \frac{b}{2}\right) \times \frac{b}{2} \times \left(a - \frac{b}{2}\right) \times \frac{b}{2}\right\}} = \frac{b}{2}\sqrt{\left\{\left(a + \frac{b}{2}\right) \times \left(a - \frac{b}{2}\right)\right\}} = \frac{b}{2}\sqrt{\left(a^2 - \frac{b^2}{4}\right)}$.



$$
\begin{aligned}
(255+221+238)\times\tfrac{1}{2} &= 357 \\
357-255 &= 102 \\
&\overline{36414} \\
357-221 &= 136 \\
&\overline{4952304} \\
357-238 &= 119 \\
\text{Ans. } &\overline{589324176}
\end{aligned}
$$

And $\sqrt{589324176} = 24276$ sq. feet $= 2$ ro. 9 per. $5\frac{1}{12}$ yds.

2. Required the area of a triangle, of which the sides are 834, 658, and 423 links.

The half sum	957·5	log. 2·981139
First rem.	123·5	log. 2·091667
Second rem.	299·5	log. 2·476397
Third rem.	534·5	log. 2·727948
		2)10·277151

Area 137586·3 links log. 5·138575
$=$ 1 acre 1 rood 20 perches 4 yards 1·6 feet.

3. Required the area of an isosceles triangle, the equal sides 156, and the third side 78 yards.
Ans. 5890·8 yds. area, $=$ 1 ac. 34 per. 22 yds. 2·8 ft.

4. Required the area of an equilateral triangle, each side 34 inches. Ans. 500·56268 square inches area.

5. Required the area of a triangle, the sides 56, 52, and 60 yards. Ans. 1344 yards.

6. Required the area of a parallelogram, the sides 432 and 263, and a diagonal 342 feet.
Ans. 89945·625 sq. feet, $=$ 2 acres 10 perch. 11·46 yards.

7. Required the area of a triangle, one side 956 links, and each of the other two 627 links.
Ans. 1·9395567 ac. $=$ 1 acre 3 roods 30 perches 10 yds.

8. Required the area of a rhomboid, the sides 57 and 83 poles, and the diagonal 127 poles.
Ans. 3661·8734 per. $=$ 22 ac. 3 ro. 21 per. 26 yds. 3·78 ft.

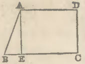
QUADRILATERALS.

Prob. VII. To find the area of a trapezoid.

Rule. Multiply half the sum of the parallel sides by the perpendicular from the one to the other.

That is, $\frac{1}{2}(AD+BC)\times AE =$ the area.

For the triangles into which it may be divided have the same perpendicular.



1. Required the area of the trapezoid ABCD, of which the parallel sides are AD 96 and BC 143, a third side AB 126 yards, and the perpendicular AE 89 yards.

$\frac{1}{2}(143+96)\times 89 = 119\cdot5 \times 89 = 10635\cdot5$ square yards $=$ 2 acres 31 perches $17\frac{3}{4}$ yards.

2. Required the area of a trapezoid, the parallels 786 and 473, another side 1230, and the perpendicular distance 986 links. Ans. 6·20687 ac. = 6 acres 33 perches 3 yards.

3. Required the area of a trapezoid, the parallels 564 and 348, a third side 452, and the perpendicular 397 feet.

Ans. 181032 sq. feet, = 4 acres 24 perches $28\frac{2}{3}$ yards.

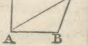
4. Required the area of a trapezoid, the parallels 93 and 157 poles, angle at the latter 62°, and the perpendicular on it 86 poles. Ans. 10750 perches, = 67 acres 30 perches.

5. Required the area of a trapezoid, the parallel sides 386 and 294 feet, an angle at the first 43°, and the perpendicular upon the latter 328 feet.

Ans. 111520 sq. feet, = 2 ac. 2 ro. 9 per. 18 yds. $7\frac{3}{4}$ ft.

Prob. VIII. To find the area of any quadrilateral.

Rule. Divide it into triangles, by drawing a diagonal. Find the areas of the triangles separately, and add them: the sum is the area of the figure.



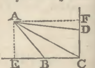
1. Required the area of the quadrilateral ABCD, of which the sides are AC 236, BD 348, AB 392, and DC 427, and the diagonal AD 473 feet.

$$\sqrt{(606\cdot5 \times (606\cdot5 - 348) \times (606\cdot5 - 392) \times (606\cdot5 - 473))} = 67003\cdot90 \text{ DAC}$$

$$\sqrt{(568 \times (568 - 236) \times (568 - 427) \times (568 - 473))} = 50259\cdot08 \text{ ABD}$$

117262·98 square feet.

Ans. 2 acres 2 roods 30 perches 21 yards $6\frac{1}{2}$ feet.



2. Required the area of the trapeze ABCD, the sides AB 218, BC 194, CD 166 yards, and the perpendiculars from A upon BC 136, and upon CD 152 yards.

Ans. 25808 yards, = 5 acres 1 rood 13 perches $4\frac{3}{4}$ yards.

3. Required the area of a trapeze ABCD, the sides AB 842, BC 938, CD 753, AD 826 links, and the angle A 78° 28′.

By trigonometry BD = 1055·05.

Ans. 683884·54 sq. links, = 6 ac. 3 ro. 14 per. $6\frac{1}{2}$ yds.

4. Required the area of a trapeze ABCD, three sides AB 543, BC 428, CD 634 links, and the angles B 74° 40′ and C 84° 20′. By trigonometry BD = 729·077.

Ans. 185392·5 links, = 1 ac. 3 ro. 16 per. 19 yds.

5. Required the area of a trapeze, the four sides 328, 456, 572, and 298, and the diagonal from the angle between the first and second 598 feet.

Ans. 150274·6 sq. ft. = 3 ac. 1 ro. 31 per. 29 yds. 3·85 ft.

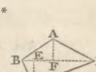
6. Required the area of a trapeze, the diagonal 1268 links, the perpendiculars from one of its extremities upon the opposite sides 784 and 672, and the length of these sides 856 and 548 links.

Ans. 519680 sq. links, = 5 ac. 31 per. 14 yds. 6·858 ft.

PROB. IX. Given a diagonal of a quadrilateral, and the perpendiculars upon it from the opposite angles; to find the area.

RULE. Add the perpendiculars together, and multiply half the sum by the diagonal.

That is, $\frac{1}{2}(AF + CE) \times BD =$ the area.*



1. Required the area of the quadrilateral ABCD, of which the sides are AB 68, BC 54, the diagonal BD 133, and the perpendiculars AF 37 and CE 44 yards.

$\frac{1}{2}(37 + 44) \times 133 = 40·5 \times 133 = 5386·5$ sq. yds. = 1 ac. 18 per. 2 yds.

2. Required the area of the trapeze ABCD, the sides AB 672, BC 834, the diagonal BD 1296, and the perpendiculars AE 418 and CF 550 links.

Ans. 627264 sq. links, = 6 ac. 1 ro. 3 per. 18¾ yds.

3. Required the area of a parallelogram, of which one of the diagonals is 486 feet, and each of the perpendiculars upon it from the opposite angle 126.

Ans. 61236 sq. feet, = 1 acre 1 rood 24 perches 28 yards.

4. Required the area of a trapeze, the diagonal 1356, the angles at one of its extremities 57° and 42°, and the perpendiculars on it 568 and 724 links.

Ans. 888180 sq. links, = 8 ac. 3 ro. 21 per. 2 yds. 6 ft.

5. Required the area of a quadrilateral, of which the diagonals cut one another at right angles, the segments of the one are 328 and 523 feet, and of the other 498 and 672.

Ans. 497835 sq. ft. = 11 ac. 1 ro. 28 per. 18 yds.

* For the quadrilateral = the triangles BAD + BCD = $\frac{1}{2}$(BD × AF) + $\frac{1}{2}$(BD × CE) = $\frac{1}{2}$(AF + CE) × BD.

Prob. X. Given the diagonals of a quadrilateral, and the angle at their intersection; to find the area.

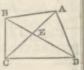
Rule I. Multiply half the product of the diagonals by the natural sine of the angle.

That is, $\frac{1}{2}(AC \times BD) \times \sin. E =$ the area.*

Or add the logarithms of one diagonal, of half the other, and the log. sine of the angle: the sum, lessened by 10 in the index, will be the logarithm of the area.

Note. If the angle made by the diagonals be a right angle, half the product of the diagonals is the area, for the sine of a right angle is 1.

1. Required the area of the quadrilateral ABCD, of which the diagonals are AC 674, BD 398 feet, and the acute angle at E 67° 30′.



Nat. sine of 67° 30′ = ·92388
674
622·69512
199

Ans. Area 123916·32888 square feet, = 2 acres 3 roods 15 perches 4¾ yards.

2. Required the area of a parallelogram, the diagonals 436 and 324 yards, and their angle 48° 38′.

Ans. 53009 yards, = 10 acres 3 roods 32 perches 11 yards.

3. Required the area of a trapeze, the sides 856 and 643, the diagonal joining their extremities 1154, and the other 1345 links, and the angle made by the diagonals 57° 30′.

Ans. 654525·76 sq. links, = 6 ac. 2 ro. 7 per. 7⅓ yds.

4. Required the area of a quadrilateral, the diagonals 72 and 48 feet, and containing a right angle. Ans. 192 yards.

5. The diagonals of a quadrilateral are 567 and 743 links, and they contain an angle of 73° 30′; the side joining their extremities opposite to this angle is 324. What is its area?

Ans. 201966·324 sq. links, = 2 ac. 3 per. 4 yds. 3¾ ft.

6. Required the area of a quadrilateral, the diagonals 924 links and 1256, and they bisect one another in an angle of 52° 30′.

Ans. Area 460358·7912 sq. links, = 4 ac. 2 ro. 16 per. 17 yds. 3·289 ft.

Rule II. If the sides be given instead of the diagonals.

* The triangle $ACD = AED + DEC = \frac{1}{2}AE \times ED \times \sin. E + \frac{1}{2}EC \times ED \times \sin. E = \frac{1}{2}AC \times ED \times \sin. E$; and $ABC = \frac{1}{2}AC \times EB \times \sin. E$.

Add the squares of each pair of opposite sides, and subtract the less sum from the greater: one-fourth of the remainder, multiplied by the natural tangent of the angle contained by the diagonals, will be the area.

That is, $\frac{1}{4}(AB^2 + DC^2 - BC^2 - AD^2) \times \tan. AED$ = the area.*

NOTE 1. This rule fails when the diagonals intersect at right angles, for then the tangent is infinite, and the difference of the aggregate of the squares is nothing.

NOTE 2. If a table of natural tangents be not at hand, multiply by the natural sine, and divide by the natural cos. Or add the log. of half the remainder to the log. tan.: the sum is the log. of the area.

RULE III. When the quadrilateral is in a circle, or its opposite angles are together 180°.

From half the perimeter subtract each side separately; multiply the four remainders successively, and the square root of the product will be the area. (El. Geom. 43.)

That is, if a, b, c, d be the four sides, and s half their sum, $\sqrt{\{(s-a) \times (s-b) \times (s-c) \times (s-d)\}}$ = the area.

7. Required the area of a quadrilateral, of which the sides are 7, 8, 9, and 10 yards, and the angle contained by the diagonals 80°.

$$10^2 + 8^2 = 164$$
$$9^2 + 7^2 = 130$$
$$4 \,|\, 34$$
$$8{\cdot}5$$
$$\text{Nat. tan. } 80° = 5{\cdot}67128$$

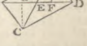
Ans. 48·20588 square yards.

8. Required the area of a trapeze in a circle, the sides 326, 438, 247, and 392 feet.

Ans. 117975·8 sq. ft. = 2 ac. 2 ro. 33 per. 10 yds. $1\frac{1}{3}$ ft.

9. Required the area of a quadrilateral in a circle, the sides 24, 26, 28, and 30 yards.

Ans. 723·98895 yards, = 23 perches $28\frac{1}{4}$ yards.



* Draw AF, CG perpendicular to the diagonal BD. Because $EF = AE \times c$ (putting c for the cosine of the angle at E), and $GE = CE \times c$; therefore $GF = AC \times c$. And because $AB^2 - AD^2 = BF^2 - FD^2$ (El. Geom. 40.) $= BG^2 + GF^2 + 2BG \times GF - FD^2$, and $DC^2 - CB^2 = DG^2 - GB^2 = DF^2 + FG^2 + 2DF \times FG - BG^2$ (El. Geom. 22.); therefore $AB^2 + DC^2 - AD^2 - CB^2 = 2FG^2 + 2FG \times (BG + DF) = 2FG \times (BG + GF + FD) = 2FG \times BD = 2BD \times AC \times c$; and the area $= \frac{1}{2} BD \times AC \times s$. ($s$ = sine AED); therefore $\frac{1}{4}(AB^2 + DC^2 - BC^2 - AD^2)$: the area :: c : s :: rad. : tan. AED. That is, $\frac{1}{4}(AB^2 + DC^2 - BC^2 - AD^2) \times \tan. AED$ = the area.

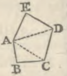
10. Required the area of a quadrilateral, of which the opposite angles are together 180°, the sides 40, 55, 60, 75 chains.
Ans. 3146·427 ch. = 314 ac. 2 ro. 22 per. 25 yds. 1·532 ft.

POLYGONS.

Prob. XI. To find the area of any rectilineal figure.

Rule. Draw diagonals so as to divide the figure into quadrilaterals and triangles, and find the areas of these figures separately, and add them: the sum is the area of the whole.

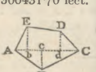
1. Required the area of the pentagon ABCDE, of which the sides are AB 354, BC 432, CD 518, DE 465, and EA 397 feet; and the diagonals AC 574, and AD 612 feet.



By Prob. VI. the triangle . .	ABC is	76338·25
	ACD	137791·11
	ADE	92302·34

Whole figure, 7 ac. 5 per. 16 yds. 6·45 ft. = 306431·70 feet.

2. In order to obtain the area of the field ABCDE, I measured along the diagonal AC; and at b, 326 links from A, I took the perpendicular bE, 97 links: then I measured to c, 543 links from A, where I took the offset cB 354 links; and measuring on to d, 749 links from A, I took the offset dD 158 links. The whole diagonal AC is 987 links. Required the area.



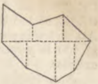
By Prob. VII. $\text{E}b d\text{D} = \frac{1}{2}(\text{E}b + \text{D}d) \times bd =$ 53932·5 links.
By Prob. IV. . . $\text{A}b\text{E} = \frac{1}{2}\text{A}b \times \text{E}b =$ 15811·0
$\text{D}d\text{C} = \frac{1}{2}dc \times d\text{D} =$ 18802·0
$\text{ABC} = \frac{1}{2}\text{AC} \times \text{B}c =$ 174699·0

Area of whole, 2 ac. 2 ro. 21 per. 5·7838 yds. = 263244·5

3. Required the area of the field ABCDEFG, of which are given the sides AB 854 and CD 927 links, the diagonals BG 1167, BF 1037, CF 1284, and CE 1342, and the perpendiculars upon BG are AH 437 and FK 384, upon CF is BL 560, and upon CE are FM 678 and DN 587 links.

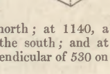
Ans. 1687388·5 links, = 16 ac. 3 ro. 19 per. 24·8534 yards.

4. Measured along a diagonal from east to west, at 230 from its east extremity, a perpendicular to it on the south side, of 356 links, reached to an angle, and at 380 from the same extremity a perpendicular on the north side, of 428 reached an angle. At 673, a perpendicular of 560 reached an angle on the south side; at 812, a perpendicular of 230 reached an angle on the north; at 1140, a perpendicular of 340 reached an angle on the south; and at the west extremity 1270, there was a perpendicular of 530 on the north side.



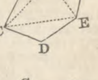
Ans. 873572 sq. lks. = 8 ac. 2 ro. 37 per. 21 yds. 5·7132 ft.

5. In a hexagon are given the sides AB 536, BC 498, CD 620, DE 580, EF 398, and AF 492 links, and the diagonals AC 918, CE 1048, and AE 652 links.



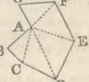
Ans. 656119·53 sq. links, = 6 ac. 2 ro. 9 per. 23 yds. 8·4173 feet.

6. In a heptagon are given the sides AB 294, BC 456, CD 572, DE 640, EF 612, FG 498, and GA 386, and the diagonals AC 540, AD 864, AE 630, and AF 490 links.



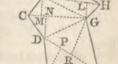
Ans. 646628·38 sq. links, = 6 ac. 1 ro. 34 per. 18·3136 yds.

7. In an octagon, the diagonals are BH 956, BG 874, GC 1078, GD 1178, and DF 1240 links; the sides AB 620, and DE 830; and the perpendiculars AK 326, GL 520, both on BH; those on GC are BM 610, DN 354; and on DF are EP 472, and GR 396 links.



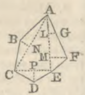
Ans. 1462144 sq. links, = 14 acres 2 roods 19 perches 13 yards.

8. Measured AB 538, and on diagonals from its extremities AG 324, and the perpendicular GF 260, AH 960, and the perpendicular HE 300; the whole diagonal AD 1240. And on the diagonal BD measured BK 460, and the perpendicular CK 350; the whole BD 1310 lks.



Ans. 823855 sq. lks. = 8 ac. 38 per. 5·082 yds.

9. The diagonals are AE 810, AC 930, CE 520; on AE at 245 is perpendicular GL 65, at 440 is perpendicular FM 198, on AC at 300 is perpendicular BN 189, on EC at 400 is perpendicular DP 125 links, all exterior.



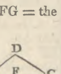
Ans. 400656·18 sq. links, = 4 acres 1 perch 1 yard 4·58 feet.

Prob. XII. To find the area of a regular polygon.

Rule. Multiply half the perimeter by the perpendicular let fall from the centre upon one of the sides.

That is, if n = the number of sides, $\frac{1}{2}n \times AB \times FG$ = the area.*

1. Required the area of the regular pentagon ABCDE, of which the side AB is 250 feet, and the perpendicular from the centre FG 172·05 feet.



Ans. $\frac{250 \times 5}{2} \times 172{\cdot}05 = 625 \times 172{\cdot}05 =$ 107531·25 square feet.

Note. The perpendicular may be found from the side by trigonometry; for 360° divided by twice the number of sides give the angle AFG, and its cotangent multiplied by AG gives FG the perpendicular.

2. What is the area of a regular octagon, the side 237 feet, and the perpendicular 286·084?

Ans. 271207·632 square feet.

3. What is the area of a regular hexagon, the side 356 yards, the perpendicular 308·305? Ans. 329269·74 yards.

4. What is the area of a regular heptagon, the side 237 links?

Ans. 204112·736 sq. links, = 2 ac. 6 per. 17 yds. 5 ft.

5. What is the area of a regular nonagon, the side 147 inches? Ans. 133582·32 sq. in. = 103 yds. 94·32 in.

6. What is the area of a regular decagon, the side 243 feet?

Ans. 454334·737 square feet, = 10 acres 1 rood 28 perches 24 yards 5·737 feet.

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* For the polygon may be divided, by drawing lines from the centre to its angles, into as many triangles as it has sides, all having equal bases and perpendiculars. And if s be the side of a polygon, p the perpendicular, and n the number of sides; then $\frac{1}{2}ps$ will be the area of one triangle, and $\frac{1}{2}nps$ the area of all the triangles, or of the whole polygon.

RULE II. Multiply the square of the side by the multiplier corresponding to the figure in the following Table: the product will be the area.*

Names.	No. of sides.	Angle centre.	Angle FAG.	Perpendiculars.	Multipliers.
Equilateral triangle,	3	120°	30°	0·2886751	0·4330127
Square, . . .	4	90	45	0·5000000	1·0000000
Pentagon, . . .	5	72	54	0·6881910	1·7204774
Hexagon, . . .	6	60	60	0·8660254	2·5980762
Heptagon, . . .	7	$51\frac{3}{7}$	$64\frac{2}{7}$	1·0382607	3·6339124
Octagon, . . .	8	45	$67\frac{1}{2}$	1·2071068	4·8284272
Nonagon, . . .	9	40	70	1·3737387	6·1818242
Decagon, . . .	10	36	72	1·5388418	7·6942088
Undecagon, . .	11	$32\frac{8}{11}$	$73\frac{7}{11}$	1·7028439	9·3656411
Dodecagon, . .	12	30	75	1·8660254	11·1961524

CONSTRUCTION OF THE TABLE. Put t for the tangent of half the angle of any regular polygon whose side is 1, and n for the number of its sides, then rad. : tan. FAG :: AG : FG; that is, $1 : t :: \frac{1}{2} : \frac{1}{2}t =$ FG, the perpendicular; hence $\frac{1}{4}tn$ = the area of the polygon: Thus the perpendicular and the area of a hexagon, whose side is 1, are $\frac{1}{2}$ tan. $60 = 0·8660254$ = the perpendicular, and $\frac{1}{4}$ tan. $60 \times 6 = 0·4330127 \times 6 = 2·5980762$ = the area.

7. Required the area of a regular heptagon, of which the side is 327 feet.

$$
\begin{array}{r}
\text{Tabular multiplier} = 3·6339124 \\
327 \\
\hline
1188·2893548 \\
327 \\
\hline
\end{array}
$$

Ans. 388570·6190196 square feet, = 8 ac. 3 ro. 27 per. $7\frac{3}{4}$ yds.

8. What is the area of an equilateral triangle, the side 436 yards? Ans. 82313·98 yards, = 17 ac. 1 per. 3·73 yds.

9. What is the area of a regular dodecagon, the side 254 poles?

Ans. 722330·968 per. = 4514 ac. 2 ro. 10 per. 29·28 yds.

10. What is the area of a regular undecagon, the side 27 yards?

Ans. 6827·5524 sq. yds. = 1 ac. 1 ro. 25 per. 21 yds. 2·7 ft.

* Regular polygons of the same number of sides being similar, are to each other as the squares of their like sides (El. Geom. 21., Cor. 3.); now the multipliers in the Table are the areas of the polygons to the side 1, whence the rule is manifest.

11. What is the area of a regular decagon, the side 197 inches?

Ans. 298604·549 sq. in. = 7 per. 18 yds. 5 ft. 128·55 in.

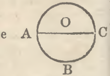
12. What is the area of a regular nonagon, the side 254 feet? Ans. 398826·57 sq. ft. = 9 ac. 24 per. 28 yds.

OF THE CIRCLE.

Prob. XIII. Given the diameter of a circle; to find the circumference.

Rule. Multiply the diameter by $3\frac{1}{7}$, or by 3·1416; or, if greater accuracy be required, by 3·141592653, &c.*

1. Required the circumference of the circle of which the diameter is 356 yards.



356	3·1416	3·1415926536
$3\frac{1}{7}$	356	356
Ans. 1118·8	1118·4096	1118·4069846816

2. Required the circumference of the circle, of which the diameter is 628 links.

Ans. 1972·9248 links, = 1 furlong 38 poles 5 yds. 1·56 in.

3. Required the circumference of a circle, of which the diameter is 7958 miles.

Ans. 25000·79434 miles, = 25000 m. 6 fur. 14 pol. 1 yd.

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* It may be shown that the arc, of which t is the tangent, is $= t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7$, &c. If $t = \frac{1}{2}$, the length of the arc is $\frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7}$, &c. = ·463647609000807, &c.; and if $t = \frac{1}{3}$, the length of the arc will be $\frac{1}{3} - \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} - \frac{1}{7 \cdot 3^7}$, &c. = ·321750554396641, &c.; and the sum of these two arcs is = ·785398163397448, &c. and the tangent of their sum is $\frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}}$ (Trig., Formulæ XXVI.) = 1, which is the tangent of 45°. Having thus found the length of the arc of 45°, multiply it by 4, and the product 3·141592653589793, &c. is the length of the arc of 180° when the radius is 1, or it is the circumference when the diameter is 1. And as the circumferences of circles are to one another as their radii or as their diameters (El. Geom. 45.); hence if we multiply the circumference of a circle, whose diameter is 1, by the diameter of any circle, the product will be the circumference of this circle. See Appendix.

4. Required the circumference of a circle, of which the radius is 512 feet.

Ans. 3216·9984 feet, = 4 furlongs 34 poles 5 yards 1 foot.

5. Required the circumference of a circle, of which the radius is 157 inches.

Ans. 986·4624 inches, = 4 pol. 5 yds. 1 ft. 2·46 inches.

6. Required the circumference of a circle, of which the radius is 38 poles.

Ans. 238·7616 poles, = 5 fur. 38 pol. 4 yds. 6·7968 in.

PROB. XIV. Given the circumference of a circle; to find the diameter.

RULE. Divide the circumference by 3·1416, or multiply it by ·318309886.*

1. Required the diameter of the circle, of which the circumference is 758 yards.

$$7580000 \div 31416 = 241{\cdot}2789$$
$$31831 \times 758 = 241{\cdot}2789$$

Ans. 1 furlong 3 poles $4\frac{3}{4}$ yards.

2. Required the diameter of the circle, of which the circumference is 984 links.

Ans. 313·21693 links, = 12 poles 2 yards $2\frac{3}{4}$ feet.

3. Required the diameter of the circle, of which the circumference is 24855·43 miles. Ans. 7911·72944 miles.

4. Required the diameter of the circle, of which the circumference is 398 ells. Ans. 126 ells 25·4 inches.

5. Required the diameter of the circle, of which the circumference is 928 poles.

Ans. 295·31968 poles, = 7 fur. 15 pol. 2 yds. 5·55 inches.

6. Required the diameter of the circle, of which the circumference is 1043 feet.

Ans. 331·9973 feet, = 20 poles 1·997 feet.

PROB. XV. Given the radius and the number of degrees in an arc of a circle; to find the length of the arc.

RULE. Find the circumference by Prob. XIII., multiply it by the degrees, and divide by 360°.

Or multiply the radius by the number of degrees in the arc, and by ·0174533.†

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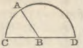
* This Prob. being the converse of Prob. 13. requires no demonstration. The number ·318309886 is the reciprocal of 3·1416.

† It has been shown that, when the radius is unity, half the circumference

1. Required the length of an arc AC of 57°, in a circle of which the radius AB is 38 feet.



$3{\cdot}1416 \times 38 = 119{\cdot}3808$ = the circumference, and $119{\cdot}3808 \times 57 \div 360 = 6804{\cdot}7056 \div 360 = 37{\cdot}80392$ feet.

Also $\cdot0174533 \times 57 \times 38 = 37{\cdot}8038478$ feet.

2. Required the length of an arc of 19° 37′, the radius being 98 yards. Ans. 33·5470317 yards.

3. Required the length of an arc of 134° 18′, the radius 9 feet 4 inches. Ans. 21·87712977 feet.

4. Required the length of an arc of 83° 24′, radius 32 poles.
Ans. 46·579367 poles = 1 fur. 6 pol. 3 yds. 6·715 in.

5. Required the length of an arc of 150°, radius 19 ells.
Ans. 49·741905 ells = 8 falls 1 ell 27·45 inches.

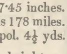
6. Required the length of an arc of 17° 50′, radius 178 miles.
Ans. 55·40259256 miles = 55 mil. 3 fur. 8 pol. $4\frac{1}{2}$ yds.

PROB. XVI. Given the chord of an arc, and its height, or the versine of its half; to find the diameter.

RULE. Divide the square of half the chord by the height, and the quotient added to the height will be the diameter.

That is, $BE^2 \div CE = AE$ (26. El. Geom., Cor. 2.)

1. Given the chord BD 287, and the height CE 78 feet; to find the diameter AC.



$287 \div 2 = 143{\cdot}5$, and $143{\cdot}5^2 \div 78 = 20592{\cdot}25 \div 78 = 264$, and $264 + 78 = 342$ the diameter.

2. Given the chord 178, and height 257 yards.
Ans. 287·821 yards.

3. Given the chord 843, and height 648 links.
Ans. 922·17 links, = 36 poles 4 yds. 2 ft. 7·5864 in.

4. Given the chord 40, and height 12 yards. Ans. $45\frac{1}{3}$ yds.

5. . . . . . . 560, and height 45 links. $1787\frac{2}{9}$ lks.

6. . . . . . . 325, and vers. sine 78 ells. 416·54 ells.

PROB. XVII. Given the chord of an arc, and its height; to find the length of the arc.

RULE. Find the diameter by Prob. XVI.; then, as the diameter is to the chord, so is radius to the sine of half the

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is 3·14159, &c.; hence $\frac{3{\cdot}14159}{180°} = \cdot01745329$, &c. is the length of an arc of 1°; therefore $r \times \cdot0174533$ = the length of 1° to radius r, and consequently if n = the number of degrees in the arc, $\cdot0174533\, rn$ = the length of that arc.

angle measured by the arc (Theor. 1. Trig.), from which find the length of the arc by Prob. XV.

1. Required the length of the arc, of which the chord is 326, and its height 97 feet.

$163^2 \div 97 = 273{\cdot}90722$; and the diameter is 370·90722, and the radius 185·45361.

326 ÷ R.	log. 12·513218
370·90722	log. 2·569265
Sin. 61° 30′ 47·2″	log. 9·943953
2	

123° 1′ 34·4″ = 123·0262°, the angle of the sector.

And 185·45361 × 123·0262 × ·0174533 = 398·2084 the arc.

2. Required the length of the arc, of which the chord is 496, and the height 654 links. Ans. 1807·787 links.

3. Required the length of the arc, of which the chord is 126, and the versed sine 14 inches. Ans. 130·10809 in.

4. Required the length of the arc, of which the chord is 78, and the versed sine 13 yards. Ans. 83·655 yards.

By Approximation. Divide the height by half the chord, and square the quotient. To 3 times this square add 15, and to the sum add 10 times the square. Then as the former sum is to the latter, so is the chord to the arc nearly.

Otherwise, having found the square as before: As $\frac{5}{9}$ of the square + 1 is to $\frac{1}{8}$ of it + 1, so is $\frac{1}{5}$ of it to a fourth number. Subtract this number from 1, multiply the remainder by the square, and to the product add 1·5: this sum, multiplied by $\frac{2}{3}$ of the chord, will produce the arc very nearly.*

5. Required the length of the arc, of which the chord is 40, and the height 6 feet.

$\frac{6}{20} = {\cdot}3$, and $\cdot 3 \times \cdot 3 = \cdot 09$, the square to be used: then $3 \times \cdot 09 + 15 = 15{\cdot}27 : 15{\cdot}27 + \cdot 9 = 16{\cdot}17 :: 40 : 42{\cdot}358$ feet the arc.

By the second approximation, $\cdot 09 \times \frac{5}{9} + 1 : \cdot 09 \times \frac{1}{8} + 1 : \cdot 09 \times \frac{1}{5} : \cdot 0173357$, and $(1 - \cdot 0173357) \times \cdot 09 + 1{\cdot}5 = 1{\cdot}58843979$, and $1{\cdot}58843979 \times \frac{2}{3} \times 40 = 42{\cdot}35843$.

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* Let x = the height, and $2y$ = the chord, then it may be shown that $2y \times (\frac{1}{2} + \frac{x^2}{3y^2} - \frac{x^4}{2.3y^4} + \frac{x^6}{5.7y^6} - \frac{x^8}{7.9y^8} + \&c.)$ = the length of the arc, or putting $v^2 = \frac{x^2}{y^2}$ the series becomes $2y \times (\frac{1}{2} + \frac{1}{3}v^2 - \frac{v^4}{2.3} + \frac{v^6}{5.7} - \frac{v^8}{7.9} + \&c.)$, which is very nearly equal to $2y \times \frac{15 + 13v^2}{15 + 3v^2}$, but more nearly equal to $\frac{4y}{3} \times (\frac{3}{2} + v^2 - v^4 \times \frac{\frac{1}{8}v^2 + 1}{\frac{5}{9}v^2 + 1})$, which are the two approximations given. See Appendix.

6. Required the length of the arc of which the chord is 184, and the height 34 feet. Ans. 200·3217 feet.

7. Required the length of the arc, of which the chord is 246, and the height 534 links. Ans. 1512·00612 links.

Note. When the height is greater than the chord, find the diameter, and from it subtract the height, to get the height of the other segment; find its arc, and subtract it from the circumference.

8. Required the length of the arc of which the chord is 128, height 216 feet. Ans. 602·7963 feet.

9. Required the length of the arc, of which the chord is 76, height 22 links. Ans. 91·98252 links.

Prob. XVIII. Given the radius and the circumference of a circle; to find its area.

Rule. Multiply the radius by half the circumference: the product is the area.*

Note. The area of a semicircle is one-half, and that of a quadrant is one-fourth of the area of a circle.

1. Required the area of the circle, of which the radius is 75, and the circumference 471·24 yards.

$471{\cdot}24 \times \frac{1}{2} \times 75 = 17671{\cdot}5$ square yards, $=$ 3 acres 2 roods 24 perches $5\frac{1}{2}$ yards.

2. Required the area of the circle, of which the diameter is 10, and the circumference 31·416. Ans. 78·54.

3. Required the area of the circle, of which the diameter is 7958, and the circumference 25001 miles.

Ans. $49739489\frac{1}{2}$ miles.

4. Required the area of the circle, of which the diameter is 223, and the circumference 700 yards.

Ans. 39025 sq. yards, $=$ 8 acres 10 perches $2\frac{1}{2}$ yards.

5. Required the area of the circle, of which the diameter is 751, and the circumference 2485 feet.

Ans. $466558\frac{3}{4}$ feet, $=$ 10 ac. 2 ro. 33 per. 21 yds. $5\frac{1}{2}$ ft.

6. Required the area of the circle, of which the diameter is 169, and the circumference 532 inches.

Ans. 22477 inches, $=$ 17 yards 3 feet 13 inches.

Prob. XIX. Given the radius or diameter of a circle; to find the area.

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* The circle is the limit of the polygons inscribed in it and described about it, the circumference is the limit of their perimeters, and the radius the limit of the perpendiculars; and as any polygon is $=$ perpendicular $\times \frac{1}{2}$ perimeter, therefore the circle is $=$ radius $\times \frac{1}{2}$ circumference. (El. Geom. 44.)

RULE. Multiply the square of the radius by 3·1416, or that of the diameter by ·7854.*

1. Required the area of a circle, of which the radius is 78.
Ans. $3{\cdot}1416 \times 78 \times 78 = 19113{\cdot}4944$.

2. Required the area of a circle, of which the diameter is 234 yards.
Ans. 43005·3624 yds. = 8 ac. 3 ro. 21 per. 20·11 yds.

3. Required the area of a circle, of which the diameter is 563 links.
Ans. 248947·4526 links, = 2 ac. 1 ro. 38 per. 9 yds. 5 ft.

4. Required the area of a circle, of which the diameter is 7·5 feet. Ans. 44·17875 feet.

5. Required the area of a circle, of which the radius is 193 yards. Ans. 117021·4584 yds. = 24 ac. 28 per. 14·46 yds.

6. Required the area of a circle, of which the diameter is 9 feet 6 inches. Ans. 70·88235 ft. = 7 yds. 7 ft. 127·06 in.

7. Required the area of a circle, of which the radius is 59 poles.
Ans. 10935·9096 per. = 68 ac. 1 ro. 15 per. 27·5129 yds.

PROB. XX. Given the circumference of a circle; to find the area.

RULE I. Divide the square of half the circumference by 3·1416.

RULE II. Multiply the square of the circumference by ·0795775 to get the area.†

1. Required the area of a circle, of which the circumference is 1284 yards.
$(1284 \div 2)^2 \div 3{\cdot}1416 = 412164 \div 3{\cdot}1416 = 131195{\cdot}569$ yards, = 27 acres 17 perches $1\frac{1}{3}$ yards the area.

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* If R = radius, and D = diameter, then $3{\cdot}1416 \times R = \frac{1}{2}$ circumference; therefore $3{\cdot}1416 \times R^2 = \frac{1}{4} \times 3{\cdot}1416 \times D^2 = {\cdot}7854\ D^2$, will be the area. (El. Geom. 45., Cor. 2.)

† These rules are evident from the preceding; the nnmber ·0795775 is one-fourth of the reciprocal of 3·1416.

If D = the diameter of a circle, C the circumference, A the area, and p = 3·1416; then any two of these being given, the others may be found: Thus,

$$1.\ D = \frac{C}{p} = \frac{4A}{C} = 2\sqrt{\frac{A}{p}}.$$

$$2.\ C = pD = \frac{4A}{D} = 2\sqrt{pA}.$$

$$3.\ A = \frac{pD^2}{4} = \frac{C^2}{4p} = \frac{DC}{4}.$$

$$4.\ p = \frac{C}{D} = \frac{4A}{D^2} = \frac{C^2}{4A}.$$

2. Required the area of a circle, of which the circumference is 1386 links.

Ans. 152867·647 sq. links, = 1 ac. 2 ro. 4 per. 17·794 yds.

3. Required the area of a circle, of which the circumference is 73 feet 8 inches.

Ans. 431·8494 square feet, = 1 perch 17·733 yards.

4. Required the area of a circle, of which the circumference is 625 yards.

Ans. 31084·961 yards, = 6 acres 1 rood 27 per. 18·2 yards.

5. Required the area of a circle, of which the circumference is 1448 feet.

Ans. 166850 feet, = 3 acres 3 roods 12 per. 25 yards 8 ft.

6. Required the area of a circle, of which the circumference is 627 poles. Ans. 31284·223 per. = 195 ac. 2 ro. 4·223 p.

7. Required the area of a circle, of which the circumference is 178 inches. Ans. 2521·33 in. = 1 yd. 8 ft. $73\frac{1}{3}$ in.

PROB. XXI. To find the area of a sector of a circle.

RULE I. If the length of the arc be known, multiply half the arc by the radius.

RULE II. If the angle of the sector be given, find the length of the arc, and work as before. Or find the area of the circle: then, as 360° to the angle of the sector, so is the area of the circle to the area of the sector.*

1. Required the area of a sector, of which the arc is 79, and the radius of the circle 47 yards.

$\frac{79}{2} \times 47 = 1856{\cdot}5$ yards, = 1 rood 21 perches $11\frac{1}{4}$ yards.

2. Required the area of a sector, of which the arc is 17 feet 5 inches, and the radius 22 feet.

Ans. $191{\cdot}58\dot{3}$ square feet, = 21 yards $2{\cdot}58\dot{3}$ feet.

3. Required the area of a sector, of which the angle is 127° 16′, and the radius 133 feet.

The area of the circle is 55571·63245; and this, multiplied by $127\frac{4}{15}$, and divided by 360, gives 19645·601175 sq. feet, = 1 rood 32 perches 4 yards 7·6 feet the area of the sector.

4. Required the area of a sector, of which the angle is 137° 20′, and the radius 456 links.

Ans. 249202·968 links, = 2 acres 1 ro. 38 per. 21·92 yds.

5. Required the area of a sector, of which the angle is 27°, and the radius 97 miles. Ans. 2216·94858 miles.

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* These rules are evident from those for finding the area of the circle.

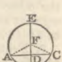
6. Required the area of a sector, of which the arc is 156 yards, the radius 478 feet.

Ans. 37284 feet, = 3 roods 16 perches 28 yards 6 feet.

PROB. XXII. To find the area of a segment.

RULE I. Find the area of the sector which has the same arc with the segment, and from it subtract the area of the triangle contained by the chord and the radii drawn to its extremities, when the segment is less than a semicircle. Otherwise, add these areas, and the remainder or the sum will be the area of the segment.

1. Required the area of the segment ABC, of which the height BD is 6, and the diameter of the circle BE 32 feet.



$\sqrt{26 \times 6} \div 16 = 12{\cdot}49 \div 16 = {\cdot}780625 =$ sin. $51{\cdot}3175°$, and $(51{\cdot}3175 \div 180) \times 3{\cdot}1416 \times 256 = 229{\cdot}289$ sector, and $229{\cdot}289 - 12{\cdot}49 \times 10 = 104{\cdot}389$ square feet the segment.

2. Required the area of the segment, of which the chord is 12, and the diameter 36 yards.

$\frac{6}{18} = {\cdot}33333$ the sine of $19{\cdot}47122°$. Ans. 8·284 yards.

3. Required the area of the segment, of which the chord is 20, and the height 2.

The diameter is 52, the angle 45·2397°. Ans. 26·87885.

4. Required the area of the segment, of which the height is 18, and the radius 56 yards.

Ans. 1024·057 sq. yards, = 33 perches 25·807 yards.

5. Required the area of the segment, of which the chord is 257, the diameter 824 feet. Ans. 3539·4216 sq. feet.

6. Required the area of the segment, of which the chord is 540, and the height 29 links.

Ans. 10464·818 links, = 16 perches 22 yards 4·475 feet.

RULE II. BY A TABLE OF SEGMENTS. Divide the height by the diameter. Look in the table for the quotient in the column of versed sines, and take out the number on the right hand of it in the column of areas, and multiply it by the square of the diameter, and the product will be the area of the segment.*

* This rule is founded on the property, that the versed sines of similar segments are as the diameters of their respective circles, and the areas of those segments are as the squares of the diameters, which is thus proved.

NOTE. If the height be greater than the radius, subtract it from the diameter to get the height of the other segment. Find the area of this segment by the rule, and subtract it from the area of the circle to get the area of the segment required.

7. Required the area of the segment, of which the height is 18, and the diameter of the circle 48.

$18 \div 48 = \cdot375$, opposite to which is $\cdot269014$, and $48 \times 48 \times \cdot269014 = 619\cdot80745$ the area.

8. Required the area of the segment, of which the height is 236, and the diameter 432 links.

$(432 - 236) \div 432 = \cdot4537$, opposite to which is $\cdot346465$ the other segment, and $\cdot785398 - \cdot346465 = \cdot438933$ the segment required from the table. Wherefore $432^2 \times \cdot438933 = 81915\cdot399234$ links the area, $= 3$ roods 11 perches 2 yds.

9. Required the area of the segment, of which the chord is 354, the height 18 feet.

Ans. $4258\cdot128$ feet, $= 15$ perches 19 yards $3\cdot38$ feet.

10. Required the area of the segment, of which the height is 26, and the diameter 298 yards.

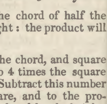
Ans. $2970\cdot2274$ yds. $= 2$ ro. 18 per. 5 yds. $6\cdot546$ feet.

11. Required the area of the segment, of which the radius is 125, and the height 36 links.

Ans. $4351\cdot5625$ links, $= 6$ perches $29\cdot116$ yards.

By Approximation. To the chord add $\frac{4}{3}$ of the chord of half the segment, and multiply the sum by $\frac{2}{3}$ of the height: the product will be the area nearly.

More accurately. Divide the height by half the chord, and square the quotient; and as 5 times the square + 11 to 4 times the square + 33, so is $\frac{1}{21}$ of the square to a fourth number. Subtract this number from 1, and multiply the remainder by the square, and to the product add 5; then multiply this sum by the chord and by the height, and $\frac{2}{15}$ of the product will be the area very nearly. See Appendix.



Let ADBA, *adba* be two similar segments cut off from the similar sectors ADBCA, *adbca* by the chords AB, *ab*, and draw the perpendicular CD to bisect them.

Then by similar triangles CA : C*a* :: CA — DP, or CP : C*a* — *dp*, or C*p* :: DP : *dp*; whence 2CA : 2C*a* : DP : *dp*.

Again, since similar sectors are as the squares of their diameters, and similar triangles as the squares of their like sides, $CA^2 : Ca^2$ :: sector CADBA :: sector C*adba* :: triangle CAB : triangle C*ab* :: segment ADBA = sector CADB — triangle CAB : segment *adba* = sector C*adb* — triangle C*ab*.

If, therefore, d be put = any diameter, and v = the versed sine, then $d : v$ :: 1 (diameter in the Table) : $v \div d$ = the versed sine of a similar segment in the Table, whose area let be called a; then $1^2 : d^2 :: a : ad^2$ = the area of the segment, whose height is v, and diameter d, which is the rule.

12. Required the area of the segment, of which the chord is 50, and the height 3.

Ans. $\sqrt{(25^2+3^2)} = 25{\cdot}1794$ the chord of $\frac{1}{2}$ the segment; then $(50+25{\cdot}1794 \times \frac{4}{3}) \times {\cdot}4 \times 3 = 100{\cdot}287$ the area nearly.

By the second method, $3 \div 25 = {\cdot}12$ and ${\cdot}12^2 = {\cdot}0144$ the square, and $5 \times {\cdot}0144 + 11 = 11{\cdot}072 : 4 \times {\cdot}0144 + 33 = 33{\cdot}0576 : : 1 \div 21 \times {\cdot}0144 = {\cdot}0006\dot{8}5714\dot{2} : {\cdot}0020473328$ the fourth number; then $(1 - {\cdot}0020473328) \times {\cdot}0144 + 5 = 5{\cdot}014370518408$; and this, multiplied by $50 \times 3 \times 2 \div 15$, gives 100·287410368 the area.

13. Required the area of the segment, of which the chord is 178, and the height 14 inches. Ans. 11 ft. 85·528 in.

14. Required the area of the segment, of which the chord is 60, the height 29 poles.

Ans. 10849·8654 perches, = 67 acres 3 roods 9·8654 per.

NOTE. If the height be greater than half the radius, find the area of the segment subtended by the chord of half the arc, and to its double add the area of the triangle contained by the chords. To find the height of this small segment: Having found the chord of half the arc for the chord of it, multiply it by half the chord of the given segment, and subtract the product from the square of the chord of half the arc: the remainder, divided by twice the height, will give the height of the small segment.

15. Required the area of the segment, of which the chord is 56, and the height 32 inches.

Ans. 3437·4741 sq. inches, = 2 yds. 5 ft. 125·474 in.

16. Required the area of the segment, of which the chord is 68, and the height 48 yards.

Ans. 2886·377 square yards, = 2 ro. 15 per. 12·627 yds.

17. Required the area of the segment, of which the chord is 24, and the height 15 poles.

Ans. 303·529 sq. poles, = 1 ac. 3 ro. 23 per. 16 yds.

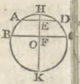
18. Required the area of the segment, of which the chord is 256, and the height 152 feet.

Ans. 32221·938 ft. = 2 roods 38 perches 10 yds. 6·44 ft.

PROB. XXIII. To find the area of a zone, or of a part of a circle intercepted between two parallels.

RULE. I. Find the areas of the segments cut off by the chords, and their difference will be the area of the zone.

RULE II. Find the area of the segment cut off by the straight line joining the extremities of the chords, and the area of the trapezoid formed by the chords; and the double of the segment added to the trapezoid will be the area of the zone.



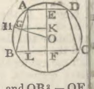
1. Required the area of the zone ABCD, of which the distance OE of the chord AD from the centre is 44, the distance OF 13, and the diameter HK 104 yards.

$(52-13) = 39 \div 104 = \cdot375$ vers. sin. to seg. $\cdot269014$
$(52-44) = \;\;8 \div 104 = \cdot076923$ . . . . $\cdot027780$

Difference of segments, $\cdot241234$
$104^2 =$ 10816

Area of the zone 2 ro. 6 per. 7·679 yds. = 2609·17905 yds.

2. Required the area of a zone, of which the chords ar AD 15 and BC 20, and their distance EF $17\frac{1}{2}$ feet.



Let O be the centre of the circle, join AB, and draw OG perpendicular to AB, meeting the circle in H. Draw GK parallel to AD, and AL parallel to EF; then $GK = \frac{1}{2}(AE + BF) = 8\frac{3}{4}$, and $BL = BF - AE = 2\frac{1}{2}$. Also, $AL : LB :: GK : KO = 1\frac{1}{4}$ (El. Geom. 19.), and $OF = FK - KO = 7\frac{1}{2}$. Now $OG^2 = OK^2 + KG^2$ (El. Geom. 22., Cor. 2.), therefore $OG = 8\cdot838834765$; and $OB^2 = OF + FB^2$ (El. Geom. 22., Cor. 2.), therefore OB or $OH = 12\cdot5$, an $GH = 3\cdot661165$, which, divided by 25, gives $\cdot1464466$ for the verse sine, for which the area is $\cdot071350$; and this, multiplied by 25^2, gives $44\cdot59346$, the area of the segment AHB, and the trapezoi $ABCD = \frac{1}{2}EF \times (AD + BC) = 306\cdot25$, which, added to twice th segment, gives the zone 395·4369 square feet.

3. Required the area of a zone, having the parallel chord 96 and 60, and their distance 26 yards.

Ans. 2136·76 sq. yards, = 1 ro. 30 per. 19·26 yds

4. Required the area of a zone, the parallels each 36, an their distance 84 feet.

Ans. 6380·828 sq. feet, = 23 per. 13 yds. 2·327 feet

5. Required the area of a zone, the parallels 136 and 68 and their distance 248 feet.

Ans. 55655·1965 sq. ft. = 1 ac. 1 ro. 4 per. 12 yds. 8·2 ft

6. Required the area of a zone, the parallels 157 and 216 and their distance 128 yards.

Ans. 15571·33794 yds. = 3 ac. 34 per. 22 yds. 7·54 feet

7. Required the area of a zone, the parallels 247 and 192 and their distance 368 feet.

Ans. 135521·597 feet, = 3 acres 17 per. 23 yds. 6·35 feet

8. Required the area of a zone, the parallels 32 and 40, and their distance 72 inches.

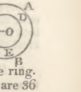
Ans. 4890·236 inches, = 33 feet $138\frac{1}{4}$ inches.

PROB. XXIV. To find the area of a ring contained by two concentric circles.

RULE I. Multiply the sum of the diameters by their difference, and then by ·7854.

RULE II. If the circumferences or similar arcs of the circles be given, multiply half their sum by the difference of the radii: the product will be the area of the ring, or of the part of it contained by the similar arcs.*

1. Required the area of the ring ABC — DEF, of which the diameters are 10 and 6, or OC 5 and OF 3.



$(10+6)(10-6)\times{}$·7854 = 50·2656 the area of the ring.

2. Required the area of the ring, of which the radii are 36 and 24 feet. Ans. 2261·952 sq. feet, = 8 per. $9\frac{1}{3}$ yds.

3. Required the area of the ring, of which the radii are 10 and 6, and similar arcs 15 and 9. Ans. 48.

4. Required the area of the ring, of which the radii are 157 and 128 yards.

Ans. 25965·324 sq. yards, = 5 ac. 1 ro. 18 per. 10 yards ·416 feet.

5. Required the area of the ring, of which the diameters are 246 and 228 inches.

Ans. 6701·0328 inches, = 46 feet 77·0328 inches.

PROB. XXV. To find the area of a space bounded on one side by a curve-line.

RULE I. Let perpendiculars be erected upon the base, so numerous, that the part of the curve between any two nearest to one another shall differ very little from a straight line. Then add the perpendiculars at the extremities of the base, if there are any, and to half their sum add the rest of the perpendiculars. Multiply the sum by the base, and divide the product by the number of parts into which the base is divided by the perpendiculars: the quotient will be the area nearly.†

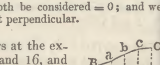
* The ring is evidently equal to the difference of the areas of the two circles; consequently let D and d be = the diameter, and $a =$ ·7854, the ring will be $aD^2 - ad^2 = a\times(D+d)\times(D-d)$, which affords the first rule. Again, the circumferences C, c are $= 4aD$, $4ad$; whence $a\times(D+d) = \frac{1}{4}C + \frac{1}{4}c$. Substituting this in the last expression we obtain $a\times(D+d)\times(D-d) = (\frac{1}{4}C+\frac{1}{4}c)\times(D-d) = (\frac{1}{2}C+\frac{1}{2}c)\times(\frac{1}{2}D-\frac{1}{2}d)$, which is the second rule.

† The rule supposes the figure to be divided into trapezoids, and would be exact if the breadths of the trapezoids were all equal. But the common rule

Rule II. If the distances between the perpendiculars be equal, the curvature, if single, may be considered as parabolical. And, taking care to have an odd number of perpendiculars, add the first and last perpendiculars into one sum, the second, fourth, &c. into another, and all the rest into a third sum; then add the first sum, twice the third, and four times the second sum together, multiply this by the base, and divide by three times the number of parts into which the base is divided. The quotient is the area.*

Note. When the offset meets the base at one end, the perpendicular there must be considered $= 0$; and when it meets the base at both ends, the first and last must both be considered $= 0$; and we must always begin with the smallest perpendicular.

1. Suppose the perpendiculars at the extremities of the base to be 10 and 16, and the other perpendiculars to be 11, 14, 16, and the base to be 20 feet.



By Rule I. $(10+16) \div 2 = 13$ and $(13 + 11 + 14 + 16) \times 20 \div 4 = 270$ square feet the area.

By Rule II. $\{(10 + 16) + (11 + 16) \times 4 + 14 \times 2\} \times 20 \div 12 = 162 \times 20 \div 12 = 270$ square feet the area.

2. A curve-lined space meets the base at one of its extremities, and the perpendicular at the other extremity is 96; the other perpendiculars are 83, 70, 64, 51, 38, 25, and the base 325 links. What is the area?

Ans. $17596\frac{3}{7}$ square links; by Rule II. 16250 square links.

3. An offset meets the base at both extremities; the base is 252 links, and the perpendiculars are 24, 36, 42, 54, 67, 76, 58, 49, 33, and 19. Required the area.

Ans. $10492\frac{4}{11}$ sq. links; and by Rule II. 10416 sq. links.

4. Perpendiculars were raised from the base to a curve; those at the ends were 364 and 578, the others were 396, 418, 453, 512, 554 links, the base 1260 links.

Ans. 588840 square links; by Rule II. 588980 square links.

5. A curve meets the base at one extremity, the base is 2364, the perpendicular at the other extremity 758, and the others are 642, 587, 524, 432, 417, and 335 links.

Ans. $1119860\frac{4}{7}$ sq. links; by Rule II. $1051417\frac{1}{7}$ sq. links.

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is to add all the perpendiculars, and to multiply by the base, and divide by the number of perpendiculars; which is not much easier, and gives the answer sometimes considerably erroneous. Thus the third example would come to 11541·6.

* For the demonstration of this rule see Appendix.

MENSURATION OF SOLIDS.

DEFINITIONS.

1. A PRISM is a solid of which the ends are equal, similar, and parallel rectilineals; and the other sides are parallelograms.

NOTE. If the ends are parallelograms, the prism is called a *Parallelopiped;* and when all its sides are squares, it is called a *Cube.*

2. A CYLINDER is a round solid of uniform thickness, of which the bases are equal and parallel circles.

3. A PYRAMID is a solid which has a rectilineal figure for its base, and its sides are triangles, which have a common vertex.

4. A CONE is a round solid, which has a circle for its base, and tapers uniformly to a point at the top.

5. A SEGMENT of a solid is the part cut off from the top by a plane parallel to its base.

6. A FRUSTUM is the part left at the bottom after the segment has been cut off.

7. A WEDGE has a rectangle for its base, and its opposite side is a straight line parallel to the base, called its *Edge.*

8. A PRISMOID has any dissimilar, parallel, plane figures, of the same number of sides, for its two ends, and its upright sides trapezoids.

9. A SPHERE, or GLOBE, is a solid bounded by a curve surface, every point of which is equally distant from a point within it called the centre.

NOTE. A Sphere may be conceived to be generated by a semicircle revolving about its diameter.

10. THE AXIS or DIAMETER of a Sphere is a straight line passing through the centre, and both ends terminating in the surface.

11. A CIRCULAR SPINDLE is a solid generated by the revolution of a segment of a circle about its chord.

12. An UNGULA, or HOOF, is a part of a solid cut off by a plane inclined to the base.

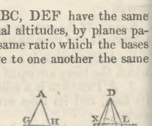
13. The **Solid Content** of a body is the number of cubical inches, feet, &c. which the body contains.

14. A **Cubical Inch** is a solid contained by six square inches; or it is a solid, of which the length, breadth, and thickness, are each of them an inch. And the same is to be understood respecting a cubical foot, yard, &c.

TABLE OF CUBICAL MEASURE.*

1728	cubical inches	make	1 cubical foot.
27	. . feet	. .	1 yard.
$166\frac{3}{8}$	. . yards	. .	1 pole.
64000	. . poles	. .	1 furlong.
512	. . . furlongs	.	1 mile.

Theorem I. If two solids, ABC, DEF have the same height, and if their sections, at equal altitudes, by planes parallel to the bases, have always the same ratio which the bases have to one another, the solids have to one another the same ratio which their bases have.



Let the section GH be at the same height with XL, and MN with OP. Upon their planes make the prisms or cylinders GQ, MS, and XR, OT. These solids have the same altitude, and therefore GQ : XR : : base GH : XL ; that is, : : base BC : EF. For the same reason, MS : OT : : base BC : EF. In the same way it may be proved, that any series of prisms inscribed in ABC, is to a like series in DEF, as the base BC to EF, and the same of the circumscribed prisms. But the inscribed series may be taken of so small altitudes, that they will differ from the circumscribed by less than any given magnitude. The ratio of the prisms is there-

* Formerly 231 cubical inches made a wine gallon, 282 cubical inches made an ale gallon, 2150·42 cubical inches made a malt bushel, and 104·2 such inches made a Scotch pint.

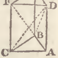
All these measures are now laid aside by act of Parliament, and the only legal standard for measuring both liquid and dry goods is declared to be the imperial gallon, containing 10 pounds avoirdupois weight of distilled water weighed in air at the temperature of 62 degrees of Fahrenheit's thermometer, the barometer being at 30 inches; each avoirdupois pound containing 7000 troy grains. It is declared that this gallon is to contain 277·274 cubic inches of rain water. A pint is the eighth part of a gallon, 8 gallons make a bushel of 4 pecks, and 8 bushels make a quarter. Hence a wine gallon is 0·8331109 imperial gallon, an ale gallon 1·017045 imperial gallon, a Winchester bushel 0·969448 imperial bushel, a Scotch wheat firlot 0·998256 imperial bushel, a Scotch barley firlot 1·4562794 imperial bushel, and a Scotch pint 0·375814 imperial gallon.

fore the ratio of the solids. Hence the solids are to one another as their bases.

Cor. 1. If two pyramids or two cones be upon equal bases and of the same altitude, they are equal.

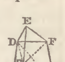
Cor. 2. A cone is equal to a pyramid of equal base and altitude with it.

THEOREM II. Every triangular prism ABCDEF may be divided into three equal triangular pyramids.



Join FB, BD, DC; and because the triangle ADC = FDC, the pyramid ADCB = FDCB, but because the triangle EBF = FBC, the pyramid EBFD = FBCD, or FDCB; therefore the prism ABCDEF is divided into three equal pyramids, ADCB, FDCB, and EBFD.

Cor. 1. Hence a pyramid is the third part of a prism of equal base and altitude with it.

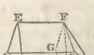


Cor. 2. The frustum of a triangular pyramid may be divided into three triangular pyramids, which are in continued proportion. For ADCB : FDCB : : ADC : FDC : : AC : DF; that is, : : BC : EF : : BCF : BEF, or : : BCFD = FDCB : BEFD.

Cor. 3. The frustum of a pyramid is equal to two pyramids upon its two bases, and a pyramid of which the base is a mean proportional between the bases of the frustum, and all of the same altitude with the frustum.

Cor. 4. If A and a be similar sides of the bases, and A^2p the area of the one, a^2p will be the area of the other, and Aap the area of the mean; and if h be the height, the content of the frustum will be $(A^2 + Aa + a^2)\,ph = \{(A + a)^2 - Aa\}\,ph$.

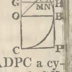
THEOREM III. A wedge ABCDEF, of which the edge EF is equal to the length AD of the base, is a triangular prism, and if the edge and length be unequal, the difference between the wedge and the prism is a pyramid DGHCF, of which the base is a parallelogram, and the altitude is the perpendicular from the edge upon the base.



Cor. 1. Hence, if $AB = a$, $EF = BC = b$, and $CH = d$, and the perpendicular from E upon the base $= p$, the wedge or prism $ABCDEF = a \times \frac{1}{2}bp$, and the pyramid $CDGHF = a \times \frac{1}{3}dp$, and therefore the wedge $ABHGEF = ap \times (\frac{1}{2}b \mp \frac{1}{3}d) = \frac{1}{6}ap \times (3b \pm 2d) = \frac{1}{6}\,ap \times \{b + 2 \times (b \mp d)\}$.

Theorem IV. A sphere is two-thirds of its circumscribing cylinder.

Let ABC be a semicircle, AC the axis, OB perpendicular to AC, describe the parallelogram ADPC, and join DO. Draw EF, GH parallel to OB, and let EF meet the circumference in L, and OD in K, and complete the rectangles GMKF, and GNLF. If the figure revolve about AC, the semicircle ABC will describe a sphere, ADPC a cylinder, ADO a cone. Also the figures GE, GL, and GK, will describe cylinders. Now, $AF \times FC = FL^2$, and $FK^2 = FO^2$; therefore $FL^2 + FK^2 = AO^2 = EF^2$; therefore the cylinder described by GL and GK are together = cylinder described by GE. In the same manner, every cylinder in the hemisphere, with the corresponding cylinder about the cone, is equal to the corresponding part of the cylinder described by AB, and the number of these cylinders may be increased, so that altogether they will not differ from the hemisphere and cone; therefore the hemisphere and cone are, together, equal to the circumscribing cylinder, and the cone is $\frac{1}{3}$ of the cylinder; therefore the sphere is $\frac{2}{3}$ of its circumscribing cylinder.



Cor. 1. Hence any part of the sphere, with the corresponding part of the cone, is equal to the corresponding part of the cylinder. Thus the segment described by ALF, together with the frustum described by ADKF, is equal to the cylinder described by ADEF. Let $AC = a$, $AF = h$, $FL = c$, and $FO = \frac{1}{2}a - h = FK$. Then the cylinder described by $FD = a^2hp$ ($p = \cdot 7854$), and the conical frustum described by $ADKF = (3a^2 - 6ah + 4h^2) \times \frac{1}{3}hp$, and taking their difference, we have the segment described by ALF $= (3a - 2h) \times \frac{2}{3}h^2p$.

And because $(a - h)\,h = c^2$; therefore $3a - 2\,h = \frac{3c^2 + h^2}{h}$.

By substituting this expression, the segment becomes $(3c^2 + h^2) \times \frac{2}{3}ph$.

Again, the zone described by OFLB, together with the cone described by OFK, is equal to the cylinder described by OE; therefore making $OF = FK = m$, the cylinder $= a^2mp$, and the cone $= \frac{1}{3}m^2 \times mp$; therefore the zone described by $OFLB = (a^2 - \frac{1}{3}m^2)\,mp$, or if $a^2 - m^2 - FL^2 = d^2$, the zone is $(2\,a^2 + d^2)\,\frac{1}{3}mp$.

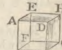
Again, from the zone described by $OFLB = (r^2 + \frac{2}{3}\,h^2) \times ph$ (where $r = FL$, $h = OF$, and $p = 3\cdot 1416$), subtract the zone described by OGNB $= (R^2 + \frac{2}{3}H^2) \times pH$, (where $R = GN$ and $H = OG$), the remainder will be the zone described by GFNL, which, when reduced by putting $m = FG = h - H$, and considering that $r^2 + h^2 = R^2 + H^2$, will become $(3R^2 + 3r^2 + m^2) \times \frac{1}{6}mp$.

Cor. 2. The sphere may be considered as a cone, of which the base is the surface of the sphere, and its vertex the centre; therefore, putting S = surface, the sphere is $= \frac{1}{3}rS$, but the sphere is = a cone upon one of its great circles, of which the height is $4r$, and is therefore $= \frac{4}{3}\, r \times r^2p$, $(p = 3{\cdot}1416)$; so that $\frac{4}{3}r \times r^2p = \frac{1}{3}rS$; therefore $S = 4r^2p = 4$ times the area of one of its great circles.

Prob. I. To find the surface of a prism.

Rule. Find the area of one of its ends, and to its double add the sum of the areas of the parallelograms.*

1. Required the surface of a cube, upon a line of 37 inches.

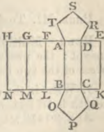


Ans. $37 \times 37 = 1369$ sq. in. area of one face, and $1369 \times 6 = 8214$ square inches whole surface.

2. Required the surface of a rectangular parallelopiped, of which the length is 11 feet, and each side of the base 27 inches. Ans. 109·125 sq. feet.

3. Required the surface of a pentagonal prism, the length 14 feet, and each side of the base 33 inches. Ans. 218·5222 ft.

TO FORM A PRISM WITH PASTEBOARD.



Let ABCD be one of the parallelograms of which the sides are compounded, AB the length, and AD a side of the base. Extend AD and BC, and make the parallelograms DK, AL, FM, &c. each equal to AC, and upon AD and BC make figures equal to the bases.

Then if the figure thus formed be cut out of the pasteboard, and folded at the sides of the parallelograms till they meet, the prism will be formed, and its surface is the figure cut out.

4. Required the surface of a chest, of which the length is 7 feet 8 inches, the breadth 4 feet 7 inches, and the depth 2 feet 9 inches. Ans. 137 feet 7 inches 10 parts.

* The truth of this rule is manifest from the first definition.

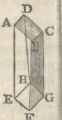
5. Required the surface of a triangular prism, of which the length is 13 feet, and the sides of the base 23, 34, and 19 inches. Ans. 85·224091 square feet.

Prob. II. To find the solid content of a prism.

Rule. Find the area of one of the ends, and multiply it by the length or perpendicular height.*

1. Required the solid content of a triangular prism, of which the height is 9 feet, and each side of the base 34 inches.

Ans. Tabular Mult. $0·4330127 \times 34^2 \times 9 \div 144 = 4505·0641308 \div 144 = 31·2851676$ cubic feet the content.



2. Required the solid content of a rectangular cistern, of which the length is 3 feet 2 inches, the breadth 2 feet 8 inches, and the depth 2 feet 6 inches. Ans. 21 feet 1 inch 4 parts.

3. Required the solid content of a heptagonal prism, of which the length is 21 feet, and each side of the base 43 inches. Ans. 979·8693346 cubic feet.

4. Required the solid content of a pentagonal prism, the length 23 feet, and each side of the base 54 inches. Ans. 801·312349 cubic feet.

5. Required the solid content of a quadrilateral prism, the length 19 feet, the sides of the base 43, 54, 62, and 38, and the diagonal between the first and second 70 inches. Ans. 306·04744 cubic feet.

Prob. III. To find the surface of a cylinder.

Rule. Multiply the circumference of the base by the height: the product is the curve surface, to which add the areas of the two bases.†

1. What is the curve surface of a cylinder, of which the length is 16 feet, and the diameter of the base 27 inches?

Ans. $3·1416 \times 2\frac{1}{4} \times 16 = 113·0976$ square feet the surface.

---

* If the height be one foot, it is evident that the solid will contain as many cubical feet as there are square feet in the base; if the height be two feet, the solid will contain twice as many cubical feet; if the height be three feet, it will contain three times as many, and so on.

† The truth of this rule is evident; for, if the circumference of the base be supposed to move in a direction parallel to itself, it will thus generate the convex surface of the cylinder.

2. Required the whole surface of a cylinder 13 feet long, and the circumference of its base 57 inches.

Ans. 65·3409347 square feet.

3. Required the whole surface of a cylinder, the length 12 feet, and the radius of the base 23 inches. Ans. 24133·7664 in.

4. Required the curve surface of a cylinder, the length 15 feet, and the diameter of the base 33 inches.

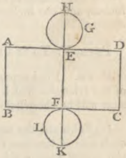
Ans. 129·591 square feet.

5. How often must a cylinder, 5 feet 3 inches long, and the diameter of its base 21 inches, revolve, to roll an acre?

Ans. 1509·18 times.

TO FORM A CYLINDER WITH PASTEBOARD.

Find the circumference of the base, and make the rectangle ABCD, of which AD is the circumference, and AB the length of the cylinder; and draw EF parallel to AB, and make EH, FK, each the diameter of the base, and describe the circles EGH and FKL. The figure thus formed being cut out of the paper, and bended round, so that AB meet CD, will form the cylinder. The area of the figure is the surface of the cylinder.



Prob. IV. To find the solid content of a cylinder.

Rule. Find the area of the base, and multiply it by the perpendicular height or length.*

1. Required the solid content of the cylinder, of which the length is 9 feet, and the circumference of the base 6 feet.

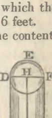
Ans. $0·795775 \times 36 \times 9 = 25·7831$ cubic feet the content.

2. Required the solid content of the cylinder, of which the length is 11 feet, and the diameter of its base 38 inches.

Ans. $·7854 \times 3\frac{1}{6} \times 3\frac{1}{6} \times 11 = 86·63398$ cubic feet.

3. Required the solid content of an oblique cylinder, the axis of which makes an angle of 75° with the base, the axis and the circumference of the base being each 20 feet.

Sin. $75° = ·965926 \times 20 = 19·31852$ the perpendicular height. Ans. 614·9278 cubic feet.



* This is proved the same way as in Prob. II.

4. An upright cylinder 20 feet high, and the diameter of the base 3 feet, is cut by a plane parallel to the axis, and 12 inches from it. Required the content of each of its segments.

Ans. 15·48738 and 125·88462 cubic feet.

5. Required the solid content of an upright cylinder 24 feet high, and the diameter of the base 27·713 inches.

Ans. 100·532253 cubic feet.

6. Required the solid content of an oblique cylinder, of which the axis inclines in an angle of 60°, the length 25 feet, and the diameter of the base 30 inches.

Ans. 106·2775055 cubic feet.

7. Required the solid content of an oblique cylinder, of which the length is 18 feet, the diameter of the base 31·305 inches, and the inclination of the axis 56°.

Ans. 79·7632337 cubic feet.

Prob. V. To find the surface of a pyramid.

Rule. Find separately the area of the base, and the areas of the triangles which constitute its sides, and add them: the sum will be the whole surface.

1. Required the surface of a triangular pyramid, of which each side of the base is 32 inches, and the perpendicular from the vertex upon a side of the base $11\frac{1}{2}$ feet.

Ans. Tabular Mult. $\cdot 4330127 \times 32^2 \div 144 = 3\cdot 0792$ feet area of the base; and 11 ft. 6 in. $\times$ 1 ft. 4 in. $\times 3 = 46$ feet area of the sides; then $3\cdot 0792 + 46 = 49\cdot 0792$ square feet whole surface.

2. What is the surface of a square pyramid, each side of the base 28 inches, and the perpendicular upon a side from the vertex 9 feet? Ans. $47\frac{4}{9}$ square feet.

3. What is the surface of a pentagonal pyramid, the slant perpendicular from the vertex 10 feet, and a side of the base 26 inches? Ans. 62·24335 square feet.

4. What is the whole surface of a triangular pyramid, of which the slant height is 18 feet, and each side of the base 42 inches? Ans. 99·80425 square feet.

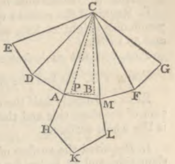
5. What is the whole surface of a hexagonal pyramid, each side of the base being 36 inches, and the slant height 20 feet?

Ans. 203·383 feet.

6. What is the whole surface of a rectangular pyramid, the sides of the base 40 and 30 inches, and the slant height upon the greater side 20·04, and upon the less side 20·07 feet?

Ans. $125\cdot 308\dot{3}$ feet.

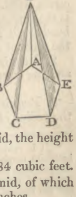
TO FORM A PYRAMID WITH PASTEBOARD.



Draw AB, and BC perpendicular to it; make AB the radius of the circle circumscribing the base, and PB the radius of the inscribed circle. Then if the axis of the pyramid be given, make BC equal to it; or if the slant perpendicular be given, make PC equal to it; or if the slant side be given, make AC equal to it, and from C describe an arc through A; in this arc place AD, DE, AM, MF, &c. each equal to a side of the base, then join CD, CE, CM, &c. and upon AM make the base AHKLM. This figure being cut out, and folded along the lines till the sides meet, will form the pyramid, and its area is therefore the surface.

Prob. VI. To find the solid content of a pyramid.

Rule. Find the area of the base, and multiply it by the height, and one-third of the product will be the content. (Theorem II. Cor. 1. page 199.)



1. Required the content of a square pyramid, of which the perpendicular height is 14 feet, and a side of the base 43 inches.

	F.	I.	
	3	7	
	3	7	
	12	10	1
$14 \times \frac{1}{3} =$	4	8	
Ans. Content	59	11	

2. Required the content of a pentagonal pyramid, the height 12 feet, each side of the base 24 inches.

Ans. 27·5276384 cubic feet.

3. Required the content of a hexagonal pyramid, of which the axis is 9 feet, and each side of the base 29 inches.

Ans. $2{\cdot}5980762 \times 29 \times 29 \times 9 \times \frac{1}{3} \div 144 = 45{\cdot}52046$ cub. feet.

4. Required the content of an octagonal pyramid, the axis 3 feet, and each side of the base 35 inches.

Ans. 177·9923684 cubic feet.

5. Required the content of a triangular pyramid, the height 22 feet, and each side of the base 39 inches.

Ans. 33·540442 cub. ft. = 33 cubic feet 933·884 inches.

6. Required the content of a triangular pyramid, the perpendicular height 24 feet, and the sides of the base 34, 42, and 50 inches.

Ans. 39·2354 cubic feet = 39 cubic feet 406·7712 inches.

Prob. VII. To find the surface of a cone.

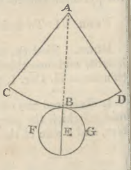
Rule. Multiply half the circumference of the base by the sum of the slant side and the radius of the base: the product is the whole surface.*

1. Required the surface of a cone, which has 10 feet for its slant side, and 32 inches for the diameter of the base.

Ans. $3.1416 \times 1\frac{1}{3} = 4{\cdot}1888$ half the circumference of the base, and $(1\frac{1}{3} + 10) \times 4{\cdot}1888 = 47{\cdot}4731$ sq. feet surface.

TO FORM A CONE WITH PASTEBOARD.

Multiply 180° by the radius of the base, and divide it by the slant side to get the angle at the vertex. Draw AB, and make BAC and BAD each equal to the angle at the vertex. Make AB the slant side, and from A describe the arc CBD. Make BE the radius of the base, and from E describe the circle BFG. The figure thus formed is the surface of the cone; and if it be bended till AC meet AD, it will give the form of the cone.



2. Required the surface of a cone, the slant side 14 feet, and the circumference of the base 92 inches.

Ans. 58·3440553 square feet.

3. Required the surface of a cone, the slant side 10 feet, and the radius of the base 2 feet 5 inches. Ans. 94·2698 sq. ft.

* It is evident that, if the circumference of the base be divided into an indefinite number of equal parts, and straight lines be drawn to the vertex through each point of division, the cone becomes a pyramid of an indefinite number of faces, the perpendicular height being the slant height of the cone, and the limit of the sum of the sides of its base, equal to the circumference of the base of the cone. Now, the sum of the areas of the triangles which constitute the pyramid (or the curve surface of the cone) is equal to their height, multiplied by half the sum of their bases; that is, the slant height of the cone multiplied by half the circumference of its base; but the area of the base is also equal to half the circumference multiplied by the radius; whence the whole surface is equal to half the circumference of the base multiplied by the sum of the slant side and the radius of the base.

4. Required the surface of a cone, the slant side 18 feet, and the diameter of the base 42 inches. Ans. 108·58155 sq. ft

5. Required the surface of a cone, the slant side 9 feet, and the diameter of the base 36 inches. Ans. 49·4802 square feet.

PROB. VIII. To find the solid content of a cone.

RULE. Multiply the area of the base by the perpendicular height, and one-third of the product will be the content. (Theorem I. Cor. 2. and Theorem II. Cor. 1. p. 199.)

1. Required the content of the cone ABC–D, of which the perpendicular height DO is 14 feet, and the diameter AC of the base 43 inches.

Ans. $\cdot 7854 \times 43^2 \div 144 = 1452\cdot 2046 \div 144 = 10\cdot 084754$ sq. feet area of the base; then $10\cdot 084754 \times 14 \div 3 = 141\cdot 186556 \div 3 = 47\cdot 062185$ cubic feet content.

2. Required the content of a cone, of which the axis is 9 feet, and the circumference of the base 7 feet 10 inches.

Ans. 14·6488914 cubic feet.

3. Required the content of a cone, the slant side 15 feet, and the radius of the base 19 inches.

The axis is 178·994413 inches. Ans. 39·1591 cubic feet.

4. Required the content of a cone, the axis 18 feet, and the diameter of the base 42 inches. Ans. 57·7269 cub. feet.

5. Required the content of a cone, the diameter of the base 12·7324 feet, and the perpendicular height 107·923 feet.

Ans. 4580·40809 cubic feet.

PROB. IX. To find the surface of a frustum of a pyramid or cone.

RULE. Add the perimeters or circumferences of the two bases together, and multiply half the sum by the slant height for the upright or curve surface, to which add the areas of the two bases to get the whole surface.*

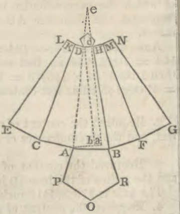
1. Required the surface of a frustum of a square pyramid, the sides of the bases being 40 and 26 inches, and the slant height 10 feet.

* This rule is evident, for the surface is composed of a number of equal trapezoids, the sums of whose parallel sides are equal to the perimeters of the ends of the frustum, and whose common height is the slant height of the frustum.

Ans. First $(40 + 26) \times 2 \times 10 = 1320$ in. surface of slan sides. Then $40 \times 40 \div 12 = 1600 \div 12 = 133\cdot\dot{3}$ inche the one base, and $26 \times 26 \div 12 = 676 \div 12 = 56\cdot\dot{3}$ inche the other; hence $(1320 + 133\cdot\dot{3} + 56\cdot\dot{3}) \div 12 = 1509\cdot$ $\div 12 = 125\cdot80\dot{5}$ sq. feet whole surface.

2. Required the whole surface of a frustum of a pentagona pyramid, the perpendicular height 11 feet, and the sides o the bases 18 and 34 inches. Ans. 137·06818 square feet

TO FORM A FRUSTUM WITH PASTEBOARD.



Make A*a* and *ab* equal to the radii of the circles described about the bases, and draw *ad* and *b*D perpendicular to A*a*; make either *b*D the axis, or AD the slant side of the frustum, and produce *ad* and AD till they meet in *e*. From *e* describe circles through A and D, and in them place straight lines AB, AC, &c. and DH, DK, &c. equal to the sides of the bases; join BH, CK, &c.; or if the frustum be that of a cone, make *ae*E, *ae*G, the angle at the vertex. Lastly, upon AB, DH, make the bases. Then the figure will be the surface; and if it be folded along the lines, or bended, it will form the frustum.

3. Required the surface of a frustum of a cone, the diameters of the bases being 43 and 23 inches, and the slant height 9 feet. Ans. 90·72446 square feet.

4. From a cone, of which the circumference of the base is 10 feet, and its slant height 30 feet, a cone has been cut off, of which the slant side is 8 feet. Required the curve surface of the remaining frustum. Ans. $139\frac{1}{3}$ square feet.

5. Required the surface of a frustum of a cone, the perpendicular height of the frustum 13 feet, and the radii of the bases 15 and 24 inches. Ans. 150·4284 square feet.

PROB. X. To find the solid content of a frustum of a pyramid or cone.

GENERAL RULE. Find the areas of the two ends, and take the square root of their product: this added to the two areas, and the sum multiplied by a third of the perpendicular height, will give the solid content. (Theor. II. Cor. 4. p. 199.)

That is, if A be the area of the greater end, a that of the less, and h the height, then $(A + a + \sqrt{Aa}) \times \frac{1}{3}h$ is the solidity.

Particular Rule. If the base be a circle, or a regular polygon, add a diameter, or a side of the greater base, to one of the less, and from the square of the sum subtract the product of these diameters or bases: the remainder, multiplied by the number belonging to the figure, and by a third of the height, will give the content.*

That is, using the same letters as in the demonstration in the note, $\{(A+a)^2 - Aa\} \times \frac{1}{3}ph$ is the solidity.

1. Required the content of the frustum of a square pyramid, the sides of the bases being 15 and 6 feet, and the height 24 feet.

Ans. Here $(15 + 6)^2 - (15 \times 6) = 441 - 90 = 351$ and $351 \times 1 \times 8 = 2808$ cubic feet content.

2. Required the content of the frustum of a triangular pyramid, the height of the frustum 14 feet, the sides of the greater base 21, 15, and 12, and those of the less base 14, 10, and 8 feet.

The areas of the bases are $36\sqrt{6}$ and $16\sqrt{6}$, and the square root of their product $24\sqrt{6}$; therefore $(36\sqrt{6} + 16\sqrt{6} + 24\sqrt{6}) \times \frac{1}{3} \times 14 = 868{\cdot}75236218$ cubic feet the content.

3. Required the content of the frustum of a pentagonal pyramid, the sides of the bases being 42 and 23 inches, and the height 16 feet. Ans. 207·668 cubic feet.

4. Required the content of the frustum of a cone, the diameters of the bases being 38 and 27 inches, and the height 11 feet. Ans. 63·9756 cubic feet.

5. Required the content of a mast 57 feet high, and the girths at its ends 63 and 38 inches. Ans. 81·972 cubic feet.

6. Required the content of the frustum of a cone, the height 35 feet, and the diameters of the bases 3·127 and 1·118 feet. Ans. 133·081794 cubic feet.

Prob. XI. To find the superficial and the solid contents of a wedge.

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* If A = diameter or side of the greater base, a that of the less, h the height of the frustum, and p the proper multiplier, then the height of the complete cone or pyramid is $= Ah \div d$ (putting $d = A - a$), and therefore its content is $A^2p \times Ah \div 3d = A^3ph \div 3d$.

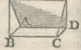
In like manner, the part of the cone which is cut off is $a^3ph \div 3d$; and therefore the content of the frustum is $(A^3 - a^3)\, ph \div 3d = \{(A + a)^2 - Aa\} \times \frac{1}{3}ph$.

Rule for the Surface. Find the areas of the rectangle, the two parallelograms or trapezoids, and the two triangles of which its surface consists, and add them together.

Rule for the solid content. To twice the length of the base add the length of the edge, and multiply the sum by the breadth of the base, and by one-sixth of the perpendicular from the edge upon the base: the product will be the content. (Theorem III. Cor. p. 199.)

That is, $(2BC + EF) \times AB \times \frac{1}{6}p$ is the content. ($p =$ the perpendicular.)

1. Required the superficial and the solid contents of a wedge ABCDEF, of which the sides of the base are BC 36 and BA 9 inches, the edge EF 44 inches, and the perpendicular height 22 inches.



Ans. First, $36 \times 9 = 324 =$ the rectangle, $22 \times 9 = 198 =$ the two triangles, and $(36 + 44) \times 22 = 1760 =$ the two trapezoids; hence $324 + 198 + 1760 = 2282$ square inches the whole surface.

Also (3 ft. + 3 ft. + 3 ft. 8 in.) × 9 in. × 22 in. ÷ 6 = 9 ft. 8 in. × 9 in. × 22 in. ÷ 6 = 13 ft. 3 in. 6 pts. ÷ 6 = 2 ft. 2 in. 7 pts. solid content.

2. Required the content of a wedge, of which the height is 25 inches, the edge 28 inches, and the sides of the base 34 and 10 inches. Ans. $2{\cdot}31\dot{4}8$ cubic feet.

3. How many solid feet are in a wedge, of which the base is 40 inches long and 10 inches broad, and each of the ends is inclined to the base in an angle of 70°, the edge being 30 inches? Ans. 1·457477 cubic feet.

4. How many solid feet are in a wedge, of which the sides of the base are 35 and 15, the length of the edge 55 inches, and the height $17\frac{3}{20}$ inches?

Ans. 5359·375 cubic inches = 3 cubic feet $175\frac{3}{8}$ inches.

Prob. XII. To find the content of any solid, of which the bases are parallel, and the greatest and least thicknesses are at its ends.

Rule. Find the areas of the two bases, and also the area of a section parallel to, and equidistant from, the bases; then to four times the middle area add the other two areas, and the sum, multiplied by one-sixth of the length, will give the solid content.*

---

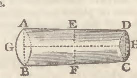
* The prismoid ABCDEFGH, see figure to example 2, may be divided into two wedges, by joining AH and BG, and if we make $EF = a$, $EH = b$, AB

That is, $(a+4b+c)\times\frac{1}{6}l =$ the content, where a and $c =$ the areas of the two bases, b the area of the middle section, and $l =$ the length of the solid.

NOTE 1. When the sides of the solid are straight between the bases, half the sum of two corresponding sides or diameters of the bases will give the corresponding side or diameter of the middle section.

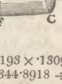
NOTE 2. When the greatest and least thicknesses are not at the ends, divide the solid into portions which shall have them at their ends. Find the contents of these portions separately, and add them: the sum will be the content of the whole.

1. A round solid ABCD, has its length GH 14 feet, the diameter of the bases AB 94, and CD 21 inches, and the diameter EF of the middle section 27 inches. Required its content.



Ans. $(94^2 + 21^2 + 54^2) \times \cdot 7854 \div 6 = 12193 \times \cdot 1309 = 1596\cdot 0637$ and $1596\cdot 0637 \times 14 \div 144 = 22344\cdot 8918 \div 144 = 155\cdot 17286$ cubic feet, content.

2. Required the content of the prismoid ABCDEFGH, of which the height is 22 feet, the upper base ABCD is a rectangle, of which the sides are AB 43, and BC 23 inches, and the under base EFGH a square, of which the side EF is 37 inches. Ans. 182·2638 cubic feet.



3. Required the capacity of a cistern $47\frac{1}{4}$ inches deep, the inside dimensions are, at the top $81\frac{1}{2}$ and 55 inches, and at the bottom 41 and $29\frac{1}{2}$ inches.
Ans. 126340·59375 cubic inches, = 455·6525 imp. gallons.

4. Required the content of a cylindroid 10 feet long, and the diameters of the bases 35 and 31 in. Ans. 59·4686 cub. ft.

5. What is the content of a log of wood, of which the length is 19 feet, and both the bases are rectangles, of which the sides of the lower are 48 and 36 inches, those of the higher 32 and 21 inches, and the sides of the middle section 5 and 34 inches? Ans. 187·361 cubic feet.

6. What is the content of a round solid, of which the whole length is 37 feet; the greatest girt, 77 inches, is 16 feet from the greater end, of which the girt is 54, and the middle girt

= m, AD = n, $a + m = p$, and $b + n = q$, then p and q are double the sides of middle base. Now the under wedge is $= (m + 2a)\, b \times \frac{1}{6}h$ (h = height), and the upper wedge $= (a + 2m)\, n \times \frac{1}{6}h$, whence they are together $= \{(p + a)\, b + (p + m)\, n\} \times \frac{1}{6}h = (p \times (b + n) + ab + mn) \times \frac{1}{6}h = (pq + ab + mn) \times \frac{1}{6}h$, which is the rule.

67; also, the girt at the lesser end is 36 inches, and the middle girt 59 inches. Ans. 80 cubic feet 693·5615 inches.

Prob. XIII. To find the surface of a sphere, or of any segment or zone of it.

Rule. Multiply the circumference of a great circle of the sphere by the axis, or by the part of it corresponding to the segment or the zone required: the product will be the surface. (Theorem IV. Cor. 2. page 201.)

Note. The surface of a sphere, or any part of it, cut off by a plane or planes perpendicular to the axis, is equal to the curve surface of the circumscribing cylinder, which has the same axis, or to the corresponding part of it.

1. Required the surface of a globe AECD, of which the axis AC is 18 inches.
Ans. $3·1416 \times 18^2 = 1017·8784$ square inches the surface.

2. Required the surface of a segment of a sphere, the axis 54 inches, and the height of the segment 18 inches.
Ans. 21·2058 square feet.

3. Required the surface of a zone of a sphere, the axis 72 inches, and the height of the zone 24 inches.
Ans. 5428·6848 square inches.

4. Required the surface of the moon, supposing her to be a perfect sphere, of which the diameter is 2180 miles.
Ans. 14930139·84 square miles.

5. Required the surface of the earth, supposing it to be a perfect sphere, of which the axis is 7912 miles; and also the surface of each of its zones, supposing the torrid zone to extend $23\frac{7}{15}°$ on each side of the equator, the frigid zones $23\frac{7}{15}°$ round the poles, and the breadth of each of the temperate zones to be $43\frac{1}{15}°$.

Ans. The part of the axis corresponding to each of the frigid zones is 327·192848, to each temperate zone is 2053·4668612, and to the torrid zone is 3150·67708; therefore the surface of each frigid zone is 8132797·39568, of each temperate zone is 51041592·7007, and of the torrid zone is 78314115·57481, and the whole surface is 196662895·867002 square miles.

Prob. XIV. To find the solid content of a sphere.

Rule. Multiply the cube of the axis by ·5236. (Theor. IV. page 200. A sphere is $\frac{2}{3}$ of its circumscribing cylinder, and ·5236 is $\frac{2}{3}$ of ·7854.)

1. Required the solidity of a sphere, of which the axis is 16 inches.

Ans. $16^3 \times \cdot 5236 = 2144 \cdot 6656$ cubic inches solidity.

2. Required the solidity of a sphere, the axis 3 feet 6 inches.

Ans. 22·44935 cubic feet.

3. Required the solidity of a sphere, the axis 19 yards.

Ans. 3591·3724 cubic yards.

4. Required the solidity of the moon, supposing her a perfect sphere, the axis 2180 miles.

Ans. 5424617475·2 cubic miles.

5. Required the solidity of the earth, supposing it to be a perfect sphere, and its axis 7912 miles.

Ans. 259332805349·80493 cubic miles.

Prob. XV. To find the solid content of a segment of a sphere.

Case I. When the axis and the height of the segment are given.

From three times the axis subtract twice the height; multiply the remainder by the square of the height, and by ·5236: the product will be the content. (Theor. IV. Cor. 1. page 200.)

That is, if $a =$ the axis and $h =$ the height of the segment, then $(3a - 2h) \times \cdot 5236 h^2$ is the solidity.

1. Required the content of a segment 13 inches high, cut off from a sphere, of which the axis is 48 inches.

$(3 \times 48 - 2 \times 13)\ 13^2 \times \cdot 5236 = 10441 \cdot 6312$ cubic inches.

2. Required the content of the frigid zone of the earth, the height 327·2, and the axis 7912 miles.

Ans. 1293874454·1815 cubic miles.

3. Required the content of a segment, of which the height is 57, and the axis 153 inches.

Ans. 586905·858 cub. in. = 339 cub. ft. 1113·858 inches.

4. Required the content of a segment, of which the height is $\frac{3}{8}$ of the axis. Ans. ·16567 cubes of the axis.

Case II. When the height and the radius of the base of the segment are given.

To three times the square of the radius add the square of the height; multiply the sum by the height, and by ·5236: the product is the content. (Theor. IV. Cor. 1. page 200.)

That is, if $r =$ BE the radius of the base, and $h =$ CE the height, then $(3r^2 + h^2) \times \cdot 5236 h$ is the solidity.

5. Required the content of the segment BCD, of which the height CE is 13, and the radius BE of the base 21 inches.

Ans. $(3 \times 21^2 + 13^2) \times 13 \times {\cdot}5236 =$ 10155·7456 cubic inches.

6. Required the content of the segment, of which the height is 3, and the diameter of the base 9 feet. Ans. 109·5633 cubic feet.

7. Required the content of the segment, of which the height is 12, and the radius of the base 48 inches.

Ans. 44334·2592 cub. in. = 25 cub. ft. 1134·2592 inches.

8. Required the content of the segment, of which the height is 7, and the diameter of the base 84 yards.

Ans. 19575·8332 cubic yards.

PROB. XVI. To find the solid content of the middle zone of a sphere.

From the square of the axis, or greatest diameter, subtract one-third of the square of the height, then multiply the remainder by the height, and by ·7854. (Theorem IV. Cor 1. page 200.)

That is, $(a^2 - \frac{1}{3}h^2) \times {\cdot}7854h$ is the solidity, where $a =$ the axis of the sphere and h the height of the zone.

NOTE. Instead of subtracting one-third of the square of the height from that of the axis, we may add two-thirds of the square of the height to the square of the least diameter.

1. Required the content of the middle zone of a sphere, of which the axis is 44, and the height of the zone 14 inches.

$(44^2 + \frac{1}{3} \times 14^2)\ 14 \times {\cdot}7854 = 20569{\cdot}1024$ cubic inches.

2. Required the content of the middle zone of a sphere, of which the height is 4, and the least diameter 3 feet.

Ans. 61·7848 cubic feet.

3. Required the content of the middle zone of a sphere, of which the height is 24, and the least diameter 18 inches.

Ans. 13345·5168 cubic inches.

4. Required the content of the middle zone of a sphere, of which the height is 3, and the least diameter 5 yards.

Ans. 73·0422 cubic yards.

5. Required the solidity of the torrid zone of the earth, the axis being 7912, and the height of the zone 3150·68104 miles. Ans. 146717436810·847 cubic miles.

PROB. XVII. To find the solid content of any zone of a sphere.

Add the squares of the radii of the two ends to one-third of the square of the height; then multiply the sum by twice the height, and by ·7854. (Theorem IV. Cor. 1. page 201.)

That is, if R and r = the radii of the two ends, and h = the height, then $(R^2 + r^2 + \frac{1}{3}h^2) \times 2h \times \cdot 7854$ is the solidity.

1. Required the solid content of a spherical zone, of which the height is 10, and the diameters at its ends 12 and 8 feet.

$(6^2 + 4^2 + \frac{1}{3} \times 10^2) \times 2 \times 10 \times \cdot 7854 = 1340 \cdot 416$ cub. feet.

2. Required the solid content of a spherical zone, of which the height is 14, and the diameters at its ends 16 and 12 inches. Ans. 3635·8784 cub. in. = 2 cub. ft. 179·8784 inches.

3. Required the solid content of a spherical zone, of which the height is 9, and the radii at its ends 14 and 10 yards.

Ans. 4566·3156 cubic yards.

4. Required the solid content of a spherical zone, of which the height is 11, and the diameters 18 and 13 feet.

Ans. 2826·5237 cubic feet.

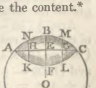
5. Required the solid content of a spherical zone, of which the height is 23, and the radii 27 and 18 inches.

Ans. 44413·8464 cub. in.

6. The height of the temperate zone of the earth is 2053·46624 miles, and the squares of the greatest and least radii are 13168239 and 2481697 square miles. Required its content. Ans. 55013866370·2 cubic miles.

PROB. XVIII. To find the solid content of a circular spindle.

RULE. Multiply the area of the generating segment by half the central distance, and subtract the product from one-third of the cube of half the length of the spindle, then four times the remainder, multiplied by 3·1416, will give the content.*



That is, if d = AE the half-length of the spindle, c = EO the central distance, a = the area ABE, and $p = 3 \cdot 1416$, then $(\frac{1}{3}b^3 - ac) \times 4p$ = the solidity of the spindle ABCF.

1. Required the content of the circular spindle ABCF, of which the length AC is 40, and its greatest diameter BF 30 inches.

Ans. $20^2 \div 15 + 15 = 41\frac{2}{3}$ diam. of the circle, $20\frac{5}{6}$ the radius, $20\frac{5}{6} - 15 = 5\frac{5}{6}$ central distance, and $2\frac{1}{1}\frac{1}{2}$ half the

* For the demonstration of this and the following rule, see Appendix.

central distance; then $15 \div 41\frac{2}{3} = \cdot 360$ versine of which the tabular area is $\cdot 25455$; now $\cdot 25455 \times 41\frac{2}{3}^2 \times 2\frac{1}{12}$ $= 1288\cdot 95399$, then $20^3 \div 3 = 2666\cdot\dot{6}$ and $(2666\cdot\dot{6} - 1288\cdot 95399) \times 4 \times 3\cdot 1416 = 17312\cdot 8884963$ the solid content.

2. Required the content of a circular spindle, of which the length is 24, and the greatest diameter 18. Ans. 3739·584.

3. Required the content of a circular spindle, of which the length is 32, and the greatest diameter 24 inches.

Ans. 8864·1989 cubic inches.

4. Required the content of a circular spindle, of which the length is 48, and the greatest diameter 18 inches.

Ans. 6770·97195 cubic inches.

5. Required the content of a circular spindle, of which the length is 60, and the greatest diameter 12 inches.

Ans. 3653·42525 cubic inches.

Prob. XIX. To find the solid content of the middle zone of a circular spindle.

Rule. From the square of half the length of the spindle subtract one-third of the square of half the length of the zone, and multiply the remainder by half the length of the zone; next find the area of the space which generates the zone; multiply it by the central distance, and subtract this from the former product; then twice the remainder, multiplied by 3·1416, will give the solid content.

That is, if $b =$ EH half the length of the zone, $d =$ AE half the length of the spindle, $c =$ EO the central distance, and $a =$ the generating area HNMG; then $\{(d^2 - \frac{1}{3}b^2)\, b - ac\} \times 2p =$ the solidity of the zone NMLK.

1. The length GH of the middle zone of the spindle ABCF is 40, and its diameters are BF 32 and KN 24 inches. Required its content.

Ans. $\frac{1}{2}(32 - 24) = 4$ and $20^2 \div 4 + 4 = 104$ diameter of circle, 52 radius and $52 - 16 = 36$ central distance; then $(104 - 16) \times 16 = 1408$ square of half the length of spindle, and $(1408 - \frac{1}{3}$ of $400) \times 20 = 25493\cdot\dot{3}$ first product. Also $4 \div 104 = \cdot 0384\frac{8}{13}$ versine of which the tabular area is $\cdot 00994$ and $\cdot 00994 \times 104^2 + (12 \times 40) = 587\cdot 51104$ generating space, which, multiplied by the central distance 36, gives $21150\cdot 39744$ second product; whence $(25493\cdot 33333 - 21150\cdot 39744) \times 2 \times 3\cdot 1416 = 4342\cdot 93589 \times 6\cdot 2832 = 27287\cdot 534784$ inches solid content.

2. Required the content of the middle zone of a circular spindle, the length 20, and the diameters 18 and 8 feet.

Ans. 3657·160776 cubic feet.

3. Required the content of the middle zone of a circular pindle, the length 36, and the diameters 24 and 16 inches.
Ans. 13090·39586778 cubic inches.

4. Required the content of the middle zone of a circular pindle, the length 60, and the diameters 50 and 30 inches.
Ans. 91302·75 cubic inches.

5. Required the content of the middle zone of a circular indle, the length 80, and the diameters 80 and 40 inches.
Ans. 298353·77264 cubic inches.

OF THE REGULAR BODIES.

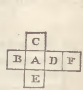
A Regular Body is a solid bounded by similar and regu- plane figures. Of these there can be only five.

Prob. XX. To form the five regular bodies with paste- ard.

1. The Tetraedron, bounded by four equi- eral triangles.

Make the equilateral triangle A, and upon h side of it make an equilateral triangle. e figure, cut out of the paper, and folded at lines, will form the tetraedron.

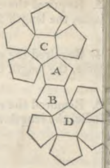
2. The Hexaedron, bounded by six ares.



Make the square A, and upon its sides squares B, C, D, E, and on the outer- st side of D make the square F. The re, cut out and folded, will form the aedron.

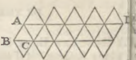
The Octaedron, bounded by eight ilateral triangles.

Take the equilateral triangle ABC, and ugh A draw AK parallel to BC, and e CE, EF, AD, AG, GH, and HK, equal to BC, and join the points as he figure. When folded, this figure will form the octae- .



4. The DODECAEDRON, bounded by twelve pentagons.

Make two regular pentagons A and B on the same straight line, and on the most distant sides of these make the pentagons C and D; then make a pentagon on each of the sides of C and D; and the figure, when folded, will form the dodecaedron.



5. The ICOSAEDRON, bounded by twenty equilateral triangles.

Make the equilateral triangle ABC, and through A draw AD parallel to BC, and lay BC five times on each of the parallels, and join the points as in t figure. This figure, when folded, will form the icosaedron.

PROB. XXI. To find the surface and the solidity of t five regular bodies.

RULE I. TO FIND THE SURFACE. Multiply the squa of the linear side by the proper number in the table und *Surface:* the product will be the surface.

RULE II. TO FIND THE SOLIDITY. Multiply the cu of the linear side by the proper number under *Solidity:* t product will be the solid content.*

TABLE

OF THE SURFACES AND SOLIDITIES OF REGULAR BODIES

No. of faces.	Name.	Surface when the side is 1.	Solidity when the side is 1.
4	Tetraedron, . .	1·7320508	0·1178511
6	Hexaedron, . .	6·0000000	1·0000000
8	Octaedron, . .	3·4641016	0·4714045
12	Dodecaedron, .	20·6457788	7·6631189
20	Icosaedron, . . .	8·6602540	2·1816950

CONSTRUCTION OF THE TABLE. The solid content of any R gular Body is equal to its surface multiplied by ⅓ of the radius

* The truth of these rules is evident; for the surfaces of similar solids as the squares, and their solidities as the cubes of their corresponding lin sides.

he inscribed sphere, for it is manifest that any regular solid may be divided into as many equal pyramids as it has faces, the common vertex of the pyramids being the centre of the body, which is also hat of the inscribed sphere.

In the Tetraedron, which is a triangular pyramid, let $e =$ the dge, $p =$ the perpendicular from the vertex to the centre of the ase, $d =$ the distance from the foot of the perpendicular to one of he edges, and $r =$ the radius of the inscribed sphere; then, since he square of the side of an equilateral triangle is equal to three imes the square of the radius of the circumscribed circle, we have $e^2 = 3d^2$ and $e^2 - d^2 = p^2$, therefore $p^2 = 2d^2 = \frac{2}{3}e^2$. Or, when he edge is $= 1$, $p = \sqrt{\frac{2}{3}} = \cdot 81649658$. The area of each face is $= \frac{1}{4}\sqrt{3}$ (Cor. 2. p. 174), therefore the whole surface $= \sqrt{3} = 1\cdot 7320508$; nd as $r = \frac{1}{4}\sqrt{\frac{2}{3}}$, the solidity is $\sqrt{3} \times \frac{1}{3} \times \frac{1}{4}\sqrt{\frac{2}{3}} = \frac{1}{12}\sqrt{2} = \cdot 1178511$.

The Octaedron is evidently composed of two equal square pyramids, the area of whose bases $= e^2$ and $p =$ half the diagonal of the ase $= \frac{1}{2}\sqrt{2}$, each of the faces $= \frac{1}{4}\sqrt{3}$; hence the whole surface $= 2\sqrt{3} = 3\cdot 4641016$, and the solidity $= \frac{2}{3} \times \frac{1}{2}\sqrt{2} = \frac{1}{3}\sqrt{2} = \cdot 4714045$.

The Dodecaedron is composed of 12 equal pentagonal pyramids, ach of whose faces $= \frac{5}{4}\sqrt{(1 + \frac{2}{5}\sqrt{5})}$, whence the whole surface $= 15\sqrt{(1 + \frac{2}{5}\sqrt{5})} = 20\cdot 6457788$; and as $r = \sqrt{\left(\frac{25 + 11\sqrt{5}}{40}\right)} = \frac{1}{20}\sqrt{(250 + 110\sqrt{5})}$, therefore the solidity is $= 5\sqrt{\left(\frac{47 + 21\sqrt{5}}{40}\right)} = \frac{1}{4}\sqrt{(470 + 210\sqrt{5})} = 7\cdot 6631189$.

The Icosaedron is composed of 20 equal triangular pyramids, ach of whose faces $= \frac{1}{4}\sqrt{3}$, hence the whole surface $= 5\sqrt{3} = 8\cdot 6602540$; and as $r = \frac{1}{2}\sqrt{\left(\frac{7 + 3\sqrt{5}}{6}\right)} = \frac{1}{12}\sqrt{(42 + 18\sqrt{5})}$, consequently the solidity $= \frac{5}{6}\sqrt{\left(\frac{7 + 3\sqrt{5}}{2}\right)} = \frac{5}{12}\sqrt{(14 + 6\sqrt{5})} = 2\cdot 1816950$.

1. Required the surface and the solidity of an octaedron, which the side is 16 inches.

Ans. $16 \times 16 \times 3\cdot 4641016 = 886\cdot 81$ square inches surface.
$16^3 \times \cdot 4714045 = 1930\cdot 8728$ cubic inches solidity.

2. Required the surface and the solidity of a dodecaedron, which the side is 12 feet.

Ans. Surface 2972·992 sq. ft., solidity 13241·8694592 cub. ft.

3. Required the surface and the solidity of a tetraedron, of hich the side is 2 feet.

Ans. Surface 6·9282032 sq. feet, solidity 0·9428104 cub. ft.

4. Required the surface and the solidity of a hexaedron, of hich the side is 27 inches.

Ans. Surface 4374 sq. inches, solidity 19683 cub. inches.

5. Required the surface and the solidity of an icosaedron, which the side is 15 inches.

Ans. Surface 1948·55715 sq. inches, solidity 7363·22062 cubic inches.

PROB. XXII. To find the convex surface of a solid ring.

RULE. To the thickness of the ring add the inner diameter, to get the axis; multiply this by the thickness, and by $3·1416^2 = 9·8696$, to get the surface.*

1. Suppose the thickness of the ring 3 inches, and the inner diameter 12 inches. Required its surface.

Ans. $(12 + 3) \times 3 \times 9·8696 = 444·132$ square inches.

2. Suppose the thickness 2, and the inner diameter 18 inches. Required the surface. Ans. 394·784 square inches.

3. Suppose the thickness 3, and the inner diameter 14 inches. Required the surface. Ans. 503·3496 square inches.

4. Suppose the thickness 5, and the inner diameter 18 inches. Required the surface. Ans. 1135·004 square inches.

5. Suppose the thickness 6, and the inner diameter 24 inches. Required the surface. Ans. 1776·528 square inches.

PROB. XXIII. To find the solidity of a ring.

RULE. Multiply the axis by 3·1416 to get the length, and then multiply the length by the square of the thickness, and by ·7854: the product is the content.

Or multiply the axis by the square of the thickness, and by 2·4674.

1. Required the solidity of a ring 2 inches thick, of which the inner diameter is 18 inches.

$18 + 2 = 20$ axis, $20 \times 3·1416 = 62·832$ length.

Ans. $62·832 \times 4 \times ·7854 = 197·393$ cubic inches.

2. Required the solidity of a ring, the thickness 3, and the inner diameter 8 inches. Ans. 244·2726 cubic inches.

3. Required the solidity of a ring, the thickness 4, and the inner diameter 16 inches. Ans. 789·5720448 cubic inches.

4. Required the solidity of a ring, the thickness 5, and the inner diameter 12 inches. Ans. 1048·65 cubic inches.

5. Required the solidity of a ring, the thickness 6, and the inner diameter 18 inches. Ans. 2131·8445 cubic inches.

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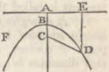
* It is manifest that, as solid rings are bent cylinders, the rules for finding their surface and solidity are the same as those already given for the cylinder.

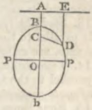
CONIC SECTIONS.

DEFINITIONS.

1. If a point D move in a plane, and its distances from a fixed point C, and from a straight line AE, both in that plane, have always the same ratio to one another, the moving point will describe a *curve*, called a *line* of the *second order*, or a *conic section*.



2. The fixed point C is called the *focus;* the straight line AE is called the *directrix ;* and the constant ratio of CD to DE is called the *ratio* of the curve.

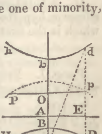


3. The straight line CA, drawn through the focus C, perpendicular to AE, is called the *axis*, or the *transverse axis*, and the point B, in which it cuts the curve, is called the *principal vertex*.

Cor. Hence CB : BA : : CD : DE, or is the ratio of the curve.

4. If CB be equal to BA, or the ratio of the curve be that of equality, the curve is called a *parabola*, as DBF.

5. If CB be less than BA, or the ratio be one of minority, the curve is called an *ellipse*, as DBP.



Cor. If AC be produced beyond C to b so that A*b* : *b*C : : AB : BC, the point b will be in the ellipse, which, therefore, contains a space.

6. If CB be greater than BA, or the ratio be one of majority, the curve is called a *hyperbola*, as DBH.

Cor. If CA be produced beyond A, so that C*b* : *b*A : : CB : BA, the point *b* will be in a hyperbola, similar, and equal to DBH, and described in the same way; it is called the *opposite hyperbola*.

7. The straight line Bb in the ellipse and hyperbola is properly the axis, B and b its vertices, and the point O in which it is bisected is called the *centre*.

8. A straight line Pp, drawn through the centre O, perpendicular to the transverse axis, is called the *conjugate* axis; and the points Pp, in the ellipse in which it meets the curve, are called its vertices. But in the hyperbola, the vertices Pp are the points in which it meets the circle described from B with the radius OC.

9. Every straight line which is perpendicular to the directrix of a parabola, or which passes through the centre of an ellipse or a hyperbola, is called a *diameter;* and the point in which it meets the curve is its vertex.

10. A straight line which meets the curve, and does not cut it, is called a *tangent;* and if the straight line from the point of contact to the focus be parallel to the directrix, the tangent is called the *focal tangent*.

11. A straight line parallel to a tangent, is said to be *ordinately applied* to the diameter which passes through the point of contact, and the part of it between the curve and that diameter is called an *ordinate*.

12. The segments of a diameter intercepted between an ordinate and its vertices, are called *abscissas* to that ordinate.

13. Straight lines drawn through the centre of a hyperbola parallel to the straight lines which join the vertices of the axes, are called *asymptotes*

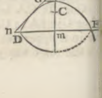
14. Two diameters of the ellipse or hyperbola, each of which is parallel to the tangent in the vertex of the other, are called *conjugate diameters*.

15. Four times the segment of a diameter of the parabola between its vertex and the directrix, is called the *parameter* of that diameter.

16. A third proportional to two conjugate diameters of the ellipse or hyperbola, is called the *parameter* of that diameter which is the first of the three proportionals.

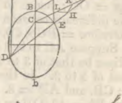
PROPOSITIONS.

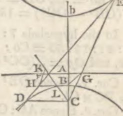
PROP. I. Problem. To find the point or the points in which a given straight line, DE, meets a conic section.



If DE be parallel to the directrix AF, draw BG parallel to AF, and make BG = BC, and join AG, and let it meet DE in n, and let the axis meet DE in m. From C, with the radius mn, describe a circle meeting DE, in the points D and E. Because the perpendicular from D upon AF is equal to Am, CD : perpendicular : : nm : mA : : GB = CB : BA; therefore the point D is in the curve, and for the same reason E is in the curve.

If DE meet the directrix in F, join FC. If DE be parallel to AC, make FH = AB. If not, draw BG parallel to DE, and make FH = BG. Then with BC for a radius, from H, cut CF in K and L, and through C draw CD and CE, parallels to HK and HL, the points D and E are in the curve. Draw DM perpendicular to the directrix. The triangles DMF, BAG are similar; therefore MD : DF : : AB : BG, and DF : DC : : FH : HK : : GB : BC; hence MD : DC : : AB : BC. Therefore D is in the curve, and for the same reason E is in the curve.





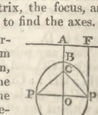
In the parabola AB = BC; and therefore, if DE be perpendicular to the directrix, the point K will fall on F; in which case the straight line DE will meet the curve only in one point D.

In the hyperbola, where AB is less than BC, the point K may fall above F; in which case DE meets each of the opposite hyperbolas in one point.

In the ellipse in which CB is less than BA, the circle described from H may not meet CF; in which case DE will not meet the curve. And a straight line may be drawn between the opposite hyperbolas, so as not to meet either of them, but two other hyperbolas which have the conjugate diameter of the former for their transverse, and the transverse for their conjugate.

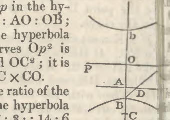
Prop. II. Problem. Given the directrix, the focus, and the ratio of an ellipse, or of an hyperbola, to find the axes.

Having drawn AC from the focus C perpendicular to the directrix, make the sum of the terms of the ratio to the first term, as AC to CB, and their difference to the first, as AC to Cb. Then B and b are the extremities of the transverse axis. And because AB : BC : : Ab : bC; therefore AB : BC : : $\frac{1}{2}$ (AB + Ab) : $\frac{1}{2}$ (BC + bC), or : : $\frac{1}{2}$



(Ab — AB) : $\frac{1}{2}$ (bC — BC), that is AB : BC : : AO : O
or : : OB : OC; therefore OB and OC are given.

Join Cp in the ellipse, and Bp in the hyperbola. Then pF = AO : Cp : : AO : OB; therefore Cp = OB, and in the hyperbola Bp = OC; hence in both curves Op^2 is the difference between OB^2 and OC^2; it is therefore = BC × Cb, or = AC × CO.



Suppose AC to be 14, and the ratio of the ellipse be that of 3 to 4, or of the hyperbola that of 4 to 3. In the ellipse 7 : 3 : : 14 : 6 = CB, and AB = 8. Also 1 : 3 : : 14 . 42 = Cb; there fore Bb = 48, OB = 24, OC = 18 OA = 32, and Op = $\sqrt{(OB^2 - OC^2)}$ = 15·8745.

In the hyperbola 7 : 4 : : 14 : 8 = CB and AB = 6. Al 1 : 4 : : 14 : 56 = Cb; therefore Bb = 48, OB = 24, OC = 32, and Op = $\sqrt{(OC^2 - OB^2)}$ = 21·166.

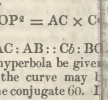
Cor. 1. Hence OB^2 = AO × CO, and OP^2 = AC × C = BC × Cb.

Cor. 2. Hence AC : CB : : Cb : CO, and AC : AB : : Cb : BO.

Cor. 3. If the axes of an ellipse or an hyperbola be given the focus, the directrix, and the ratio of the curve may b found. Let the transverse axis be 80, and the conjugate 60. In the ellipse OC = $\sqrt{(OB^2 - OP^2)} = \sqrt{(40^2 - 30^2)} = 10\sqrt{}$ = 26·4575, OA = OB^2 ÷ OC = 60·4743, AB = 20·4743 and BC = 13·5425, and the ratio of the curve that of 10$\sqrt{}$: 40, or of $\sqrt{7}$: 4, or of 2·64575 : 4.

In the hyperbola OC = BP = $\sqrt{(40^2 + 30^2)}$ = 50, OA = 32, and the ratio of the curve that of OC : OB, or of 5 to 4.

Prop. III. Problem. Given the directrix, and the focus of a conic section, to draw a straight line which shall touch the curve at a given point.



Let D be the given point in the curve. Draw DC to the focus, and perpendicular to it draw CE, and let it meet the directrix in E, and join DE; it will touch the curve in D.

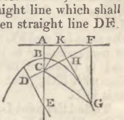
Take any point G in DE, and draw GK parallel to DC, and join GC, and draw DF, GH perpendicular to the directrix. GK : GH : : CD : DF, but GC $>$ KG; therefore the ratio of GC to GH is greater than the ratio of the curve, and so G is without the curve, and DE touches it.

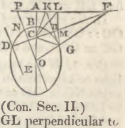
If B be the given point, so that CL perpendicular to CB, is parallel to AF, then BM parallel to AE touches the curve.

or CM $>$ CB and MN $=$ BA; therefore CM : MN $>$ CB : BA.

Cor. If CL be parallel to AF, then AL touches the section, and is the focal tangent.

Prop. IV. Problem. Given the directrix AF, and the focus C of a conic section, to draw a straight line which shall touch the curve, and be parallel to a given straight line DE.





From the focus C, draw CD perpendicular to DE, and let it meet the directrix in F, draw the diameter FG, and through its vertex G draw GH parallel to DE, and it will touch the curve at G. Join GC and CK.

In the parabola. Since the angles at H are right angles, HCG $+$ CGH $=$ a right angle $=$ GFK, of which GCH $=$ GFH, because GC $=$ GF; therefore CGK $=$ CFK, and the four points C, G, F, K, are in the circumference of a circle, of which GK is the diameter; therefore (El. Geom. 34.) GCK is a right angle, and GK touches the curve. (Con. Sec. II.)

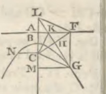
In the ellipse and hyperbola. Draw GL perpendicular to the directrix, and let it meet CF in M, then GM : GL : : OC : OA : : $OC^2 : OB^2$ (Con. Sec. I.), or : : $CG^2 : GL^2$; therefore the triangles CGM, CGL, are similar, and the angle GCM $=$ GLC; hence CGH $=$ CLK, and the four points C, G, L, K, are in the circumference of a circle, of which GK is the diameter; therefore GCK is a right angle and GK a tangent.

Cor. 1. A straight line FC, drawn to the focus from the intersection F of a diameter, with the directrix, is perpendicular to the ordinates to that diameter.

Cor. 2. Tangents at the vertices of the same diameter are parallel to one another.

Cor. 3. Two diameters OG, ON, one of which ON is parallel to the tangent GK in the vertex of the other, are conjugate. Because in the triangle OFP, FC and OC are perpendicular to the sides OP and PF; therefore since the perpendiculars from the angles of a triangle upon the opposite sides all pass through the same point, PC will be perpendicular to OF; that is, OF is parallel to the tangent in N.

Prop. V. Problem. Given the axis, the directrix, and the focus, to find the point in which a tangent GK meets the axis.



Through G draw the diameter GF, meeting the directrix in F. Join GC, CK, and CF, and draw FL parallel to CG, and GM parallel to AF. CF is perpendicular to the tangent GK, and CK to CG. And because in the triangle LFC, FK and CK are perpendicular to the sides LC and FL; consequently LK is perpendicular to the third side CF, and is therefore in the same straight line with KG; that is, GK meets the axis in L.

To find the point L. *In the parabola*, LC = FG = AM, and AB = BC; therefore LB = BM, and LC = CG.

In the ellipse and hyperbola. OL : OC : : OF : OG : : OA : OM; therefore LO × OM = OA × OC = OB², and OB² ÷ OM = OL.

Cor. 1. If the tangent meet the conjugate axis of the el-lipse or hyperbola in N, and GR be parallel to BO, the rect-angle NO × OR = OP². For the triangle AFC is simila[r] to LON, and OAF to OMG; hence LO : ON : : FA : A[C] and MO : MG : : OA : AF; therefore LO × OM : NO × M[G] = NO × OR : : OA : AC, or : : OB² : OP², and LO × OM = OB²; therefore NO × OR = OP².

Cor. 2. The rectangle OM × ML = BM × M*b*.

For LO × OM = OB², take OM² from each, and OM × ML = BM × M*b*.

Prop. VI. Problem. Given the abscissa and the para-meter of a parabola, to find GM the ordinate to the axis.

The triangles LMG, FAC, are similar (see last figure) hence LM : MG : : AF = MG : AC; therefore MG² = LM × AC = BM × 2AC = BM × parameter.

Let AB be 10, and the abscissa BM 22½, the parameter i[s] 40, and 40 × 22½ = 900 = MG²; therefore MG is 30.

Prop. VII. Problem. Given the two axes of an ellips[e] or of an hyperbola, and the abscissa, to find the ordinate GM.

Because the triangles FAC, LCH, are similar (see las[t] figure), the angle AFC = CLH, and therefore the triangl[e] FAC is similar to LGM, and LM : MG : : FA : AC, an[d] OM : MG : : OA : AF; hence LM × OM : MG² : : OA : A[C] : : OB² : OP², and LM × OM = BM × M*b*; therefore BO[²] : OP² : : BM × M*b* : MG².

Let the axes of an ellipse be 210 and 150, and the absciss[a] cut off from the vertex of the first be 42. What is the ordinate?

Ans. $(210 - 42) \times 42 = 7056$, and $210 : 150 :: \surd 7056 = 84 : 60$ the ordinate.

The following formulæ exhibit the rules for finding any of the quantities concerned.

Let the ratio of the curve be that of 1 to n, or in the parabola of n to n, AC the distance of the focus from the directrix $= d$, the abscissa BM $= x$, the ordinate MG $= y$, the subtangent ML $= t$, and in the ellipse and hyperbola, let OB the semi-transverse axis be $= a$, OP the semi-conjugate $= b$, OC the distance from the focus to the centre $= c$, and the parameter $= p$.

In the parabola. 1. AB $=$ BC $= \frac{1}{2}d$. 2. AM $= \frac{1}{2}d + x$ $=$ CG. 3. LM $= 2x$. 4. MG $= \surd\{(x + \frac{1}{2}d)^2 - (x - \frac{1}{2}a)^2\}$ $= \surd 2dx = \surd px = y$. 5. LG $= \surd\{2x \times (d + 2x)\}$. 6. CH $= \frac{1}{2}\surd\{(2x + d) \times d\} = \frac{1}{2}\surd(y^2 + d^2)$. 7. CN $=$ $\surd(2d \times \frac{1}{2}d) = d$.

In the ellipse and hyperbola. 1. BC $= \frac{d}{1+n} = a \times \frac{1-n}{n}$. 2. AB $= \frac{nd}{1+n} = a \times (1 - n)$ or $a \times (n - 1)$. 3. BO $= a =$ $\frac{nd}{1-n^2} = \frac{nd}{r^2}$ (putting $r^2 = 1 - n^2$ in the hyperbola, or $= n^2 - 1$ in the ellipse.) 4. CO $= \frac{d}{r^2} = \frac{a}{n} = \surd(a^2 - b^2)$ in the ellipse, and $= \surd(a^2 + b^2)$ in the hyperbola. 5. OP $= \frac{d}{r} = \frac{ar}{n} = b$. 6. OA $= \frac{n^2 d}{r^2} = \frac{a^2}{\surd(a^2 \mp b^2)}$. 7. OM $= a \mp x = \frac{nd}{r^2} \mp x$. 8. OL $=$ $\frac{a^2}{a \mp x} = \frac{n^2}{r^2} \times \frac{d^2}{nd \mp r^2 x}$. 9. ML $= \frac{2\,ndx \mp r^2\,x^2}{nd \mp r^2\,x} = \frac{2\,ax \mp x^2}{a \mp x}$. 10. OM $\times$ ML $=$ BM $\times$ Mb $= \frac{2\,ndx}{r^2} \mp x^2 = (2\,a - x) \times x$. 11. MG the ordinate $= \frac{\surd(2ndx \mp r^2 x^2)}{n} - \frac{b}{a} \times \surd(2ax + x^2) = \frac{r}{n}$ $\times \surd(2\,ax + x^2)$.

EXAMPLES.

1. In the parabola is given the parameter $p = 4$ to find the distance of the focus from the directrix, and from the principal vertex. Ans. AC $= d = \frac{1}{2}p = 2$, and BC $= \frac{1}{4}p = 1$.

2. In the parabola are given the distance of the focus from the directrix $d = 2$ and the absciss BM $= x = 9$, to find the distance of the ordinate from the directrix, and from the tangent at the extremity of the ordinate LM.

Ans. $AM = \frac{1}{2}d + x = 1 + 9 = 10$, $LM = 2x = 18$. Henc
$CM = x - \frac{1}{2}d = 8$.

3. In the parabola are given the distance of the focus fron the directrix $= 2$, or the parameter and the absciss 9, to fin the ordinate MG. Here $MG = \surd(x + \frac{1}{2}d + x - \frac{1}{2}d) \times (x + \frac{1}{2}d - x + \frac{1}{2}d) = \surd 2dx = \surd px = \surd(4 \times 9) = 6$.

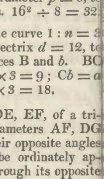
Again, let the parameter be 9 and the abscissa 16, the $\surd(9 \times 16) = \surd 144 = 12$ the ordinate.

Again, let $p = 54$ and $x = 6$, the ordinate is $\surd(6 \times 54) = 18$.

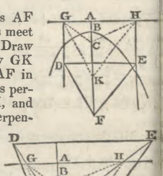
4. Given the ordinate $y = 16$ and the parameter $p = 8$, to find the abscissa. Ans. $16^2 \div 8 = 32$.

5. In the ellipse are given the ratio of the curve $1 : n = 3$ and the distance of the focus from the directrix $d = 12$, to find their distances from the principal vertices B and b. BC $= d \div (n + 1) = 12 \div 4 = 3$; $AB = 3 \times 3 = 9$; $Cb = d \div (n - 1) = 12 \div 2 = 6$, and $Ab = 6 \times 3 = 18$.

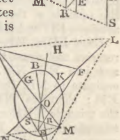
PROP. VIII. Theorem. If two sides, DE, EF, of a triangle DEF be ordinately applied to the diameters AF, DG of a conic section, which pass through their opposite angles F and D, the third side DF shall also be ordinately applied to the diameter EH, which passes through its opposite angle E.



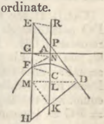
First, let one of the diameters AF be the axis, and let the diameters meet the directrix in A, G, and H. Draw GC, HC to the focus, and draw GK perpendicular to CH, meeting AF in K, and join HK. Because GK is perpendicular to CH, or CH to GK, and CA to GH; consequently GC is perpendicular to KH, that is, HK is an ordinate to GD, and is therefore parallel to EF. Hence in the parabola KF = EH = DG; and therefore DF is parallel to GK, which is an ordinate to EH. In the ellipse and hyperbola OK : OF : : OH : OE or : : OG : OD; therefore DF is parallel to GK, an ordinate to EH.



Next, let none of the diameters be the xis. Let DE and EF meet the axis in P nd R. Draw DL and MEN perpendicu- r to the axis, and let FK meet them in and N, and let DG meet MN in M. oin PL, PN, and MR, and let MR meet K in S. Because EN, EP are ordinates the diameters PR, KN, therefore PN is ordinate to EH. For the same ason MR is an ordinate to EH, d PL an ordinate to GD; there- re PL is parallel to EF and PN to S. Wherefore *in the parabola*, SN PR = LF and SF = LN = DM, d DF is therefore parallel to MS. nd *in the other curves* OS : ON : : R : OP, that is, : : OF : OL, and alter- tely OS : OF : : ON : OL, that is, : : OM : OD; therefore F is parallel to SM, and is an ordinate to the diameter EH.



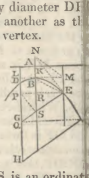
PROP. IX. Theorem. If a tangent to a conic section DE eet a diameter EF, and from the point of contact D, an dinate DH be applied to that diameter, then, in the rabola, the segment of the diameter EH between the tan- nt and the ordinate is bisected in the vertex F. And in e ellipse and hyperbola the semidiameter OF is a mean oportional between the segments of it OE and OH from the ntre, intercepted by the tangent and the ordinate.



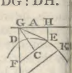
Let the axis meet the tangent in P, and e ordinate in K. Draw DM parallel to e directrix, and FN to touch the curve F, and let the axis meet them in L d N. Join FC, GN, they are parallel on. Sec. III.), and draw ER parallel FC. Because DM and DK are ordi- tes to the diameters through K and M, erefore MK is an ordinate to the diame- through D, and it is therefore parallel DE. Wherefore *in the parabola*, EM PK, and GM = AL = PC; therefore = EG = RN, and NK = CR, that HF = FE.

In the ellipse and hyperbola. OK : OP OM : OE; but because OB is a mean oportional between OL and OP, and also ween OC and OA, therefore OP : OC OA : OL : : OG : OM. Wherefore, *by* *inverse equality*, OK : OC : : OG : OE : : ON : OR, ar *alternando* OK : ON : : OC : OR, that is, OH : OF : : O : OE.

Prop. X. Theorem. If from two points E and F of parabola ordinates EG, FH be applied to any diameter DI the squares of the ordinates will be to one another as th abscissas DG and DH between them and the vertex.



Draw LBM, EP, FQ, parallel to the directrix, and draw the tangents DK, EN, and join BE, NM. Because ER is an ordinate to NB, therefore NB = BR = EM; therefore NM is parallel to BE, the triangle NBE = MBE, and the whole triangle NRE = BREM. In the same manner it may be proved, that BLPR = KDPR. And because PE, ES, are ordinates to the diameters through S and P, PS is an ordinat to the diameter through E, and is parallel to EN, therefor NR : RS : : ER : RP, and NR : RS : : triangle NRE : RES and ER : RP : : parallelogram RM : RL, and the triangl NRE = RM, therefore the triangle RSE = RL = KDPR and by adding PRSG, the triangle EPG = KDGS. In th same manner it may be proved, that the triangle FQH = parallelogram KDHT. And the triangles are similar; therefore $GE^2 : FH^2$: : EPG : FQH : : KG : KH : : DG : DH.

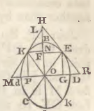


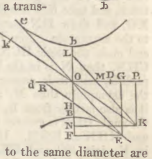
Cor 1. If the ordinate EF to the diameter DG pass through the focus C, EF is ½ the parameter of DG. Let DG meet the directrix in G, join GC, it is perpendicular to EF, and DG = DF. Also GE will touch the curve at E. Draw EH parallel to DG, the triangles GCE, GHE, ar equal, and the angle GEC = GEH = FGE; therefore FE = FG = ½ parameter.

Cor. 2. If KL be another ordinate to DG, $LK^2 = DL \times$ parameter. For $EF^2 : LK^2$: : DF : DL : : DF × par. : to DL × par. and $EF^2 = DF \times 2\,FG = DF \times$ par.; therefore KL^2 = DL × parameter.

Prop. XI. Theorem. If from any point E of the ellipse or hyperbola, an ordinate be applied to any diameter B*b* the square of the diameter D*d*, which is parallel to the ordinate, is to the square of the ordinate EF, as the square of the diameter B*b*, to which the ordinate is applied, to the difference between the square of this semidiameter and the square of the segment of it between the centre O and the ordinate EF.

Let the tangent at E meet the diameters Bb, and Dd in H and R, and draw EG parallel to Bb, it is an ordinate to Dd. Therefore $OB^2 = FO \times OH$, and $OD^2 = OG \times OR$. Also $OD^2 : OG^2 :: OR : OG = EF$, that is, $:: OH : HF$. But because $OB^2 : OF^2 :: OH : OF$, therefore (by conversion, when Bb is in the ellipse, or a transverse of the hyperbola, and by composition when OB is a conjugate) $OB^2 : OB^2 \mp OF^2 :: OH : HF$, that is, $:: OD^2 : EF^2$.





Cor. 1. When Bb is a transverse diameter, the rectangle $HF \times FO = BF \times Fb$ is $= OB^2 - OF^2$ (Con. Sec. III.); therefore $Bb^2 : Dd^2 :: BF \times Fb : EF^2$.

Cor. 2. The squares of ordinates to the same diameter are to one another as the rectangles contained by the abscissas between them and the vertices.

Prop. XII. Theorem. If from the vertices E and K of two conjugate diameters of the ellipse or hyperbola ordinates EF, and KN be applied to any other diameter Bb, the rectangle $BF \times Fb$ contained by the abscissas of that diameter between one of the ordinates and its vertices is equal to the square of ON the segment between the other ordinate and the centre. (See figure to Prop. XI.)

Let the tangents at E and K meet the diameter Bb in H and L. Because HE is parallel to OK, KL to OE, and KN to EF, the triangles KON, FEH are similar, and likewise OKL, HEO; therefore $FH : HE :: NO : OK$, and $HE : HO :: KO : OL$, and, *by equality*, $FH : HO :: NO : OL$, and multiplying the two first by OF, and the other two by ON, the rectangle $HF \times FO : HO \times OF :: ON^2 : LO \times ON$, but $HO \times OF = OB^2 = LO \times ON$, therefore $ON^2 = HF \times FO = BF \times Fb$. And in the same way we prove that $OF^2 = BN \times Nb$.

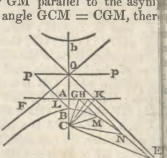
Cor. 1. $Bb : Dd :: ON : EF$, and $:: OF : KN = OP$.

Cor. 2. In the ellipse $OF^2 + ON^2 = OB^2$, but in the hyperbola $OF^2 - ON^2 = OB^2$.

Cor. 3. If KP be parallel to Bb, then FP is parallel to BD or Bd.

Prop. XIII. Theorem. The asymptotes and the hype bola continually approach, and at length come nearer to o another than by any given distance, but they never meet.

Join BP, Bp the vertices of the axes, and parallel to the draw OE, OF, these are the asymptotes. Let G be an point in the directrix, and draw GM parallel to the asym tote, and join GC, and make the angle GCM = CGM, ther fore M is in the hyperbola. Let GK be any given distance, and take KH less than KG, and draw HN parallel to the asymptote. Join HC, and make the angle HCN = CHN, then N is in the hyperbola, and it is nearer to the asymptote than M, and it is also farther from B, for

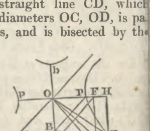


the angle HCN is greater than GCM, because CHK $>$ CGK and KHN = KGM. If the hyperbola meet the asymptot in E, join EC and CK, then ECK = EKC = a righ angle, which is impossible; therefore they never meet.

That CK is perpendicular to the asymptote may be prove thus: The triangles OPB, OAK, are similar; hence KO OA : : PB = OC : OB : : OB : OA; therefore OK = OB and the angle OKC = OAK = a right angle.

Prop. XIV. Theorem. The straight line CD, which joins the vertices of two conjugate diameters OC, OD, is parallel to OL, one of the asymptotes, and is bisected by the other OK.

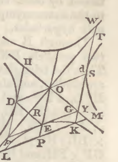
Draw CE, CF, DG, DH, parallel to the axes OB, OP, and join BP, FG. They are parallel to one another, and BP is bisected by the asymptote OK; therefore FC, GD will meet one another in OK, let it be at K. Then (Con. Sec. XI. Cor. 1.) OE : OG : : OH : OF, that is, FC : FK : : DG : GK; therefore CD is parallel to FG or BP, and because OK bisects BP, it also bisects FG and CD.



Prop. XV. Theorem. If a straight line, FEG, touch the hyperbola in E, the segments of it between the point of contact and the asymptotes will be equal; and if a straight line MN cut the hyperbola, or opposite hyperbolas, in K and L.

the segments of it, MK, LN, between the hyperbola and the asymptotes will be equal.

Draw the diameter OE, and its conjugate OD, and join DE, meeting the asymptote ON in R. Then EGOD is a parallelogram, and ER = RD; therefore EF = DO = EG.



Bisect KL in P, and draw the diameter OP, and through its vertex E draw FG parallel to KL, it touches the hyperbola in E; therefore FE = EG, consequently MP = PN. But the ordinate KL is bisected in P, ∴ KP = PL; therefore MK = LN.

Cor. 1. The tangent FG = the diameter D*d* parallel to it.

Cor. 2. The rectangles MK × KN, ML × LN, MK × ML and KN × NL, are all equal.

Cor. 3. The subtangent FR = OR, the distance from the centre.

Cor. 4. FD touches the adjacent hyperbola in D.

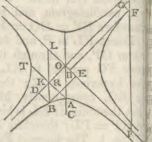
Prop. XVI. Theorem. If a straight line which cuts the hyperbola, or the opposite hyperbolas, meets the asymptotes, the rectangle contained by the segments of it between a point in the hyperbola and the asymptotes, is equal to the square of the semidiameter parallel to it.

Let MN (see last figure) cut the hyperbola in K, and meet the asymptotes in M and N, and let DO be the semidiameter parallel to it. Draw OE the diameter conjugate to DO, it bisects KL; draw also FEG parallel to MN, it touches the hyperbola, and EG = OD. But $OE^2 : OD^2 = EG^2 :: OP^2 : PM^2$, also $OE^2 : OD^2 :: OP^2 - OE^2 : PK^2$; therefore $OE^2 : OD^2 :: OE^2 : PM^2 - PK^2 = MK \times KN$, consequently $OD^2 = MK \times KN$.

Again, let KW cut the opposite hyperbolas, and meet the asymptotes in T and Y, and be parallel to the diameter OE, and let OD be its diameter which meets it in S, and let FH be the tangent parallel to it. Then $OD^2 : DF^2 = OE^2 :: OS^2 : ST^2$, also $OD^2 : OE^2 :: OD^2 + OS^2 : KS^2$; therefore $OD^2 : OE^2 :: OD^2 : KS^2 - ST^2 = TK \times KY$, consequently $OD^2 = TK \times KY$.

Cor. The rectangles under segments of parallels between points in the hyperbola and the asymptotes are equal.

Prop. XVII. Theorem. The rectangle contained by any two straight lines, BD, BE, drawn from a point B in the hyperbola to the asymptotes, is equal to the rectangle contained by other two lines, FG, FH, parallel to them, drawn to the same asymptotes from any point F of the four conjugate hyperbolas.



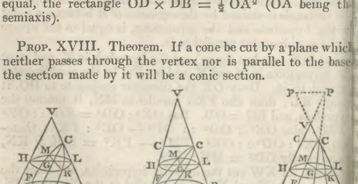
Through B and F draw any two parallels BKL and MFP. Then the triangles DBK, FGM, are similar, and also the triangles BEL, FHP, and therefore BK : BD : : MF : FG, and BL : BE : : FP : FH. Wherefore $BK \times BL : BD \times BE : : MF \times FP : GF \times FH$, and $BK \times BL = MF \times FP$; therefore $BD \times BE = GF \times FH$.

Cor. 1. If BD, FG be parallel to the asymptote, the rectangle $DB \times DO = OG \times GF$, and if BE, FH be also parallel to the asymptote, the parallelogram $DE = HG$.

Cor. 2. If AR be the line which joins the vertices of the axes, and C the focus, $AR = RO = \frac{1}{2}$ OC; therefore the rectangle $OD \times BD = AR^2 = \frac{1}{4} OC^2$.

Cor. 3. If the hyperbolas be equilateral, or have their axes equal, the rectangle $OD \times DB = \frac{1}{2} OA^2$ (OA being the semiaxis).

Prop. XVIII. Theorem. If a cone be cut by a plane which neither passes through the vertex nor is parallel to the base, the section made by it will be a conic section.



Let the cone AEBV, of which the base is the circle AEBF and vertex V, be cut by a plane, which forms the section ECF, this is a conic section. Let it meet the base in the line EDF, and draw the diameter AB perpendicular to EF, and join AV and BV, and let the plane ABV cut the section ECF in CD. Let a plane parallel to the base cut the cone in the circle HKL, and the planes ABV and ECF in HL and KGM. The base AEB is perpendicular to AVB;

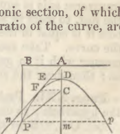
Therefore DE and the plane ECF are perpendicular to AVB, and the angles EDC, KGC are right angles. And because AVB bisects the cone, $ED = DF$, $KG = GM$, and the rectangle $AD \times DB = DE^2$, and $HG \times GL = GK^2$ (El. Geom. 19).

First, let CD be parallel to AV, then $AD = HG$ and by similar triangles $CD : CG :: DB : GL :: AD \times DB : HG \times GL :: DE^2 : GK^2$, which is the property of the parabola.

Next, let CD meet AV in P. Then by similar triangles $PD : PG :: AD : GH$ and $CD : CG :: DB : GL$; therefore $PD \times DC : PG \times GC :: AD \times DB : HG \times GL :: DE^2 : GK^2$, which is the property of the ellipse or hyperbola, viz. of the ellipse, if P be below V, and of the hyperbola, if P be above V.

Cor. In the ellipse and hyperbola. If Cc and Pp be parallel to AB, then $\sqrt{Cc \times Pp}$ = conjugate axis.

Prop. XIX. Prob. To describe a conic section, of which the directrix AB, the focus C, and the ratio of the curve, are given.



Draw CA perpendicular to AB, and divide it in D, so that CD be to DA in the ratio of the curve, by Prob. IX. Practical Geometry.) Draw BP at right angles to BA, and draw CP, so that $CP : BP :: CD : DA$, and let CP revolve about C, and at the same time let BP move perpendicular to AB, still retaining the same ratio; then their intersection P will describe the curve.

Or by points. Draw DE parallel to AB, and make it equal to DC; join AE, and produce it. Draw a great many parallels to AB, meeting AC in m, and AE in n. Take mn on any of them, and from the centre C cut that parallel in P and p; these are two points in the curve. (Con. Sec. I.) In the same manner two points may be found in every parallel, and the curve made to pass through them all.

Prop. XX. Prob. Given the transverse and conjugate axes of a hyperbola or ellipse; to describe the curve.

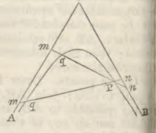
Add the squares of the two semiaxes in the hyperbola, or subtract them in the ellipse, and take the square root of the sum or remainder: this root has to the transverse semiaxis the ratio of the curve, with which the curve may be described as before; for the difference between the root and the transverse semiaxis is the distance of the focus from the principal vertex (Con. Sec. VII. formula 4.); and a fourth proportional to the root, the transverse semiaxis, and their difference, will give the distance of the directrix from the principal vertex.

Otherwise, let Bb and Pp be the axes, bisecting one another at right angles in the centre O. Lay BP in the hyperbola from O to C and c, or lay BO in the ellipse from p to C and c; then C and c are the foci. (Con. Sec. II.) Take any point m in Bb (produced in the hyperbola), and with the distance Bm describe two arcs n, n, from each of the foci C and c. With bm for a radius, from the foci cut these arcs in n, n, n, n; then, since in the ellipse the transverse axis is equal to the sum of two lines drawn from the foci, to meet in any point of the curve, and in the hyperbola it is equal to their difference, these will be four points of the curve. Take another point m, and proceed in the same manner with it to get other four points of the curve, and so on; then draw the curve through all these points.

Prop. XXI. Prob. Given the asymptotes and a point in the hyperbola; to describe the curve.

Let OA, OB be the asymptotes, and P the point in the curve.

Through P draw any straight line meeting the asymptotes in m and n. Make mq equal to nP, then q is a point in the curve. (Con. Sec. XV.) In this way any number of points in the curve may be found, and the curve drawn through them all will be the hyperbola.



MENSURATION OF CONIC SECTIONS AND THEIR SOLIDS.

DEFINITIONS.

1. A SPHEROID is a solid generated by the revolution of an ellipse about one of its axes. It is called a *Prolate Spheroid* when the revolution is made about the transverse axis, and an *Oblate Spheroid* when made about the conjugate.

NOTE. The axis about which the ellipse revolves is called the axis of the Spheroid, and the other its *Greatest Diameter*.

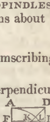
2. A PARABOLIC CONOID, or a PARABOLOID, is a solid generated by a parabola about its axis.

3. A HYPERBOLIC CONOID, or a HYPERBOLOID, is a solid generated by a hyperbola about its axis.

4. ELLIPTIC, PARABOLIC, and HYPERBOLIC SPINDLES, are solids formed by the revolution of these sections about a double ordinate.

THEOREM I. A spheroid is two-thirds of its circumscribing cylinder.

Let ABC be a semi-ellipse, AC the axis, OB perpendicular to AC; describe the parallelogram ADPC, and join DO. Draw EF, GH parallel to OB, and let EF meet the circumference in L, and OD in K, and complete the rectangles GMKF, and NLF. If the figure revolve about AC, the semi-ellipse ABC will describe a spheroid, ADPC a cylinder, and ADO a cone. Also the figures GE, GL, and GK will describe cylinders.



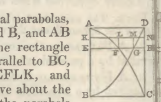
Now $AF \times FC : FL^2 :: AO^2 : OB^2 = AD^2 :: OF^2 : FK^2$; therefore $AF \times FC + OF^2 : FL^2 + FK^2 :: AO^2 : AD^2$, and $AF \times FC + OF^2 = AO^2$; therefore $FL^2 + FK^2 = AD^2 = EF^2$; hence the cylinder described by GL and GK are together = cylinder described by GE.

In the same manner, every cylinder in the hemispheroid, with the corresponding cylinder about the cone, is equal to the corresponding part of the cylinder described by AB, and the number of these cylinders may be increased, so that altogether they will not differ from the hemisphere and cone; consequently the hemispheroid and cone are together equal to the circumscribing cylinder, and the cone is one-third of the cylinder, therefore the spheroid is two-thirds of its circumscribing cylinder.

Cor. 1. Hence any part of the spheroid, with the corresponding part of the cone, is equal to the corresponding part of the cylinder. Thus the segment described by ALF, together with the frustum described by ADKF = cylinder described by ADEF.

Cor. 2. The sphere and its portions are to the spheroid and its corresponding portions as $AO^2 : OB^2$, (Theorem IV, p. 200.)

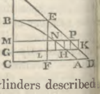
Theorem II. A parabolic conoid is one-half of its circumscribing cylinder.



Let BAC and ABD be two equal parabolas, which have their vertices at A and B, and AB their common axis. Complete the rectangle ABCD, and draw EH, KN parallel to BC, and complete the rectangles EFLK, and EGMK, and let the whole revolve about the axis AB. By the property of the parabola $EF^2 : EG^2 :: AE : EB$, and $EF^2 : EF^2 + EG^2 :: AE : AB :: EF^2 : BC^2 = EH^2$; hence $EF^2 + EG^2 = EH^2$ and therefore the cylinders described by EL and EM are, together, equal to the cylinder described by EN. And thus one of the paraboloids with cylinders, which, together, are greater than the other paraboloid, is greater than the cylinder described by BD, and with cylinders less than the paraboloid, it is less than that cylinder; therefore the two paraboloids are equal to the cylinder, or the paraboloid is half the cylinder.

Cor. The paraboloid described by BEG, with the frustum described by BEFC is equal to the cylinder described by BH. If, therefore, $BC = y$, $BE = x$, and $EF = z$, then $EG^2 = y^2 - z^2$, and the conoid described by $BEG = (y^2 - z^2)\,\frac{1}{2}px$, and the cylinder $= y^2 \times px$; therefore the frustum described by $BEFC = \frac{1}{2}\,(y^2 + z^2)\,px$.

Theorem III. The hyperbolic conoid is equal to the difference between the corresponding frustum of the asymptotic cone, and the cylinder of the same altitude, which has the conjugate axis for the diameter of its base.



Let BCA be a hyperbola, of which OBC is the transverse axis, and OD the asymptote, draw the tangent BE, it is = the conjugate semi-axis. Draw any two straight lines GK, MP, parallel to CD, and complete the rectangles MH, MK, GN, GP, and CE, and let the whole revolve about BC. Because $GH^2 + BE^2 = GK^2$, the cylinders described by the rectangles ML and MH are equal to that described by MK. And for the same reason, the cylinders described by GN and ML are equal to that described by GP. Therefore the cylinder described by CE, together with any series of cylinders about the hyperboloid, is greater than the frustum described by BEDC, and with any series in the hyperboloid, it is less than the frustum; therefore the cylinder and hyperboloid are equal to the frustum.

Cor. 1. If $OB = a$, $BE = c$, $BC = x$, and $CA = y$, then $CD = \frac{c}{a}(a + x)$. And the conic frustum made by BCDE $= (a^2 + ax + \frac{1}{3}x^2)\frac{c^2xp}{a^2}$, and the cylinder made by CE $= \frac{a^2c^2xp}{a^2}$, and taking their difference, the hyperboloid $= (ax + \frac{1}{3}x^2) \times \frac{c^2xp}{a^2}$, or putting $\frac{y^2}{2ax + x^2}$ instead of $\frac{c^2}{a^2}$, it becomes $\frac{ax + \frac{1}{3}x^2}{2ax + x^2} \times y^2xp = \frac{2a + \frac{2}{3}x}{2a + x} \times \frac{1}{2}y^2xp$.

Cor. 2. If $CG = x$, and $BG = m$, the content of the hyperboloid described by CBA will be $\frac{c^2p}{a^2} \times \{a \times (m + x)^2 + \frac{1}{3}(m + x)^3\}$, and the content described by GBH will be $\frac{c^2p}{a^2} \times (am^2 + \frac{1}{3}m^3)$, and their difference $= \frac{1}{2}\frac{c^2px}{a^2} \times (4am + 2ax + 2m^2 + 2mx + \frac{2}{3}x^2)$ will be the content of the frustum described by CGHA. But $\frac{1}{2}pxy^2 = \frac{\frac{1}{2}c^2px}{a^2}(2am + 2ax + m^2 + 2mx + x^2)$, and putting $GH = v$, $\frac{1}{2}pxv^2 = \frac{\frac{1}{2}c^2px}{a^2} \times (2am + m^2)$, and the sum of these two $\frac{1}{2}px \times (y^2 + v^2) = (4am + 2ax + 2m^2 + 2mx + x^2) \times \frac{\frac{1}{2}c^2px}{a^2}$, which exceeds the content by $\frac{\frac{1}{2}c^2px}{a^2} \times \frac{1}{3}x^2$, wherefore the content of the frustum is $= \frac{1}{2}px\left(y^2 + v^2 - \frac{c^2x^2}{3a^2}\right)$.

Prob. I. To find the area of an ellipse.

Rule. Multiply one of the semiaxes by the other, and by 3·1416; or one of the axes by the other, and by ·7854.

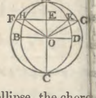
Or if the circle upon either axis be given: As that axis is to the other, so is the circle to the ellipse, and so is any sector or segment of the circle to the sector or segment of the ellipse which has the same chord perpendicular to the first-mentioned axis.*

1. Required the area of the ellipse ABCD, of which the semiaxes are OA 436, and OB 254 feet.

Ans. $3 \cdot 1416 \times 436 \times 254 = 347913 \cdot 3504$ square feet, = acres 3 roods 37 perches 27 yards 7 feet.

2. Required the area of an ellipse, of which the axes are 526 and 354 inches.

Ans. $146244 \cdot 6216$ sq. in. = 112 yards 7 feet 84·62 inches.



3. Required the area of the sector OHAK of an ellipse, the chord HK being perpendicular to the greater axis AC; the axes AC 72, BD 54, and the versed sine AE 18 feet.

The angle FOG is 120°. The circle $= 4071 \cdot 50408$, and $\frac{1}{3}$ of it $\times \frac{3}{4} = 1017 \cdot 87601536$ square feet the area of the sector.

4. Required the area of the segment of an ellipse, the chord being perpendicular to the less axis, the versed sine 12, and the axes 80 and 60 yards.

Ans. 536·7504 square yards, = 17 perches $22\frac{1}{4}$ yards.

5. Required the area of the segment of an ellipse, the chord being perpendicular to the greater axis, the height 25 feet and the axes 156 and 120 feet.

Ans. 1521·936 square feet = 5 perches 17 yards 7·686 feet.

6. Required the area of the segment of an ellipse, the chord being perpendicular to the less axis, the height 110, and the axes 246 and 180 yards.

Ans. 22267·92492 square yards = 4 acres 2 roods 16 perches 3 yards 8·32338 feet.

Prob. II. To find the circumference of an ellipse.

Rule. Add the squares of the two axes, and take the square root of half the sum, and to the half of this root add a fourth of the sum of the axes, and then multiply by 3·1416; the product will be the circumference nearly.†

* If any two straight lines be drawn perpendicular to AC, and the points be joined in which they meet the circle and the ellipse, these trapezoids are to one another as EG to EK, and their number may be multiplied, until their sum, either in the circle or ellipse, shall be more nearly equal to it than by any given difference. Therefore the circle and ellipse, which are their limits, are in that ratio; that is, the circle is to the ellipse as EG to EK, or AC : BD, or as $AC^2 \times \cdot 7854 : AC \times BD \times \cdot 7854$. See also Appendix.

† Let $t =$ the transverse axis, $c =$ the conjugate $d = 1 - (c^2 \div d^2)$, and $p =$ the periphery of the circumscribing circle, then it will be shown in the

That is, using the same symbols as in the demonstration
$\times \left\{\frac{t+c}{4} + \frac{1}{2}\sqrt{\left(\frac{t^2+c^2}{2}\right)}\right\}$ is the circumference.

1. Required the circumference of the ellipse, of which the
es are 24 and 18.

Ans. $\sqrt{\frac{24^2+18^2}{2}} = 21{\cdot}2132$, and $\frac{24+18}{2} = 21$, then $(21{\cdot}2132$
$21) \div 2 \times 3{\cdot}1416 = 66{\cdot}3085$ the circumference.

2. Required the circumference of the ellipse, of which the
es are 60 and 40 feet.
Ans. 158·6354 feet = 9 poles 3 yards 1 foot 1·6248 inches.

3. Required the circumference of the ellipse, of which the
es are 256 and 196 feet. Ans. 713·1156 feet.

4. Required the circumference of the ellipse, of which the
es are 320 and 240 yards. Ans. 884·1133 yards.

5. Required the circumference of the ellipse, of which the
es are 16·6 and 12·8 inches. Ans. 46·3736 inches.

6. Required the circumference of the ellipse, of which the
es are 27 and 18 poles. Ans. 71·385917 poles.

PROB. III. To find the area of a parabola.

RULE. Multiply the base by the perpendicular height,
d $\frac{2}{3}$ of the product will be the area.*

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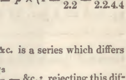
pendix that the circumference of the ellipse is $= p \times (1 - \frac{d}{2.2} - \frac{3d^2}{2.2.4.4} - \frac{3.3.5.d^3}{2.2.4.4.6.6} - \&c.)$.

But $\sqrt{(1-\frac{1}{2}d)} = 1 - \frac{d}{2.2} - \frac{d^2}{2^3.4} - \frac{3d^3}{2^6.4.6} - \&c.$ is a series which differs
m the former only by the small series $-\frac{d^2}{64} - \frac{3d^3}{256} - \&c.$; rejecting this dif-
nce, therefore, we have $p \times \sqrt{(1-\frac{1}{2}d)} =$ the circumference of the el-
e. Now as the one series gives the circumference nearly as much too
e as the other gives it too small, their arithmetical mean, or $\frac{1}{2}p \times$
$\frac{t+c}{2} + \sqrt{\left(\frac{t^2+c^2}{2}\right)}\}$, which is the rule, gives the circumference very ac-
ately.

* If EG bisect AD, the triangle AFG = $\frac{1}{2}$
B, or it is $>$ $\frac{1}{2}$ trilineal AFBK. Also,
e GK = KH, the triangle PLG = $\frac{1}{2}$ ALG,
$>$ $\frac{1}{2}$ trilineal AGBK; and every triangle
s formed cuts off more than the half of what
left by the preceding; therefore the trilineal
BK is the limit of the sum of the triangles.
w the triangle AFG = $\frac{1}{4}$FD, and the triangle
L = $\frac{1}{4}$AGB, or of AFG, and so on; therefore



1. Required the area of the parabola ABC, of which the base AC is 54, and the height BD 36 feet.

Ans. $\frac{2}{3} \times 54 \times 36 = 1296$ square feet area.

2. Required the area of the parabola, of which the base is 42, and the height 63 yards. Ans. 1764 square yards.

3. Required the area of the parabola, of which the base is 482, and the height 320 feet. Ans. $102826\frac{2}{3}$ square feet.

4. Required the area of the parabola, the base 126, and the height 210 inches. Ans. 17640 sq. in. = 13 yds. 5 ft. 72 in.

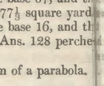
5. Required the area of the parabola, the base 67, and the height 98 yards. Ans. $4377\frac{1}{3}$ square yards.

6. Required the area of the parabola, the base 16, and the height 12 poles. Ans. 128 perches.

Prob. IV. To find the area of a frustum of a parabola.

Rule. Find a third proportional to the sum of the bases and one of them, to which add the other base: the sum, multiplied by two-thirds of the height, gives the area.

That is, $\left(A + \frac{a^2}{A+a}\right) \times \frac{2}{3}b$, or $\left(a + \frac{A^2}{A+a}\right) \times \frac{2}{3}b =$ the area; A, a being the two ends, and b the height.*



1. Required the area of the frustum of a parabola, of which the bases are 64 and 32, and the height 26 feet.

Ans. $64 + 32 : 32 :: 32 : 10\frac{2}{3}$, and $(10\frac{2}{3} + 64) \times 26 \times \frac{2}{3} = 74\frac{2}{3} \times 17\frac{1}{3} = 1294\frac{2}{9}$ sq. feet = 4 per. 22·8 yds. the area.

---

the sum of them is FD $\times \left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3}, \text{\&c.}\right)$, and the limit of this geometrical series is FD $\times \frac{1}{4-1} = \frac{1}{3}$FD $= \frac{1}{3}$BD $\times$ AD, and therefore AKBL $= \frac{2}{3}$FD. See also Appendix.

* Let A = AC, a = FG, and b = ED, then by the property of the parabola $A^2 - a^2 : b :: A^2 : \frac{bA^2}{A^2 - a^2} :: a^2 : \frac{ba^2}{A^2 - a^2} =$ the altitudes DB and ED of the two complete segments whose bases are the ends A, a of the frustum; hence the difference of the areas of these segments = the area of the frustum AFGC. That is, $\frac{2}{3}b \times \left(\frac{A^3}{A^2 - a^2}\right) - \frac{2}{3}b \times \left(\frac{a^3}{A^2 - a^2}\right) = \frac{2}{3}b \times \frac{A^3 - a^3}{A^2 - a^2} = \frac{2}{3}b \times \frac{A^2 + Aa + a^2}{A + a} = \frac{2}{3}b \times \left(A + \frac{a^2}{A+a}\right) = \frac{2}{3}b \times \left(a + \frac{A^2}{A+a}\right)$ which affords the rule.

2. Required the area of the frustum of a parabola, of which ıe bases are 16 and 54, and the height 46 yards.

Ans. 1768·15238 sq. yds. = 1 ro. 18 per. 13·65 yds.

3. Required the area of the frustum of a parabola, of which ıe bases are 364 and 186, and the height 280 feet.

Ans. 79688·33$\frac{3}{3}\frac{1}{3}$ square feet.

4. Required the area of the frustum of a parabola, of which ıe bases are 424 and 268, and the height 318 inches.

Ans. 111891·8828 square inches.

5. Required the area of the frustum of a parabola, of which ıe bases are 63 and 22, and the height 44 poles.

Ans. 2015·024 perches.

6. Required the area of the frustum of a parabola, of which e bases are 18 and 12, and the height 20 yards.

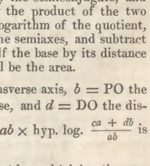
Ans. 304 square yards.

PROB. V. To find the area of a hyperbola.

RULE. Multiply half the base by the semitransverse axis, d its distance from the centre by the semiconjugate, and vide the sum of the products by the product of the two miaxes, and take the hyperbolic logarithm of the quotient, d multiply it by the product of the semiaxes, and subtract e product from the product of half the base by its distance om the centre: the remainder will be the area.

That is, if a = BO the semitransverse axis, b = PO the miconjugate, c = AD half the base, and d = DO the distance from the centre; then $cd - ab \times$ hyp. log. $\frac{ca + db}{ab}$ is e area.*

NOTE. The hyperbolic logarithm is got by multiplying the comn logarithm by 2·30258509.



1. Required the area of the perbola ABC, of which the base C is 24, the altitude BD 10, e transverse axis Bb 30, and the njugate Pp 18 feet.

Ans. $\frac{12 \times 15 + 25 \times 9}{15 \times 9} = 3$, of ich the logarithm 0·477121 2·30258509 = 1·0986117 the perbolic logarithm of 3; and

* For the demonstration of this rule see Appendix.

this logarithm, multiplied by 15×9, gives 148·3125795, which taken from 25×12, leaves 151·6874205 sq. feet the area.

2. Required the area of the hyperbola, of which the base is 208, the height 70, and the transverse semiaxis 105 yards.

$\sqrt{\{(210+70) \times 70\}} : 104 :: 105 : 78$ the semiconjugate.

Ans. 9202·36772 sq. yds. = 1 ac. 3 ro. 24 per. 6·3677 yds.

3. Required the area of the hyperbola, of which the base is 384, the height 250, and the axis 176 feet.

Ans. 55686·0453 square feet.

4. Required the area of the hyperbola, of which the base is 156, height 196, and axis 248 yards. Ans. 18449·697 sq. yds.

5. Required the area of the hyperbola, of which the base is 48, height 22, and axis 36 inches. Ans. 647·2532 sq. in.

6. Required the area of the hyperbola, of which the base is 96, height 110, and axis 124 poles. Ans. 6324·6852 perches.

SOLIDS.

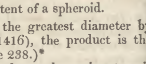
Prob. VI. To find the solid content of a spheroid.

Rule. Multiply the square of the greatest diameter by the axis, and by ·5236 (or $\frac{1}{6}$ of 3·1416), the product is the content. (Theorem I. Cor. 2. page 238.)*

That is, if t = the transverse, and c = the conjugate axis of the generating ellipse; then $\cdot 5236 \times tc^2$ = the oblate, and $\cdot 5236 \times t^2c$ = the solidity of the oblong spheroid.

1. Required the solid content of an oblong spheroid, the axes of the generating ellipse being 54 and 36 inches.

Ans. $36^2 \times 54 \times \cdot 5236 = 36643 \cdot 6224$ cubic in. the content.



2. Required the content of the oblate spheroid ABCD, the axes of the generating ellipse being 42 and 30 feet.

Ans. 27708·912 cubic feet.

3. Required the content of an oblong, and also of an oblate spheroid, the axes of each ellipse being 48 and 36 inches.

Ans. The oblate 43429·4784, and the oblong 32572·1088 cubic inches.

4. Required the content of an oblong spheroid, of which the axes are 50 and 30 yards. Ans. 23562 cubic yards.

5. Required the content of an oblong, and also of an oblate spheroid, the axes of each ellipse being 25 and 15 inches.

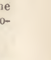
Ans. Oblong 2954·25 cubic in. oblate 4908·75 cubic in.

* If a circle be described upon either axis of an ellipse, and both revolve about that axis, the spheroid generated by the ellipse will be to the sphere described by the circle, as the circle described by the revolving axis of the ellipse to the circle described by the diameter of the circle; and so is any segment or frustum of the spheroid to the corresponding segment or frustum of the sphere.

PROB. VII. To find the solid content of a segment of a heroid.

RULE. Find the spherical segment which has the same ight and the same axis; then, if the base be perpendicular the fixed axis, the square of that axis is to the square of the her as the spherical to the spheroidal segment. But if the volving axis be perpendicular to the base, that axis is to the ed one as the spherical to the spheroidal segment. (Theo- m I. Cor. 2. page 238.)

1. The height CG of the segment ECF the oblong spheroid ABCD, of which the se is perpendicular to the fixed axis, is , the axes are AC 48 and BD 38 feet. equired the content.



Ans. $\{(48\times 3) - (16\times 2)\}\times 16^2\times \cdot 5236 = 112\times 256$ $\cdot 5236 = 15012\cdot 6592$; then $48^2 : 38^2 :: 15012\cdot 6592 :$ 08·97564 cubic feet the content.

2. Required the content of a segment of an oblate spheroid, e base perpendicular to the fixed axis, the height 12, and e axes 44 and 30 inches. Ans. 10704·562176 cubic inches.

3. Required the content of a segment of an oblong spheroid, e base parallel to the fixed axis, the height 14, and the axes and 45 inches. Ans. 13177·12704 cubic inches.

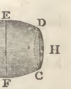
4. Required the content of a segment of an oblate spheroid, e base parallel to the fixed axis, the height 18, and the axes and 42 feet. Ans. 16245·339264 cubic feet.

PROB. VIII. To find the solid content of the middle zone a spheroid.

RULE. To twice the area of the greater base add the ea of the less, and multiply the sum by one-third of the ngth or height: the product will be the solid content. Theorem I. Cor. 1. p. 238, and Theorem IV. Cor. 1. p. 200.)

That is, if D = the diameter of the greater end, and d that of the less, a = the altitude, and $n = \cdot 7854$; then $(D^2 + d^2)\times \frac{1}{3}an$ = the solidity of the zone.

1. Required the content of the middle e ABCD of an oblong spheroid, the ses being perpendicular to the fixed is, the height GH 48, the greater ameter EF 42, and the less AB 32 hes.



Ans. $(42^2\times 2 + 32^2)\times 16\times \cdot 7854 = 4552\times 16\times \cdot 7854 =$ 202·2528 cubic inches the content.

2. Required the content of the middle zone of an oblong

spheroid, the bases parallel to the fixed axis, the height the diameters of the greater base 54 and 42, and those of less 35 and 25 inches. Ans. 39664·7944 cubic inch

3. Required the content of the middle zone of an obl spheroid, the bases perpendicular to the fixed axis, the heig 19, the diameter of the greater base 46, and of the less 38 fe

Ans. 28233·5592 cubic fe

4. Required the content of the middle zone of an obla spheroid, the bases parallel to the fixed axis, the height the diameters of the greater base 35 and 50, and those of t less base 20 and 28 feet. Ans. 12754·896 cubic fe

5. Required the content of the middle zone of an oblo spheroid, the bases perpendicular to the fixed axis, the leng 40, and the diameters 30 and 18 inches.

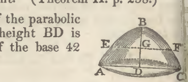
Ans. 22242·528 cu. in. = 12 cubic feet 1506·528 inche

6. Required the content of the middle zone of an obla spheroid, the bases parallel to the fixed axis, the length 4 inches, the diameters of the greater base 50 and 30, and of t less 30 and 18 inches. Ans. 37070·88 cubic inche

Prob. IX. To find the solid content of a parabolic conoid.

Rule. Multiply the area of the base by half the height the product will be the content. (Theorem II. p. 238.)

1. Required the content of the parabolic conoid ABC, of which the height BD is 36, and the diameter AC of the base 42 inches.



Ans. $42^2 \times 18 \times {}\cdot7854 = 24938{\cdot}0208$ cubic in. the content.

2. Required the content of a parabolic conoid, of which the height is 54, and the diameter of the base 40 feet.

Ans. 33929·28 cubic feet.

3. Required the content of a parabolic conoid, of which the height is 16, and the diameter of the base 36 inches.

Ans. 8143·0272 cubic inches.

4. Required the content of a parabolic conoid, of which the height is 30, and the diameter of the base 40 inches.

Ans. 18849·6 cubic inches.

5. Required the content of a parabolic conoid, of which the height is 27, and its parameter 12 inches.

Ans. 13741·3584 cubic inches.

Prob. X. To find the solid content of a frustum of a paraboloid.

Rule. Multiply the sum of the squares of the diameters of the bases by half the height, and by ·7854 : the product will be the content. (Theorem II. Cor. page 238.)

1. Required the content of the frustum EACF (see last figure) of a paraboloid, of which the height DG is 12, and the radii of the bases EG 20, and AD 28 inches.

Ans. $(28^2 + 20^2) \times 6 \times 3{\cdot}1416 = 1184 \times 6 \times 3{\cdot}1416 =$ 22317·9264 cubic inches the content.

2. Required the content of the frustum of a paraboloid, of which the height is 38, and the diameters of the bases 32 and 0 feet. Ans. 21249·7824 cubic feet.

3. Required the content of a cask consisting of two frustums of a parabolic conoid joined at their greatest ends, the greatest diameter 34 inches, the least 27, and the whole length 42 inches.

Ans. 31090·059 cubic inches, = 112 imperial gallons 1 pint.

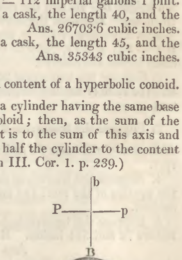
4. Required the content of a cask, the length 40, and the diameters 32 and 26 inches. Ans. 26703·6 cubic inches.

5. Required the content of a cask, the length 45, and the diameters 40 and 20 inches. Ans. 35343 cubic inches.

Prob. XI. To find the solid content of a hyperbolic conoid.

Rule. Find the content of a cylinder having the same base and altitude with the hyperboloid; then, as the sum of the transverse axis and the height is to the sum of this axis and two-thirds of the height, so is half the cylinder to the content of the hyperboloid. (Theorem III. Cor. 1. p. 239.)

1. Suppose the height BD to be 10, the radius of the base AD 12, and the transverse axis Bb 30 inches. Required the content.



$$\begin{array}{r} 3{\cdot}1416 \\ 144 \\ \hline 452{\cdot}3904 \\ 5 \\ \hline \end{array}$$

40 : 2261 ·952 : : $36\frac{2}{3}$: 2073·456 cubic inches.

2. Suppose the height 14, the radius of the base 48, and the transverse axis 60 feet. Required the content.

Ans. 47472·4629 cubic feet.

3. Suppose the height 22, the radius of the base 60, and the transverse axis 96 feet. Required the content.

Ans. 116675·829 cubic feet.

4. Suppose the height 49, the radius of the base 78, and the transverse axis 124 inches. Required the content.

Ans. 424069·1484 cubic inches.

5. Suppose the height 55, the radius of the base 96, and the transverse axis 84 inches. Required the content.

Ans. 691191·778 cubic inches.

Prob. XII. To find the content of a frustum of a hyperboloid.

Rule. Find a fourth proportional to the transverse, the conjugate, and the altitude, and subtract a third of its square from the sum of the squares of the radii of the bases: the remainder, multiplied by twice the altitude, and by ·7854 will give the content. (Theorem III. Cor. 2. p. 239.)

1. Suppose the transverse B*b* 270, the conjugate P*p* 108, the height DG 10, and the radii of the bases AD 24 and EG 16 inches. Required the content of the frustum.

Ans. $270 : 108 :: 10 : 4$, and $4^2 \div 3 = 5\frac{1}{3}$; then $(24^2 + 16^2 - 5\frac{1}{3}) \times 20 \times {\cdot}7854 = 826\frac{2}{3} \times 20 \times {\cdot}7854 = 12985{\cdot}28$ cubic inches the content.

2. Suppose the transverse 200, conjugate 350, height 14, and the radii of the bases 36 and 20 feet. Required the content.

Ans. 32897·0026 cubic feet.

3. Suppose the transverse 270, conjugate $\frac{108}{\sqrt{10}}$, height 40, diameters of the bases 32 and 24 inches. Required the content.

Ans. 24596·6336 cubic inches.

4. Suppose the transverse 30, conjugate 18, height 5, and the squares of the radii 144 and 194·4 inches. Required the content.

Ans. 2634·2316 cubic inches.

5. Suppose the transverse 45, conjugate 27, height 9, diameters 72 and 544 inches. Required the content.

Ans. 1064111·002416 cubic inches.

Prob. XIII. To find the solid content of an elliptical spindle.

Rule. Divide three times the area of the generating segment by the length of the spindle, and from the quotient subtract the greatest diameter; multiply the remainder by four times the central distance, and subtract the product from the square of the greatest diameter: the remainder, multiplied by the length and by ·5236, will give the content.*

* For the demonstration of this and the three following rules see Appendix.

1. Suppose the length AC of the spindle to be 40, the greatest diameter BF 12, the central distance OE 9 inches, and the area of the elliptic segment ABC 167·7345 square inches. Required the content.

Ans. $167·7345 \times 3 \div 40 - 12 = ·5801$, then $12^2 - (·5801 \times 4 \times 9) = 144 - 20·88316 = 123·1169$, and $123·1169 \times 40 \times ·5236 = 2578·55931$ cubic inches the content.

2. Let the length of the spindle be 48, its greatest diameter 18, and the central distance 24 inches. Required the content. The elliptic segment is 296·89885. Ans. 6801·10457 cu. in.

3. Required the content of an elliptical spindle, the length 60, the greatest diameter 24, and the central distance 32 inches. Ans. 15113·986 cubic inches.

4. Required the content of an elliptical spindle, the length 36, the greatest diameter 16, and the central distance 20 inches. Ans. 4039·5446784 cubic inches.

5. Required the content of an elliptical spindle, the length 30, the greatest diameter 14, and the central distance 20 inches. Ans. 2565·4321308 cubic inches.

Prob. XIV. To find the content of the middle zone of an elliptical spindle.

Rule. Find the area of the elliptical segment, of which the chord is equal to the length of the zone, divide three times this area by its length, and from the quotient subtract the difference between the greatest and least diameters of the zone, and multiply the remainder by eight times the central distance. Subtract the product from the sum of twice the square of the greatest diameter and the square of the least; the remainder, multiplied by the length and by ·2618, will give the content.

Note. The rules for an elliptical spindle and its zones will give the content of a hyperbolical spindle and of its zones, if the product be added to the squares of the diameters instead of subtracting it.

1. Suppose the length GH of the zone (see last figure) to be 40, its greatest and least diameters FB 32, and KN 24, the central distance OE 4 inches, and the area of the elliptical segment cut off by the straight line KL 109 square inches. Required the content of the zone.

Ans. $(109 \times 3 \div 40 - 8) \times 4 \times 8 = 5·60$; then $\{(40^2 \times 2 + 24^2) - 5·6\} \times 40 \times ·2618 = (2624 - 5·6) \times 10·472 = 2618·4 \times 10·472 = 27419·8848$.

2. Suppose the length of the zone to be 60, its greatest and least diameters 40 and 30, and the central distance 20 inches. Required the content of the zone.

Ans. 64063·6178 cubic inches.

3. Suppose the length of the zone to be 48, its diameters 36 and 28, and the central distance 16 inches. Required the content of the zone. Ans. 42264·795495 cubic inches.

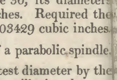
4. Suppose the length of the zone to be 30, its diameters 20 and 14, and the central distance 12 inches. Required the content of the zone. Ans. 7757·1034754 cubic inches.

5. Suppose the length of the zone to be 36, its diameters 30 and 24, and the central distance 18 inches. Required the content of the zone. Ans. 22316·03429 cubic inches.

PROB. XV. To find the solid content of a parabolic spindle.

RULE. Multiply the square of the greatest diameter by the length and by ·7854, and $\frac{8}{15}$ of the product will give the content. Or multiply the square of the greatest diameter by ·418879 to get the content.

1. Suppose the length AC to be 80, and the greatest diameter BD 32 inches. Required the content.



Ans. $32^2 \times 80 \times ·7854 \times \frac{8}{15} = 81920 \times ·418879 =$ 34314·56768 cubic inches the content.

2. Suppose the length to be 64, and the greatest diameter 20 inches. Required the content. Ans. 10723·328 cu. in.

3. Suppose the length to be 84, and the greatest diameter 36 inches. Required the content. Ans. 45600·95232 cu. in.

4. Suppose the length to be 72, and the greatest diameter 42 inches. Required the content. Ans. 53200·984 cu. in.

5. Suppose the length to be 108, and the greatest diameter 38 inches. Required the content. Ans. 65325·017808 cu. in.

PROB. XVI. To find the content of the middle zone of a parabolic spindle.

RULE. To twice the square of the greatest diameter add the square of the least, and from the sum subtract $\frac{4}{10}$ of the square of the difference of these diameters; multiply the remainder by the length and by ·2618, to get the content.

1. Suppose the length FG to be 40, the greatest diameter BD 32, and the least HK 24 inches. Required the content.

Ans. $(32^2 \times 2 + 24^2) - \{(32 - 24)^2 \times ·4\} \times 40 \times ·2618$ $= 2624 \times 25·6 \times 48 \times ·2618 = 103936 \times ·2618 = 27210·4448$ cubic inches the content.

2. Suppose the length to be 42, and the diameters 34 and 27 inches. Required the content. Ans. 33222·10584 cu. in.

3. Suppose the length to be 48, and the diameters 36 and 30 inches. Required the content. Ans. 43700·91264 cu. in.

4. Suppose the length to be 44, and the diameters 34 and 28 inches. Required the content. Ans. 35497·56672 cu. in.

5. Suppose the length to be 38, and the diameters 30 and 24 inches. Required the content. Ans. 23494·14144 cu. in.

OF UNGULÆ.

PROB. I. To find the contents of the parts into which a frustum of a rectangular or square pyramid is cut, by a plane passing through one of the sides of the base.

RULE. One of the parts cut off will be a wedge, of which the content may be found by Prob. XI. MENSURATION OF SOLIDS; and this subtracted from the content of the whole will give the other part.

1. Let the perpendicular height of the frustum of a square pyramid be 287·9649 inches, and the sides of its bases 15 and 6 inches; and let a plane pass through one of the sides of the less base, and cut the side of the frustum at the perpendicular height of 119·98536 inches from that base: the length of the section it makes is 9·75 inches. Required the contents of the parts.

$\{(15+6)^2 - 15 \times 6\} \times \frac{1}{3} \times 287{\cdot}9649 = 33691{\cdot}8933$ cu. in. the frustum.
$(12 + 9{\cdot}75) \times 6 \times \frac{1}{6} \times 119{\cdot}98536 \;\; = \;\; 2609{\cdot}6816$ cu. in. the wedge.

Ans. $\overline{31082{\cdot}2117}$ cu. in. the remaind.

2. Let the height of the frustum of a rectangular pyramid be 30 inches, the sides of the greater base 48 and 36, and those of the less base 36 and 27; and let a plane pass through the less side of the greater base, and cut the opposite at the height of 20 inches; the length of the section it makes with that side is 30 inches. Required the contents of the parts.

Ans. Wedge 15120, remainder 24840 cubic inches.

3. Required the contents of the parts of the frustum of a square pyramid, the sides of the bases 30 and 20, a plane through the greater base passes through the less base, the height 72 inches.

Ans. Wedge 28800, remainder 16800 cubic inches.

4. Required the contents of the parts of the frustum of a rectangular pyramid, the sides of the under base 40 and 30, and of the upper base 24 and 18, and the plane passes through the greater sides of the two bases, the height 42 inches.

Ans. Wedge 21840, remainder 11088 cubic inches.

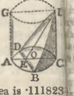
5. Required the contents of the parts of the frustum of a rectangular pyramid, the height 60 inches, the sides of the under base 36 and 28, and of the upper 30 and 23$\frac{1}{3}$; a plane passes through the greater side of the lower base, and cuts the opposite side at the height of 30 inches; the section it makes is 33 inches.

Ans. Wedge 14700, remainder 36260 cubic inches.

PROB. II. To find the content of the hoof of a cylinder.

RULE. Find the area of the base of the hoof, and multiply it by the difference between the radius and the versed sine or height of the base, and add the product to $\frac{1}{12}$ of the cube of the chord of the base, if the height of the base be greater than the radius; otherwi[se] subtract them: the sum or difference, multiplied by the height [of] the hoof, and divided by the height of the base, will give the content.

NOTE. If the cutting plane pass through the centre of the base, multip[ly] the square of the diameter by $\frac{1}{6}$ of the height of the hoof to get the content.



1. Suppose the diameter AC of the base of the cylinder to be 50, the height CF of the hoof 120, and the height or versed sine of its base CE 10 inches. Required the content of the hoof.

Ans. $10 \div 50 = \cdot 200$ versine, of which the tabular area is $\cdot 111823$, then $\cdot 111823 \times 50^2 = 279\cdot 5575$ circular segment, and $\sqrt{(40 \times 10)} \times 2$ $= 40$ the chord. Now $\{(40^3 \div 12) - (279\cdot 5575 \times (40 - 25)\} \times 12$ $= 5333\cdot 3333\dot{3} - 4193\cdot 3625 \times 12 = 1139\cdot 9708\dot{3} \times 12 = 13679\cdot 65$ cubic inches the content.

2. Suppose the versed sine of the base to be 40, the rest as before. Required the content.

Here the chord is 40, the base 1683·9359. Ans. 91777·1875 cu. in.

3. Suppose the cutting plane to pass through the centre, the rest as before. Required the content. Ans. 50000 cubic inches.

4. Suppose the diameter of the cylinder 48, the versed sine of the hoof 30, and its height 36 inches. Required the content.

Ans. 18604·98 cubic inches.

5. Suppose the diameter of the cylinder 36, the height of the hoof 42, and its versed sine 12 inches. Required the content.

Ans. 5167·07117 cubic inches.

PROB. III. To find the content of the hoof of the frustum of a cone.

CASE I. When the cutting plane passes through the extremities of the two bases.

RULE. Take the square root of the product of the diameters at the base and the top of the hoof, and multiply it by the diameter at the top, then take the difference between this product and the square of the diameter of the base, and divide it by the difference of the diameters: the quotient, multiplied by the diameter of the base, by the height, and by ·2618, will give the content.

1. Suppose the diameter of the base of the hoof to be 30, and the diameter of the frustum at the top of the hoof to be 19·2, and the height 18 inches. Required the content.

Ans. $\sqrt{(19\cdot 2 \times 30)} \times 19\cdot 2 = 24 \times 19\cdot 2 = 460\cdot 8$, and $30^2 - 460\cdot 8 \div (30 - 19\cdot 2) = 439\cdot 2 \div 10\cdot 8 = 40\cdot\dot{6}$, then $40\cdot\dot{6} \times 30 \times 18 \times \cdot 2618 = 21960 \times \cdot 2618 = 5749\cdot 128$ cubic inches the content.

2. Suppose the diameter at the base 19·2, that at the top 30, and the height 18 inches. Required the content.

Ans. 2943·553536 cubic inches.

3. Suppose the diameter of the base 24, the diameter of the top 18, and the height 36 inches. Required the content.

Ans. 7610·6089 cubic inches.

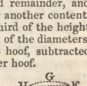
* For the demonstration of this and the following Problem see Appendix.

4. Suppose the diameter of the base 20, that at the top 28, and the height 14 inches. Required the content.
Ans. 2406·21259648 cubic inches.

5. Suppose the diameter of the base 15, that at the top 12, and the height 16 inches. Required the content. Ans. 1340·481136 cu. in

CASE II. When the plane cuts off a part of the base.

RULE. Find the tabular area answering to the quotient of the height of the base by its diameter, and multiply it by the cube of that diameter for the first content. From the height of the base subtract the difference between the diameters at the top and the base of the hoof; take the tabular area answering to the quotient of the remainder divided by the diameter at the top, and multiply it by the cube of the diameter at the top, and by the quotient of the height of the base divided by the said remainder, and also by the square root of the same quotient, for another content. Multiply the difference of these contents by one-third of the height of the hoof; the product, divided by the difference of the diameters, will give the content of the under hoof; and this hoof, subtracted from the content of the frustum, will give the other hoof.



1. Suppose the height of the hoof to be 18, the diameter AC of the lower base 30, the diameter GH at the top 19·2, and that the plane cuts off BE 20 inches height from the lower base. Required the content.

The tabular area of $\frac{20}{30}$ is ·556226, which, multiplied by 27000, gives 15018·102 the first content; and the tabular area of $9·2 \div 19·2 = ·4791\frac{2}{3}$ is ·371872, which, multiplied by $19·2^3$, and by $20 \div 9·2$, and by $\sqrt{(20 \div 9·2)}$, gives 8136·4657, which, subtracted from the former content, leaves 6581·6363; then this, multiplied by 6, and divided by 10·8, gives 3656·4646 cubic inches the content.

2. Suppose the plane to cut 15 inches for the height of the base, the rest as before. Required the content. Ans. 2517·8613 cu. in.

3. Suppose the height of the base 10·8 inches, the rest as before. Required the content. Ans. 1606·41 cubic inches.

NOTE. In this example, where the height of the base is equal to the difference of the diameters at the base and top, the tabular versed sine for the second area is nothing. Therefore, multiply the first tabular area by the cube of the diameter at the base, and divide the product by the height of the base, for the first content. Also, multiply the height of the base by the diameter at the top, and multiply the square root of the product by the same diameter, and to the product add one-third of itself, for the second content. The difference of these contents, multiplied by one-third of the height of the hoof, gives its content.

4. Suppose the diameter of the base 36, that at the top 27, the height 24, and the versed sine 18 inches. Required the content.
Ans. 4945·166936 cubic inches.

5. Suppose the diameter of the base 24, that at the top 32, the height 42, and the versed sine 16 inches. Required the content.
Ans. 11447·9264 cubic inches.

SURVEYING.

SURVEYING is the method of determining the magnitude, position, and shape of lines, fields, &c. on the surface of the earth.

For this purpose, various instruments are used for measuring lines and angles.

OF INSTRUMENTS USED FOR MEASURING LINES.

Straight lines are measured by applying to them a line of known length, as a foot, a yard, a chain, &c. a number of times.

The CHAIN used in surveying consists of 100 links, and is distinguished at the end of every 10 links by a small piece of brass cut into points to facilitate the counting of the odd links. Thus, at 10 links from either end the piece of brass has 1 point; at 20 links, it has 2 points; and so on to the middle of the chain, which is marked by a circular piece. Ten chains in length, and one in breadth, make an acre.

If the length of a pendulum vibrating seconds at Greenwich Observatory be taken 23 times, and the amount divided into 25 equal parts, each of these parts will be nearly an English yard, or 22 of them an English chain; therefore a link of it will be 7·92 inches.*

The Scotch chain was 74·1196 English feet long, and each link of it 8·89 inches. The Scotch ell was 37 Scotch inches = 37·0598 English inches; and 6 ells made a fall.

The OFFSET-STAFF is a pole of 10 links in length. It is divided into 10 parts, and the last of them subdivided into 10 smaller parts. Its use is for examining the chain, which is liable to stretch with long usage or the roughness of the ground. It is also used for measuring short distances, such as perpendiculars or offsets from the principal straight line to the enclosures.

The CROSS consists of two pair of sights fixed on a pole, at

* The length of the pendulum vibrating seconds in a vacuum at the level of the sea in the latitude of London is 39·1393 inches; 23 times the length of this pendulum is = 900·204 inches; and 25 yards = 900 inches; so that 23 times the length of the pendulum is only about $\frac{1}{5}$ of an inch more than 25 yards.

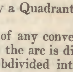
ght angles to one another. Its use is to determine the point which a perpendicular from a corner would meet the prinpal line that is measured. It is moved backwards or forards along the line, keeping its extremities in view through ne pair of the sights, till the corner from which the perpenicular comes is seen through the other pair of sights: the oss is then at the foot of the perpendicular.

The PERAMBULATOR is sometimes used for measuring ads, &c. It turns upon a wheel, of which the circumferce is 8·25 feet; so that 8 revolutions make an English chain length. The distance measured is pointed out by an index oved by machinery. As, however, by its entering into holws, and going over small eminences, it must give the disnce too great, much reliance cannot be placed on the result.

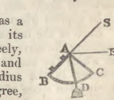
OF THE INSTRUMENTS USED FOR TAKING ANGLES.

Angles in the field are taken either in a vertical or in a rizontal plane. The former are measured by a Quadrant, d the latter by a Theodolite or Circle.

A QUADRANT is the fourth part of a circle of any conveent radius. It is made of brass or wood, and the arc is dided into 90 degrees, and each degree is subdivided into aller parts. The degrees are numbered from one extremity, lled the beginning of the arc, to the other extremity or end it.



The most simple quadrant, ABC, has a e with a plummet suspended from its ntre, as AD, which, when hanging freely, always perpendicular to the horizon; and *ghts*, or a telescope, is affixed to the radius B, which passes through the 90th degree, end of the arc, to direct the eye in a raight line towards the object.



Sometimes an index AD, with lescopic sights, is made to revolve und the centre A; in which case a irit-level is fixed to the radius AC, hich passes through the beginning the arc. The telescope is placed ong AD. But sometimes the derees are numbered from B, and a lescope is fixed at D, perpendicular the index AD.

The THEODOLITE is the most complete instrument for surying. It consists of a circular brass plate, the circumference which is divided into 360 degrees, or twice 180 degrees, and each degree is subdivided into smaller parts. An index with a compass on it is fixed to the centre, and revolves round it; and on it is erected a semicircle, perpendicular to the plane of the instrument, furnished with a telescope perpendicular to the index of it, which moves round its centre. The use of the circle is for taking horizontal angles, and that of the semicircle is for taking vertical ones. The instrument is furnished with two spirit-levels for placing the plate, and the telescope, when at the top of the semicircle, in a horizontal direction; in subservience to which, the tripod upon which the instrument stands has four screws, &c. A more particular description of this instrument, in its most improved state, would scarcely be intelligible to a learner, without seeing and using it; and it is therefore omitted here.

The CIRCUMFERENTER is a circle, on the centre of which is a large compass; and the circumference is divided, not only into points and quarters, but also into degrees and parts of a degree. An index or two is moveable about the centre. Its use is the same with that of the theodolite; only, when using it, greater reliance is placed upon the compass. It is chiefly used for surveying mines, or large tracts of land where great accuracy is not required.

Large LEVELS, with telescopic sights, are often requisite for finding the elevation of one place above another. And the surveyor ought also to be possessed of several pocket-levels, to be applied when occasion requires them.

Each of the indices of these instruments has a NONIUS, for enabling the surveyor to read off minutes. The nonius is a scale on which the number of divisions is greater by one than the number in the same space upon the arc. If the nonius occupy the space of 29 divisions on the arc, it is divided into 30 equal parts, by which means each division will exceed one on the nonius by $\frac{1}{30}$ of a division on the arc; so that, by moving forward the index $\frac{1}{30}$ of a division of the arc, the first one on the nonius will coincide with one on the arc; and by moving another $\frac{1}{30}$, the second will coincide, and so on. Consequently, if the arc be divided into half degrees, the nonius will point out minutes.

The PLANE-TABLE is an instrument much used in surveying, when the survey is not large, because it gives the plan of the ground, as well as its quantity. It is a rectangular board fixed upon a tripod, with a ball and socket for giving it any inclination. It has a loose frame fitted to it, one side of which is divided into equal parts all around; and the other side is divided into 360 degrees, by lines directed to the centre of the table; and a compass is fastened to one of the sides of the table.

ere is a loose index to be used with it, having a telescope ced parallel to its fiducial side; and there are several plane les upon the index, for laying down the measured distances. sheet of paper, moistened equally with a sponge, is spread on the table, and the frame pressed down upon it to keep it ed. The paper will become smooth when it is dry, and it l then be fit for drawing the plan upon.

An angle may be measured with the plane-table, by placing t side of the frame uppermost which has degrees on it, and ceeding as with the theodolite. Or the angle may be drawn the table, by directing the index to marks in the sides of angle in the field; and, in like manner, a given angle y be formed in the field by the table. Also, a perpendi- ar may be drawn in the field with it, by placing the centre the instrument at the given point, and turning it, till the ex, while cutting the same divisions on opposite sides of the me, is in the direction of the given line: then, if the index made to cut similar divisions on the other sides of the table, will give the direction of the perpendicular.

OF SHIFTING THE PAPER ON THE PLANE-TABLE.

When one paper is full, and there is occasion for more, w a line in any manner through the farthest point of the station-line, to which the work can be conveniently laid wn; then take the sheet off the table and fix on another, wing a line on it, in the most convenient part for the rest the work; then fold or cut the sheet formerly used by the drawn on it, apply the edge to the line on the new sheet, l, as they lie in that position, continue the last station-line the new paper, placing on it the rest of the measure, be- ning at the point where the previous sheet left off.

When the work is finished, the different sheets used must carefully joined, so that the lines may come together in the e manner as when the lines were transferred from the old ets to the new ones.

It may be noticed, that if the joining lines on the old and v sheets have not the same inclination to the side of the le, the needle will not point to the original degree when table is rectified; and if the needle be required to point l to the same degree of the compass, the easiest way of wing the lines in the same position is to draw them both allel to the same sides of the table, by means of the equal isions marked on the other two sides.

INSTRUMENTS USED IN DRAWING PLANS.

The surveyor ought to be provided with compasses of ious sizes, some of which must have very fine points, both of steel and for ink. He ought also to have drawing-pens of different finenesses, for drawing coarse and fine lines; and a number of scales of various sizes, from one chain in an inch to 8 or 10 chains in an inch, which ought to have the divisions marked on the edges for laying down distances without compasses. He will also stand in need of lines of chords, and protractors of different radii; and, for the sake of expedition, he ought to use parallel and perpendicular rulers and reducing scales.

PROB. I. To measure a straight line in the field.

Erect poles at the extremities, and at convenient distances along the line, for showing the direction. Ten arrows of iron or wood are used for marking the spot to which the chain extends, and for preserving the number of chains. Let the leader, or the person going before, take the end of the chain and the ten arrows; and having stretched the chain, and taken notice that none of the links are involved in one another, let the follower, placing the end of the chain at the extremity of the line, direct him, by waving his hand towards the right or left, into the proper direction. And the leader having fixed an arrow at the end of the chain, let them both go forward with the chain, till the follower comes to the arrow; there let him direct the leader as before, who fixes another arrow, while the follower takes up the former one. Let them proceed thus, till all the arrows are in the hand of the follower, and the chain stretched beyond the last of them; then let the arrows be conveyed to the leader, and let him fix one of them at the end of the chain, and proceed in the same way till all the arrows are again changed, or till he has arrived at the end of the line to be measured. And at the last, let the follower reckon the number of changes, the number of arrows in his hand, and the number of links between him and the extremity of the line. Thus, 3 changes 7 arrows and 45 links make the length of the line to be 3745 links.

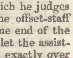
NOTE 1. The surveyor, while measuring a straight line, ought carefully to take notice of every surrounding object of which the position can be more easily determined from it than from any other line which he intends to measure. He ought to mark the distance at which the line meets a corner, or crosses a boundary, or begins or ceases to run along a hedge, a wall, or a road. He must likewise mark the distances at which perpendiculars or offsets are to be raised; and, in general, every thing which may tend to shorten his other operations in the survey, or will assist him in drawing his plans. When he has settled, by the cross or otherwise, the place of an offset or short perpendicular, it will be easiest to measure the length of it as he goes along, to save the time and trouble of returning to the place a second time.

NOTE 2. The plan ought to be drawn upon paper, with horizontal distances only; otherwise it will be impossible to join several fields together without distortion. For when several lines are to be joined together, a small error in the lengths of some of them will alter the position of others; a circumstance which has a greater tendency to distort the plan, than even the lengths of the lines themselves. It is, however, impossible for a surveyor to ascertain the exact level of every elevation and depression of his lines; but it would be of great advantage to him to take a level at that part which he judges to have a mean inclination. This may be done with the offset-staff thus:—Having laid the chain along that part, place one end of the offset-set staff at the uppermost of 10 links on it, and let the assistant take the other end, and a line and plummet hung exactly over the other end of the 10 links on the chain, and let the surveyor apply a pocket or other level to the staff; and when it is level, the line of the plummet will point out on the staff the horizontal length of the 10 links of the chain. Consequently, by using a diagonal scale of 10 to a link, it will point out how much the line is to be diminished to get the horizontal length of it.

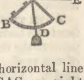
PROB. II. To take a vertical angle in the field.

Vertical angles are denominated Angles of Elevation when the object is higher than the eye, and Angles of Depression when it is lower.

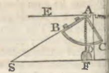
1. *To take an angle of elevation.* If the quadrant ABC have a plummet, place the eye to the limb B, and look through the sights in AB to the object S, and the line and plummet AD hanging freely, will cut off the arc CD from the end C, farthest from the sights, the degrees, &c. of which will be the measure of the angle EAS, contained by the horizontal line AE, and the visual ray AS; for DAE and CAS are right angles.



If the quadrant have a telescope fixed on the index AD, which moves about the centre A: Having levelled the radius AC, and directed the quadrant towards the object S, move the index AD till S is seen at the crossing of the wires of the telescope; then the arc CD is the measure of the angle CAD.



If the telescope be at D, perpendicular to AD, move the index, till, looking through the telescope, the object E is in the centre of the telescope; then the arc BD is the measure of the angle of elevation.



2. *To take an angle of depression.* If the quadrant ABC have a plummet, place the eye at the centre A, and look through the sights in the radius AB to the object S below, and the line of the plummet AD will cut off the arc CD, the measure of the angle of depression EAS; for EAD and BAC are right angles.

If the telescope be on the index AD, place the eye at the limb D, and look down to S through the telescope; and the arc CD is the measure of the angle of depression.

If the telescope be perpendicular to the index, depress the object-glass till the object be seen; and the arc BD between the index and the vertex is the measure of the angle of depression.

PROB. III. To measure a horizontal angle in the field.

WITH THE THEODOLITE. Having placed the instrument at the angular point, and the cipher of the index at the beginning of the degrees on the circle, turn the whole instrument about till a distant pole in one of the sides of the angle be seen in the centre of the telescope; there fix the instrument, and turn the index upon it, till a pole fixed in the other side of the angle be seen in the centre of the telescope: then the degrees, &c. moved over by the index is the measure of the angle.

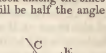
WITH THE CIRCUMFERENTER OR THE COMPASS. Having fixed the instrument, so that the north point of the compass point to the fleur-de-lis, direct the sights to a mark in one side of the angle, and mark the degrees, &c. pointed out by the needle. Then turn the sights towards a mark in the other side of the angle, and again mark the degrees cut by the needle. Their sum or difference, according as they are on different or on the same side of the north or south points, will give the quantity of the angle.

NOTE. The degrees marked show the bearing of the sides of the angle, allowance being made for the variation.

WITH THE CHAIN. Extend the chain along one of the sides, from the angular point A to B, and along the other side from A to C, and measure from C to B. Then, having drawn the triangle ABC upon paper, the angle BAC may be measured with a protractor, or with the line of chords.

NOTE. If a table of natural sines be at hand, look among the sines r BC, and the degrees, &c. answering to it will be half the angle AC.

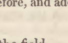
WITH THE CROSS. If the angle be ute, as BAC, place the cross at B in one the sides of the angle, so that one pair the sights may be directed along AB; d, looking through the other pair of ghts, let an assistant mark the point C of the line AC, hich is seen through them; and then the angle BAC is termined by measuring AB and BC. If the angle be obse, as CAD, it may be determined by measuring its supplent BAC, or by placing the cross at A, so that AD may be en through one pair of the sights; then let an assistant ace a distant mark at E, seen through the other pair of hts; after which measure the angle EAC as before, and add ight angle to it.



PROB. IV. To make or lay down an angle in the field.

WITH THE THEODOLITE. Having placed the instrument the point at which the angle is to be made, and fixed the lex at the beginning of the degrees, turn the theodolite til a mark is seen in the given line; there fix it, and turn e index upon it, the proper way, over the given number of grees; then, looking through the telescope, direct an astant to place a mark.

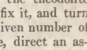
WITH THE CHAIN. The angle must first be de on paper, as ABC. Make B*b* and B*c* each , and measure *bc*. Lay 30 links on the given e on the ground from B to *b*; and having koned as many links of the chain as are in e sum of B*c* and *cb*, fix the ends of them at B and *b*, and, ing 30 links from B in your hand, go backward till both s of the chain are equally stretched, and there fix a pin in ground, which will give *c*.



PROB. V. To raise a perpendicular in the field.

WITH THE THEODOLITE, CIRCUMFERENTER, &c. At given point in the line make an angle of 90°, by the last blem.

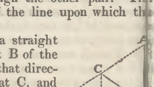
WITH THE CROSS. Having placed the ss at A, and directed one pair of the hts to a mark B in the given line, look ough the other pair of sights, and cause a rk D to be placed in that direction.



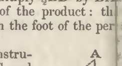
WITH THE CHAIN. Measure in the given line 30 links from A to B, and as many from A to C; and, fixing the ends of the chain at B and C, take hold of the 50th link, and go backwards till both ends of the chain are equally stretched, and there fix a pin at D; AD will be perpendicular to BC.

PROB. VI. To drop a perpendicular in the field.

WITH THE CROSS. Move the cross along the given line so that its extremities appear through one pair of the sights until the given point is seen through the other pair. The instrument is then in the point of the line upon which the perpendicular falls.

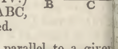


WITH THE CHAIN. Measure a straight line from the point A to any point B of the given line. Let BC be a chain in that direction. Fix one end of the chain at C, and with the other go along the given line till the chain is again stretched, and there make a mark, as at D. Measure BD, and multiply ½BD by BA, and cut off two figures from the right of the product: the rest will give BF, the distance of B from the foot of the perpendicular AF.



WITH THE THEODOLITE. Fix the instrument at any point B of the given line BC, and measure the angle ABC (by Prob. III.); then fix the instrument at A, and (by Prob. IV.) make the angle BAC the complement of ABC, and AC will be the perpendicular required.

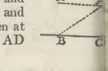
PROB. VII. To run a line in the field parallel to a given straight line BC.



Take any point B in the given line BC, and measure the angle ABC contained by BC, and the line directed to the given point A; then at A make the angle BAD equal to ABC, and AD will be the direction of the parallel.

OF HEIGHTS AND DISTANCES.

PROB. VIII. To find the height of an object A, when the point B on the level ground, directly below it, is accessible.



On the level ground measure any distance BC in a straight line, and at C take the angle of elevation ACB with a quadrant. Then rad. : tan. C : : CB : BA; and if CA be required, then sin. C : R : : BA : AC (Theor. I. Trig.)

1. In the triangle ABC. right-angled at B, are given BC 36 feet and the angle ACB 35° 48′. To find AB.

C 35° 48′ tan. — R	$\bar{1}$·858069	BA 170·208 log.+R	12·230981
CB 236 feet log.	2·372912	C 35° 48′ sine	9·767124
eight BA 170·208 feet log.	2·230981	AC 290·976 feet log.	2·463857

NOTE. The height thus obtained is that above the level of the e of the observer, and must be increased by the height of the eye, have its height above the level ground. The same is to be done all the observations on heights.

2. From the bottom of a steeple I measured upon a level ane a straight line 136 feet, and at its extremity I took the vation of the top of the steeple 47° 25′. Required the ight of the steeple. Ans. 147·985 feet.

3. The elevation of a wall, taken from the edge of the ditch feet wide, was 62° 40′. Required the height of the wall, d the length of a ladder to reach the top of it.

Ans. Height 34·8246, ladder 39·20153 feet.

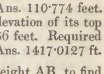
4. At 85 feet from the bottom of a tower, the angle of its vation was 52° 30′. Required its altitude.

Ans. 110·774 feet.

5. Near the bottom of a hill I took the elevation of its top ° 40′, and the altitude of the hill was 1156 feet. Required e distance of my station from its top. Ans. 1417·0127 ft.

PROB. IX. From the top of a known height AB, to find e distance of an object C, on the plane below.

Take the angle of depression AD; then, in the triangle ABC, ght-angled at B, are given AB, d the angle ACB = DAC. Then . C : R : : AB : AC, and if the rizontal distance CB be required, . C : R : : AB : BC (Theor. I. Trig.)



NOTE. If AC be given, then R : cos. C : : AC : CB, and R : cos. : AC : AB (Theor. I. Cor. 1. Trig.)

1. Suppose AB 83 feet, and the angle ACB 23° 37′, re- ired AC and CB.

AB 83 log. + R	11·919078	83 log. + R	11·919078
C 23° 37′ sin.	9·602728	23° 37′ tan.	9·640716
AC 207·181 log.	2·316350	BC 189·829 log.	2·278362

2. Let the sloping side of a hill AC be 268 feet, and the gle of depression at its top DAC be 33° 45′. Required the e BC, and its particular height AB.

Ans. BC 222·834, AB 148·893 feet.

3. From the top of a mast 80 feet high the angle of de-

pression of another ship's hull was 20°. Required their distance. Ans. 219·798 feet.

4. From the top of a tower 120 feet high I took the depression of two trees 57° and 25° 30′. Required their distances from the tower and from each other.

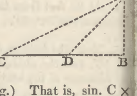
Ans. 77·93 feet, and 251·58 feet, and 173·65 feet.

5. Suppose the mean semidiameter of the sun subtends at the earth an angle of 16′ 7½″; what is his distance from the earth? Ans. 213·1946 semidiameters.

6. From the top of a lighthouse 110 feet high I observed two ships in a straight line from it, and took the angles of depression of their hulls 56° 44′ and 18° 26′. Required their distance from the lighthouse.

Ans. 72·1649 feet, and 330·031 feet.

Prob. X. To measure an inaccessible height AB.



On the level ground measure any distance CD, in a straight line towards the height, and at C and D take the angles of elevation ACB and ADB; their difference is CAD. Then sin. CAD : sin. ACD : : CD : DA (Theor. II. Trig.) and R : sin. ADB : : DA : AB (Theor. I. Trig.) That is, sin. C × sin. D × CD ÷ sin. (C — D) = AB.

Or the difference of the natural cotangents of C and D is to the radius as CD to AB.

1. Let CD be 248 yards, the angles ACB 23° 30′, and ADB 37° 24′; then CAD is 13° 54′.

37° 24′ sine	9·783458	Nat. cot.	1·307946
23° 30′ sine	9·600700	Nat. cot.	2·299843
Dist. 248 log.	2·394452	Diff. 0·991897 log.	9·996466
	21·778610	248 log. + R.	12·394452
13° 54′ sin. + R.	19·380624	AB 250·0264 log.	2·397986
AB 250·0264 log.	2·397986		

2. Sailing in a boat, a hill was observed, and the elevation of its top above the level of the sea was 27° 38′. After sailing 540 fathoms, each 5 feet, directly towards the hill, the elevation of its top was 35° 28′. Required the height of the hill above the level of the sea. Ans. 1066·268 fathoms.

3. The elevation of a hill at the bottom of it was 46°, and at 100 yards distance 31°. Required the height of it.

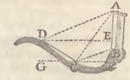
Ans. 143·1452 yards.

4. The angle of elevation of a tower was 26° 30′, and, 75 yards nearer to it, the elevation was 51° 26′. Required its height and distance. Ans. Height 61·97, dist. 49·2934 yds.

5. Measured 149 yards towards a hill, and at the extre-ities of the line the elevations of its top were 29° 17′ and 9° 25′. Required its height. Ans. 263·02 yards.

PROB. XI. To measure a height which has no level ground fore it.

Take two stations C and D, in a rtical plane, and measure CD; at C ke the elevation of D above C, or the gle GCD, and the elevations or de-essions of the top and bottom of the ight, viz. the angles ACF and BCF;



D take the elevation of the top, or the angle ADE. Since e angle EDC = DCG; therefore ADC = ADE + DCG d DAC = ACE — ADE. Hence the triangle ADC has o angles, ADC and DAC, and the side CD given to find e side AC. Then in the triangle ACB are given the angles CB = ACF ± BCF, and ABC = 90° ± BCF, and the le AC to find the side AB; wherefore sin. DAC : sin. ADC DC : DA, and sin. ABC : sin. ACB : : CA : AB (Theor. Trig.)

1. Suppose the angles GCD 31° 26′, ACF 53° 26′, BCF ° 32′, and ADE 22° 30′, and the distance CD 286 feet. quired the height AB.

Hence the angle ADC = (22° 30′ + 31° 26′) = 53° 56′ AC = (53° 26′ — 22° 30′) = 30° 56′, and ACB = (53° ± 18° 32′) = in this case (since F is below B) 34° 54′; en

ADC 53° 56′	sine 9·907590
DC 286	log. 2·456366
ACB 34° 54′	sine 9·757507
DAC 30 56 ar. co.	sine 0·289003
ABC 71 28 ar. co.	sine 0·023128
AB 271·39 ft.	log. 2·433594

NOTE 1. If DE be above A, the angle DAC is the sum of ACF ADE; otherwise it is their difference. Also, in this case ADC he difference of DCG and ADE; otherwise it is their sum. o, when F is below B, the angle ACB is the difference of ACF BCF; otherwise it is their sum.

NOTE 2. If the stations C and D cannot be conveniently taken in rtical plane, they may be taken anywhere, and then the angles C and ACD must be measured with a sextant, and the triangle D will give the side AC.

. At a considerable distance from a hill, I took the eleva-n of the top of a tower built upon it, 33° 45′; and mea-ing on level ground 300 feet directly towards the hill, I again took the elevations of the top and the bottom of th tower 51° and 40°. Required the height of the tower.

Ans. 46·666 yard

3. At a window on a level with the base of a steeple I too the elevation of its top 40°; and at another window of th same house, 18 feet higher, I took again the elevation of th top of the steeple 37° 30′. Required the height of the steepl

Ans. 210·44 fee

4. The elevation of the top of a hill at one station was 38 25′. Another station was taken 450 feet from the first, bu neither on a level with it nor in the direction of the hill. A the first station, the line from the other station to the top the hill subtended an angle of 67° 30′; and at the second, th line from the first to the top of the hill subtended an angle 74° 48′. Required the height of the hill. Ans. 441·25 fee

5. I measured directly up a hill 132 yards: there I too the depression of the hill 42°, that of the bottom of a distan object 27°, and that of its top 19°. Required the height o the object. Ans. 28·6367 yard

Prob. XII. To find the distance of a place A, from an in accessible object B.

When B is visible from A.

Choose a station C, from which both A and B can be seen. Measure AC, 650 yards, and take the angles BAC 72° 22′, and ACB 78° 37′, with the theodolite. Then ABC is 29° 1′, and sin. B : sin. C : : CA : AB = 1313·67 yards.

When B is not visible from A.

Choose a station C from which both A and B may be seen, and their distances from it measured. Take the angle ACB 75° 38′, and measure AC 358 and CB 560 feet. Then (BC + CA) 918 : (BC — CA) 20 : : tan. $\frac{1}{2}$(A + B) 52° 11′ : tan. $\frac{1}{2}$(A — B) 15° 49·7′; whence BAC is 68° 0·7′, and sin. A : sin. C : : CB : BA = 585·043

3. A straight line was measured along the bank of a river 528 feet, and at its extremities the angles contained by it, and straight lines directed to a tree upon the opposite bank were 62° 40′ and 73° 26′. Required the breadth of the river.

Ans. 676·445 feet to the nearest station, and 648·366 perp breadth.

4. Straight lines from a station to two places measured 69 and 456 yards, and the angle contained by them was 127° 16 Required the distance of the one place from the other.

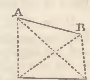
Ans. 1035·772 yards

5. To find the distance between two trees, I found the angle

subtended at a station to be 55° 40′, and measured from the ation to the trees 588 and 672 yards. Required their disnce. Ans. 592·97 yards.

PROB. XIII. To find the distance between two places, both them inaccessible.

1. To find the distance of two places A and , on the opposite side of a river, I took two tions, C and D, distant 1267 links from e another, and such, that from each of them e other station and the places A and B were en. At C I took the angles BCA 53° 38′, d BCD 34° 50′, and at D the angles ADC ° 44′, and ADB 58° 38′. Required the distance between and B.



In the triangle ADC, the angle ACD is 88° 28′, and CAD ° 48′; hence sin. A : sin. C : : CD : DA = 1709·69. In the iangle BCD, the angle CDB is 102° 22′, and CBD 42° 48′; nce sin. B : sin. C : : CD : DB = 1065·14. In the triangle DB are given AD and DB, and the angle ADB; therefore D + DB) 2774·83 : (AD — DB) 644·55 : : tan. $\frac{1}{2}$(A + B) ° 41′ : tan. $\frac{1}{2}$(A — B) = 22° 28$\frac{1}{4}$′; whence ABD is 83° ′, and sin. ABD : sin. ADB : : DA : AB = 1470·3 links.

2. To find the distance between two steeples A and B, I k two stations C and D, distant 428 yards from one aner; and at C took the angles ACB 54° 30′, and BCD 42° ; and at D took the angles CDA 40° 44′, and ADB 57° 42′. quired the distance of the steeples. Ans. 546·7 yards.

3. To find the distance between two places M and P, I took o stations A and B, distant from one another 908·36 feet; d at A took the angles PAM 14° 34′, and MAB 46° 16′; d at B took the angles ABP 96° 44′, and PBM 18° 39′. quired the distance between M and P. Ans. 674·6375 ft.

NOTE. If the distance between the objects be known, and the disce between the stations be required, assume 1 or 1000 for the tance between the stations, and with it find the distance between objects. Then, as the distance found is to the given distance, so 000 to the true distance between the stations.

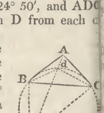
4. Suppose the distance AB 700 feet, and at the station C the angles ACB be 42° 45′, and BCD 54° 12′, and let the gles at D be ADB 50° 19′, and ADC 57° 33′. Required distance CD. Ans. 330·04 feet.

5. To find the distance between two lighthouses A and B, easured the distance between two stations M and R, 3370 ds, and at M took the angles AMB 37° 52′, and BMR

91° 27′, and at R the angles ARM 29° 56′, and ARB 40° 27′. Required the distance AB. Ans. 7063·36 yards.

6. At a station C, I took the angle ACB, subtending a line AB 3291 yards, and found it 4° 35′, and the angle BCD between B and another station D 86° 52′; and at D took the angles ADB 8° 24′, and ADC 70° 23′. Required the distance of the stations from one another. Ans. 3370·248 yards.

PROB. XIV. Given the distances of three places, A, B, C from one another, viz. AB 317, AC 308, and BC 478 feet, and the angles which these distances subtend at a station D in the same plane with them, viz. ADB 24° 50′, and ADC 27° 44′; to find the distance of the station D from each of the places.



Having drawn the triangle ABC, make at the point C, on the side of BC, opposite to that on which the station D lies, the angle BC*d* 24° 50′, and at B the angle CB*d* 27° 44′, and about the triangle BC*d* describe a circle, and join A*d*, meeting the circle again in D, and join BD and DC.

The three sides of the triangle ABC are given to find the angle ABC = 39° 25′ 14·6″; then AB*d* = ABC ± *d*BC = 67° 9′ 14″, when A and *d* are on different sides of BC, or = 11° 41′ 14·6″, when, as here, A and *d* are on the same side of BC. Also, the angles of the triangle BC*d* are given, with the side BC, to find B*d* = 252·7 feet. Again, in the triangle AB*d* are given the sides AB and B*d*, and the included angle AB*d*, to find the angles A*d*B = 131° 53′ 53″, and BA*d* = 36° 24′ 53″. Then in the triangle ABD are given the angles and the side AB, to find BD = 448·066, and AD = 661·738. And in the triangle DBC are given the angles and BC, to find DC = 591·563 feet.

2. If A be the place nearest to D, the angle BA*d* is 46° 47′ 32·2″; then BD is 550·153, AD 282·25, and CD 528·4 feet.

NOTE 1. If the given station be within the triangle, as at *d*, make the angles BCD and CBD equal to the supplements of B*d*A and A*d*C.

NOTE 2. If two of the given places, A and B, be in a straight line with the station D, the distances BC and CA subtend the same angle BDC. After finding the angle at B, work the triangle DBC.

NOTE 3. If the three places A, B, C, be in a straight line, the first operation will not be required. The rest are the same as before.

3. The three sides of the triangle ABC are AB 280, BC 314, and AC 326 yards; and from the station D without the triangle, the angle ADB was 25° 52′, and ADC 23° 6′, the

point C being the nearest to D. Required their distances from D. Ans. AD 586·163, BD 413·4114, CD 308·1078 yds.

4. Suppose AB 267 feet, BC 209, and AC 346, and at the point D, within the triangle, the angle ADC is 128° 40′, and ADB 91° 20′. Required the distances of D from the angles.

Ans. AD 195·357, BD 85·98, and CD 188·5074 feet.

NOTE. When D is in one of the sides, describe a segment on BC containing the given angle.

5. Suppose AB 122·4, BC 74, and AC 82 chains, and at D in AB, produced beyond B, the angle ADC is 22° 45′. Required the distance of D from the angles.

Ans. AD 181·79, BD 59·79, and CD 125·434 chains.

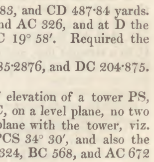
6. Suppose AB 1234, BC 873, and AC 632 yards, and at D in AB the angle ADC is 120°. Required its distance from the angles.

Ans. AD 226·117, BD 1007·883, and CD 487·84 yards.

7. Suppose AB 138, BC 224, and AC 326, and at D the angles are ADB 7° 22′, and ADC 19° 58′. Required the distance of D from the angles.

Ans. AD 510·9635, BD 385·2876, and DC 204·875.

PROB. XV. Given the angles of elevation of a tower PS, taken at three stations A, B, and C, on a level plane, no two of which are in the same vertical plane with the tower, viz. PAS 20° 10′, PBS 18° 50′, and PCS 34° 30′, and also the distances between the stations AB 324, BC 568, and AC 672 yards; to find the height of the tower.



Make the triangle ABC, of which AB is 324, BC 568, and AC 672 yards; make BE = BC, and BD = BA. Join ED, and upon it make the triangle EDF on either side of DE, so that BE : EF : : cot. PBS : cot. PAS and BD : DF : : cot. PBS : cot. PCS; or make EF = 527·495, DF = 160·792, and join BF, and make the angle BAP = BFE. Then erect PS perpendicular to the plane ABP, and in the plane passing through AP and PS make the angle PAS = 20° 10′, and PS will be the tower required.

Join PC, CS, BS, the triangles APB, FBE, being similar, AP : PB : : FE : EB : : cot. SAP : cot. SBP, therefore SBP = 18° 50′; also PB : BE = BC : : BA = BD : BF, therefore the triangles PBC and FBD are similar; and BP : PC :: BD : DF :: cot. PBS : cot. PCS, therefore PCS is 34° 30′.

In each of the triangles EBD, EFD, are given the three sides, to find the angles BED 28° 45′ 31″, and FED 6° 47′ 24″; then their difference 21° 58′ 7″, or their sum 35° 32′ 55″, is the angle BEF, from which, with the sides BE and EF, the angle BFE or BAP is found in the first case to be 89° 48′ 3″·7, and in the other 78° 48′ 18″·2. Therefore AP is 804·313 or 507·692, and PS is 295·3986 or 186·4592.

2. Let AB be 326, BC 584, and AC 683, and the angles of elevation SAP 30°, SBP 26°, and SCP 23°; to find PS.

Ans. PS is 952·161 or 168·645.

3. Let AB be 80, BC 119, and AC 140 feet, and the elevation at A 50°, at B 60°, and at C 55°. Required the height of the object D. Ans. 305·431 or 97·3602 feet.

4. Let AB be 60, BC 72, and AC 132 feet, and the elevations of S at A 30° 48′, at B 40° 33′, and at C 50° 23′. Required the height of S. Ans. 94·8328 feet.

5. Let AB and BC be each 84 feet, and the points A, B, C, in a straight line, and the elevation at A 36° 50′, at B 21° 24′, and at C 14°. Required the height of the object.

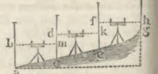
Ans. 53·9606 feet.

OF LEVELLING.

When the altitudes of the several parts of an irregular ascent are to be determined, the surveyor should be provided with a Spirit Level, with telescopic sights, and one or two square poles, which slide out to the length of 20 or 25 feet, divided into feet and hundredth parts of a foot. On each pole is fitted a moveable vane, with a strong black line drawn horizontally between two white ones. A small level is also fixed upon the top of the under-part of the pole, to assist in holding it perpendicular during the observations.

Prob XVI. To find the height of *g* above *a*.

Place an assistant with the pole *ab* at *a*, and another with the pole *cd* at *c*, and having fixed the level nearly midway between them, turn the telescope towards *a*, and direct the assistant to move the vane upwards and downwards



upon the pole till the black line on it coincide with the horizontal hair in the telescope, and then let him fix the vane. The feet and hundredth parts of a foot, cut by the under-part of the vane upon the pole, are then carefully read off, and entered into the surveyor's book. The telescope is then turned towards the pole at c, and the assistant is directed, the height read off, and entered as at the first station. The pole at a is now removed and placed at e, whilst that at c still remains; the level is again placed in the middle between them, and the observations made and registered as before; and so on till the whole is finished. The difference between the sums of the heights of the back-observations, or those taken with the telescope directed towards a, and that of the fore-observations, or those taken towards g, will show the height of g above a.*

To find the height of any point c in a regular ascent: The distance ag is to ac as the height of g above a to the height which c ought to have above a.

It is not necessary to place the poles in the same direction with ab and gh, but it is necessary to erect them perpendicular, or nearly so.

NOTE. When the distance between the poles ab and cd is very great, the line bm will differ a little from the true level; for bm is a tangent to a great circle of the earth, passing through the centre of the instrument, and the true level is the arc of that circle between the poles ab and cd. The correction may be neglected when the distance between the stations does not exceed 300 or 400 yards, and the instrument is placed in the middle between them: for a mile it is 7·96 or 8 inches; and for other distances from the instrument, the correction varies as the square of the distance.

1. To determine the height of an eminence, the following observations were taken:—

No.	Back.	Fore.	Ascent. Feet.	No.	Back.	Fore.	Ascent. Feet.
1	2·174	8·216	6·042	7	11·273	2·756	82·081
2	1·276	11·127	15·893	8	2·184	25·763	105·660
3	3·111	18·713	31·495	9	0·516	24·738	129·882
4	2·756	21·847	50·586	10	0·213	23·716	153·385
5	4·210	20·175	66·551	11	3·276	20·516	170·625
6	0·314	24·361	90·598	12	2·143	15·726	184·208

* Two poles are not necessary, for, after taking the back-observation upon the pole at a, it may be removed to c, and the fore-observation taken; then, removing the level into the second position, another back-observation is taken, and the pole removed to e for another fore-observation, and so on.

2. Determination of the height of Carnethy Hill, one of the highest peaks in the Pentland Chain, above the waste-wear of the Compensation Pond, by the method of Levelling.*

No.	Back.	Fore.	No.	Back.	Fore.	No.	Back.	Fore.
1	6·158	8·942	20	0·267	23·084	39	0·757	23·799
2	0·587	21·460	21	1·207	22·977	40	0·942	23·797
3	0·892	22·246	22	0·498	23·791	41	0·242	23·919
4	1·599	22·090	23	0·875	23·195	42	1·111	23·900
5	1·145	21·998	24	0·255	23·635	43	1·245	23·359
6	0·609	23·381	25	1·346	23·467	44	1·503	22·917
7	1·902	22·874	26	0·553	23·827	45	0·512	23·404
8	1·874	23·860	27	0·579	23·842	46	0·823	23·980
9	1·719	22·868	28	0·799	23·788	47	1·822	23·833
10	0·988	22·764	29	1·805	23·226	48	0·126	23·762
11	0·000	23·820	30	1·311	23·180	49	0·350	24·042
12	0·829	23·157	31	0·159	22·625	50	1·422	23·448
13	0·667	22·171	32	0·000	23·281	51	0·549	11·439
14	0·756	23·728	33	0·564	24·067	52	0·517	10·836
15	0·428	24·036	34	0·250	23·816	53	1·387	10·960
16	0·875	22·324	35	0·981	24·114	54	1·989	7·239
17	1·096	24·017	36	2·290	23·395	55	1·771	20·630
18	1·318	23·930	37	1·120	23·351	56	21·657	16·528
19	1·179	23·564	38	0·671	23·025		38·725	361·792
	24·621	423·230		15·530	445·686			38·725
		24·621			15·530			323·067
		398·609			430·156			430·156
								398·609
					Height in feet,			1151·832
	Height of waste-wear of Compensation Pond by barometrical measurement, ...							733·774
	Height above the level of the sea in feet,							1885·606

The Black Hill to the west of Carnethy is about 20 feet higher, and as it appears to be the highest peak of the range, the highest elevation of the Pentland Chain is therefore 1900 feet above the level of the sea.

3. Let the heights on the poles taken by looking down the eminence be 11, 8, 5, 6, 4, and those taken by looking up be 5, 3, 1, 4, 6 feet. Required the height of the eminence.

Ans. 15 feet high.

* The surveyor should only have one series of back and fore observations on each page of his field-book, reserving a broad column on the right hand for remarks.

4. Let the heights taken by looking down be 10, 11, 7, 5, 8, 4, 9, and those taken by looking up be 3, 5, 2, 6, 4, $5\frac{1}{2}$, $4\frac{1}{2}$ feet. Required the height of the eminence. And, supposing the sloping distance from the bottom to the top to be 346 feet,—Required the height in a regular slope at the distance of 136 feet from the bottom.

Ans. 25 feet high in all, and, at 136 feet, 9·8266 feet.

TO MEASURE HEIGHTS BY THE BAROMETER.

The elasticity or the density of the air is as the weight of the superincumbent atmosphere; and therefore, if the heights vary in arithmetical progression, the densities will vary in geometrical progression; that is, the height is as the logarithm of the density. It has been found by experiment, that the module of the barometrical logarithms is 10,000 times that of the common logarithms; wherefore, if B be the height of the mercury at the lower station, and b that at the higher, and h the difference of the heights of the stations, then $h = 10{,}000 \times$ (com. log. B — com. log. b) expressed in fathoms. But this formula is true only upon the supposition that the temperature of the air is 32°, and that it is the same at both stations; neither of which is exactly true.

It is found by experiment, that quicksilver expands about $\frac{1}{10000}$ part of its bulk for every degree of Fahrenheit's thermometer. Let r be the temperature at the lower station, and r' that at the higher, as indicated by the thermometer attached to the barometer, then $b + \frac{r - r'}{10000}b$ will be the height of the mercury at the higher station, when reduced to the same temperature with that at the lower station; and thus $h = 10000 \times \left(\log. \text{B} - \log. \left(b + \frac{r - r'}{10000}b\right)\right)$.

Again, the air expands nearly ·00223* of its bulk for every degree of Fahrenheit's thermometer. Let t be the temperature of the air at the lower station, and t' that at the higher, as indicated by a thermometer in the open air, then $\frac{1}{2}(t + t')$ may be taken for the mean temperature; and therefore the former formula has to be multiplied by $·00223 \times \left(\frac{t + t'}{2} - 32\right)$ for an additional correction.

PROB. XVII. To find the height of one place above another.

* General Roy's experiments gave ·00244, and Laplace's ·00222: the mean of the whole is ·00223.

From what has been shown, the complete formula will be $h = 10000 \times \left(\log. B - \log. \left(b + \frac{r - r'}{10000} b\right)\right) \times \left(1 + \cdot00223 \times \left(\frac{t + t'}{2} - 32\right)\right)$, which, expressed in words, gives the following

RULE. Divide the difference of the heights of the attached thermometer by 10000, and add 1 to the quotient, and add the logarithm of the sum to the logarithm of the height of the barometer at the highest station, and subtract the sum from the logarithm of the height of the barometer at the lower station: the remainder, multiplied by 10000, will give the approximate height. Take the difference between 32° and half the sum of the heights of the detached thermometer, and multiply it by ·00223; and if the half sum of the heights be greater than 32°, add the product to 1, otherwise subtract; and the sum or remainder, multiplied by the approximate height, will give the true height very nearly.

NOTE. This method of finding heights is more convenient, but it is not so accurate as that of levelling.

1. Suppose the height of the mercury in the barometer at the bottom of the hill to be 29·56 inches, and at the top 28·27 inches, and the temperature of the mercury 63° and 54°, and the temperature of the air 56° and 48°. Required the height of the hill.

Ans. $\frac{63 - 54}{10000} = \cdot0009$ and $10000 \times$ (log. 29·56 — log. 28·27 — log. 1·0009) $= 10000 \times$ (1·4707044 — 1·4513258 — 0·0003907) $= 10000 \times \cdot0189879 = 189\cdot879$ fathoms $=$ 1139·274 feet, the approximate height. Also, $\frac{1}{2}(56 + 48) - 32 = 20$, and $1 + 20 \times \cdot00223 = 1\cdot0446$; therefore 1139·274 $\times$ 1·0489 $=$ 1190·0856 feet, the true height.

2. Let the height of the barometer at the lower station be 29·57, and at the higher 28·7 inches, the height of the attached thermometer at the lower 55·28°, and at the higher 51·75°, and the temperature of the air at the lower 54°, and at the higher 50·5°. Required the elevation. Ans. 803·684 feet.

3. Let the heights of the barometer be 29·4 and 25·19 inches, the attached thermometer 50° and 46°, and the temperature of the air 45° and 39°. Required the elevation.

Ans. 684·3787 fathoms.

4. Let the heights of the barometer be 29·89 and 26·27 inches, the attached thermometer 56·5° and 42·75°, and the

temperature of the air 55·25° and 43°. Required the elevation. Ans. 3455·2375 feet.

PROB. XVIII. To measure distances by sound.

RULE. Multiply the time the sound takes in seconds by 1142 : the product will be the distance in feet.

NOTE. Sound in common air moves uniformly at the rate of about 1142 feet in a second. Cold, and uneven surfaces, retard its motion a little, and heat accelerates it in a small degree.*

1. I observed the flash of a gun 30 seconds before I heard the report. How far was it distant from me?

Ans. $30 \times 1142 = 34260$ feet.

2. I observed a flash of lightning, and after 6 strokes of my pulse I heard the thunder, and my pulse makes 68 strokes in a minute. How far was the thunder distant from me?

Ans. 1 mile 255·3 yards.

3. How long, after firing a gun, will it be till the report is heard at the distance of 8 miles? Ans. 37 seconds.

4. A person standing on the bank of a river heard the echo of his voice reflected from a rock on the opposite bank in 4 seconds after he uttered it. What was the breadth of the river? Ans. 2284 feet.

PROB. XIX. To measure a height by the descent of a stone, &c.

RULE. Multiply the square of the time of descent in seconds by $16\frac{1}{12}$: the product will be the height in feet.

To find the time of descending. Divide the height in feet by $16\frac{1}{12}$, and the square-root of the quotient will be the time in seconds.†

1. A stone takes 3 seconds in falling from the top of a tower to the ground. What is the height of the tower?

Ans. $3 \times 3 \times 16\frac{1}{12} = 144\frac{3}{4}$ feet.

2. In what time will a stone dropt from the height of 579 feet reach the ground? Ans. 6 seconds.

---

* A commission of the French Academy of Sciences in 1822 found by experiment that the velocity of sound, when the temperature of the air is 61° Fahrenheit, was 1118 feet per second, and that every increase or decrease of temperature of 1° of Fahrenheit caused an increase or decrease of velocity of $1\frac{14}{100}$ foot per second.

† It has been found by accurate experiments, that a heavy body in the latitude of London descends $16\frac{1}{12}$ feet in the first second of time, and the spaces descended by falling bodies are as the squares of the times.

3. What is the height of a precipice, when a stone takes 7 seconds in falling from the top to the bottom?

Ans. $788\frac{1}{12}$ feet.

4. I reckoned 7 strokes of my pulse during the falling of a stone from the top of a rock. What height did it fall, the pulse beating 70 times in a minute. Ans. 579 feet.

5. While a stone descended from the top of a tower, a pendulum 10 inches long made 8 vibrations. Required the height.*

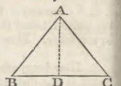
Ans. 262·995 feet.

TO SURVEY FIELDS.

Prob. XX. To survey a triangular field ABC.

With the Chain. Measure the three sides by Prob. I.

With the Chain and Cross. Measure along BC by Prob. I., and with the cross find the point D, where the perpendicular from A meets BC, by Prob. VI. Write down the measures of BD, BC, and DA.



With the Theodolite and Chain. Measure one angle ABC by Prob. III., and the containing sides AB and BC by Prob. I. Or measure BC by Prob. I., and two angles ABC and ACB by Prob. III. From these measures the plan may be easily drawn by Prob. XIX. XX. or XXI. of Practical Geometry; and the area may be found by Prob. IV. V. or VI. of Mensuration of Surfaces.

1. In a triangular field I measured the base 856 links, and found the extremity to be the foot of the perpendicular upon it, which I measured 672 links. Required the content.

Ans. 287616 sq. links = 2 ac. 3 ro. 20 per. 5 yds. 5·53 sq. ft.

2. In measuring the base of a triangular field, I found the foot of the perpendicular 256 links from its extremity, the base 927, and the perpendicular 582 links. Required the area.

Ans. 269757 sq. links = 2 ac. 2 ro. 31 per. 18 yds. 4·4 ft.

3. I measured an angle of a triangular field 73° 24′, and the sides containing it 688 and 492 links. Required the area.

Ans. 162119·4 sq. links = 1 ac. 2 ro. 19 per. 15 yds. 4 ft.

4. I measured one side of a triangular field 1268 links, and took the angles at its extremities 57° 36′ and 62° 24′. Required the area.

Ans. 694579·4 sq. links = 6 ac. 3 ro. 31 per. 9·893 yds.

Note. Add the log. of the side and of its half to the log. sin. of the two angles, and the arithmetical complement of the log. sin. of

* The number of vibrations made by pendulums in the same time is as the square roots of their lengths.

he third angle; the number answering to the sum is the area required.

5. The three sides of a triangular field are 1275, 987, and 42 links. Required the area.

Ans. 311128 sq. links = 3 ac. 17 per. 24 yds. 3·1068 ft.

PROB. XXI. To survey a field contained by four sides.

Fig. 1.

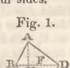


Fig. 2.

WITH THE CHAIN. Measure the four sides nd a diagonal BD by Prob. I.

WITH THE CHAIN AND CROSS. Measure long a diagonal BD by Prob. I., and, with the ross, find by Prob. VI. the points E and F, pon which the perpendiculars fall from A and , and write down the lengths of BE, BF, BD, hen measure AE, and CF.

Or measure the longest side BC, marking E nd F the places of the perpendiculars, and easure AE and DF.

WITH THE THEODOLITE AND THE CHAIN. lace the theodolite at B (fig. 1.) and take the ngles ABD and DBC by Prob. III., and measure the diagonal BD by Prob. I., and again at D take the ngles ADB and BDC. Or take the angle ABC, and measure the four sides.

If the angle ABC cannot be measured conveniently within he field, fix a pole G in the direction of either side AB, exnded beyond B, and measure the angle CBG, which, subracted from 180°, will give ABC.

WITH THE PLANE-TABLE AND THE CHAIN. Place the able at one of the angles B, from which all the other angles ay be seen, and turn it round till the needle points to the eur-de-lis, and there fix it. Fix also a pin in some part of he paper to represent B. Apply the fiducial side of the dex to the pin, and turn it till the angle A is seen through he sights. Draw a line from the pin in that direction. easure BA, and by the scale on the index lay it on that ne from B to A. Next turn the index till the angle D is een through the sights, and draw a line in that direction, and n it lay the length of BD. Then draw a line in the direcon of C, and on it lay BC, and join CD and DA. In the ame manner any field may be surveyed by the plane-table, hen an angle can be taken, from which all the other angles f the field are seen.

1. I measured along the diagonal BD (fig. 1.), and at E, 18 links from B, was the foot of the perpendicular AE 318,

and at F, 527 links from B, was the foot of the perpendicular CF on the opposite side of BD, 426 links: the whole length of the diagonal BD was 968 links. Required the plan and the area.

Ans. Area, 360096 sq. links = 3 ac. 2 ro. 16 per. 4 yds. 5·8176 feet.

2. I measured along BC the longest side of a four-sided field ABCD (fig. 2.), and at E, 125 links from B, was the foot of the perpendicular AE, which measured 624 links, and at F, 635 from B, was the foot of another perpendicular FD 462 links: the whole length of the side BC was 1274 links. Required the plan and the area.

Ans. Area, 463539 sq. lin. = 4 ac. 2 ro. 21 per. 20·0376 yds.

3. I measured an angle ABC of a quadrilateral field 128°, and the four sides AB 536 links, BC 843, CD 634, and AD 936 links. Required the plan and the area.

Ans. Area, 466592·7 sq. links = 4 ac. 2 ro. 26 per. 16 yds. 5·28 feet.

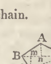
4. I measured the diagonal BD of a four-sided field 1462 links, and at its extremities I took the angles which it made with the sides, viz. ABD 48° 20′, CBD 41° 26′, ADB 29° 40′, and BDC 38° 44′. Required the plan and the area.

Ans. Area, 853086 sq. links = 8 ac. 2 ro. 4 per. 28 yds. 3·2616 feet.

5. In taking the plan of a quadrilateral field by the Plane-Table, I found the straight side AB to lie N. 73° E., and to measure 568 links; the diagonal AC to lie S. 83° E., 978 links; and the side AD to lie S 47° E., 734 links. Required the plan and the area.

Ans. Area, 323942·9 square links = 3 ac. 38 per. 9 yds. 3·02724 feet.

PROB. XXII. To survey any field with the chain.



Measure all the sides of the field, and then the diagonals BF, FC, FD. From these the field may be drawn upon paper by Prob. XXVIII. of PRACTICAL GEOMETRY, and its area may be found by Prob. XI. of MENSURATION OF SUPERFICIES.

Or divide the field by diagonals into as many trapezes as possible, and the remainder will consist of one or more triangles. Thus the field ABCDEF may be divided into two trapezes ABCF and CDEF, by joining CF. These may be surveyed as in the last Problem.

1. In a six-sided field I measured all the sides, viz. AB 583 links, BC 324, CD 456, DE 892, EF 728, and AF 477

links, and from F measured the diagonals FB 897, FC 723, and FD 948 links. Required the plan and the area.

Ans. Area, 700266·04 sq. links = 7 ac. 12·876 yds.

2. In a heptagonal field I measured along the northernmost diagonal BG, and at 207 links from B found the foot of a perpendicular above it AH, which measured 272; and at 578 from B found the foot of a perpendicular under it FK, which measured 498; the diagonal BG 928. From F, I measured along a diagonal FC, and at 488 from F was the foot of the perpendicular from B, which measured 587, and the diagonal FC 896. Then, from C, I measured along a diagonal CE, and at 498 from C was the foot of an under perpendicular ED 630, and at 688 from C was the foot of a perpendicular FM 574 links; the diagonal CE was 1093 links. Required the plan and the area.

Ans. Area, 1278242 sq. links = 12 ac. 3 ro. 5 per. 5 yds. 1·9652 feet.

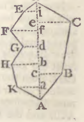
Note. 1. If a perpendicular, as Ep, upon a diagonal DF, fall without the field, and it be inconvenient to measure it in that situation, the other diagonal CE, with the perpendiculars upon it, may be taken; or the two triangles DEF, CDF, may be measured separately.

3. In a hexagonal field ABCDEF, I measured along the diagonal BF, and, at 328 links from B, I was at the foot of the perpendicular AG, which measured 286, and the diagonal BF was 536; but had to measure 127 links farther without the field, to come to the foot of the perpendicular EH on the opposite side of BF, which measured 453. Again, measuring along the diagonal EC, I found, at 386 from E, the foot of the perpendicular DK, which measured 496; and, 674 from E, found the foot of the perpendicular BL, which measured 436; the whole length of the diagonal EC was 895 links. Required the plan and the area.

Ans. 615122 sq. links = 6 ac. 24 per. 5 yds. 8·1432 ft.

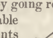
Note 2. In fields not very large it will be sufficient to measure one diagonal, and the perpendiculars upon it from all the other angles.

4. Suppose the distances of the perpendiculars from A to be 50, 145, 220, 295, 380, 475, and 655, the whole line AD being 725 links, the second and sixth distances reach to perpendiculars on the right hand, and the rest to those on the left hand. Also the perpendiculars on the right are 75 and 150, and the others in their order are 110, 135, 85, 275, and 185 links. Required the plan and the area.



Erect perpendiculars upon AD, at their proper distances from A; and, having made them of their proper length, the plan is drawn by joining their extremities. The area is found by Prob. IV. and VII. of Mensuration of Surfaces to be 178162·5 sq. links = 1 ac. 3 ro. 5 per. 1 yd. 7·335 ft.

Prob. XXIII. To take the plan of a field by going round it.



With the Plane-Table. Place the table at a corner A, and fix it when the needle points to the fleur-de-lis, and take a point A on the paper. Direct the index from the assumed point to the corner E of the field, and draw a line; then direct the index to B, and draw another line. Measure the lines in the field from A to B and from A to E, and lay these lines on the paper. Place the table at B, and, laying the index along BA on the paper, turn the table about till A is seen through the sights; the needle ought then to point to the fleur-de-lis. Direct the index to the corner C of the field, and draw a line, on which lay the length of BC. In the same manner are to be laid down the position and the lengths of the other sides CB and DE, and the last line will terminate at E on the paper, if no error has been committed.

With the Theodolite. Place the instrument at the corner A of the field, and, having turned it till the needle points to the fleur-de-lis, take the bearing of one of the sides, as AE; then observe the angle EAB, and measure AB. Again, place the theodolite at the corner B, and observe the angle ABC, and measure BC. Proceed in this way to take all the angles and to measure the sides.

Note 1. Add all the angles together, if they be interior; but if any of them be exterior, add the difference between it and 360°: the sum should be equal to 180°, multiplied by the number of sides, wanting two. The error, if any, should be equally divided amongst the angles.

Note 2. If the interior angles cannot be taken, let the exterior be taken by extending the direction of the sides. The sum of all the exterior angles should be 360°; but if any of the corners point inward, add 180° to 360° for every such angle, and the sum should be the sum of the angles.

Note 3. The things measured for laying down the plan of a field will always be sufficient for finding its content, but they will not always afford the shortest method. Thus, in taking the plan of the pentagonal field ABCDE by measuring the sides and angles, if we draw diagonals AC and CE, we can find the area of the triangle ABC from the sides AB and BC and the angle B, that of the triangle CDE from the sides CD and DE and the angle D; but then we have nothing given in the triangle ACE from which to find its area. We must therefore find, by trigonometry, in the triangle ABC, the angle ACB and the base AC, and in the triangle CDE, the angle DCE find the base CE; and these two angles, subtracted from BCD, will give the angle ACE, from which, with the sides AC and CE, we can find the area of the triangle ACE. And thus, by the help of trigonometry, we may find in every case sufficient data for computing the area from the things measured for taking the plan.

1. Let AB be 750, BC 810, CD 628, DE 598 links, and the angles at B 72°, at C 136°, and at D 122°. Required the area.

The angles are ACB 50° 58′ 11″, DCE 28° 13′ 23″, and ACE 56° 48′ 26″, and the sides AC 918·23, and CE 1072·38 links.

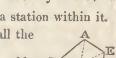
Ans. Area, 860133 sq. links = 8 ac. 2 ro. 16 per. 6 yds. 0348 ft.

2. In a six-sided field ABCDEF, let AB be 482, BC 586, CD 760, DE 812, and EF 910 links, and the angles at B 5°, at C 132°, at D 146°, and at E 106°. Required the area.

Ans. Area, 1500073·62 sq. links = 15 ac. 3 yds. 5·07 ft.

Prob. XXIV. To survey a field from a station within it.

The station must be chosen such that all the angles may be seen from it.



With the Plane-Table. Place the table at O, from which all the corners may be seen, then turn it to bring the needle to the fleur-de-lis; and on the paper take a point O, to represent the station. Direct the index from O to the corner A, and draw a straight line to represent OA in the field. Draw, in the same manner, lines to represent OB, OC, &c. Then measure from the station to A, B, C, &c. in the field, and lay them on their representatives, and join their extremities.

With the Theodolite. Place the instrument at the station O, and, putting the needle to the fleur-de-lis, take the bearing of OA. Next observe the angles AOB, BOC, &c. the sum of which should amount to 360°. Then measure straight lines from O to A, B, C, &c.

1. Suppose OA 798, OB 459, OC 434, OD 852, and OE 942 links, and the angles at O, AOB 74°, BOC 38°, COD 102°, DOE 82°, and EOA 64°. Required the area.

Ans. 1130004·3 sq. links = 11 ac. 1 ro. 8 per.

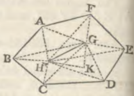
2. In a heptagonal field I found the angles at the instrument to be 67°, 43°, 84°, 56°, 27°, 51°, and 32°, and the distances of the angles from the instrument to be 528, 632, 916, 578, 732, 830, and 816 links. Required the plan and area.

Ans. Area, 1228999 square links = 12 ac. 1 ro. 6 per. 10·0516 yds.

Prob. XXV. To survey a field from two stations.

The stations must be such that all the objects to be laid down on the plan may be seen from them both, and that the angles which they make with the line joining the stations may not be too small.

The stations may be taken either within the boundaries of the field, in one of the sides, in the direction of two of the objects to be laid down, or at a distance, and without the boundaries of the field to be surveyed.



WITH THE PLANE-TABLE. Place the table at one of the stations, and the needle to the fleur-de-lis, then take a point G on the paper to represent that station, and direct the sights of the index from it to the other station; draw GH, and on it lay the distance between the stations from G to H. Direct the sights from G to the corner A; draw GA with a black-lead pencil, and upon any part of it place the letter A. Again direct the sights from G to the corner B; draw GB, and on it write B. In the same manner draw GC, GD, &c.

Remove the table to the second station, and turn it till the needle points to the fleur-de-lis; then the index, laid on HG of the paper, will point to the former station. Direct now the sights from H to the corner A; draw HA, which will meet the line GA in the point representing that corner, at which place A, and erase the former A. In the same manner draw HB, meeting GB in B, and so on; then join AB, BC, &c. In the same way the position of any other thing, as the house K, may be determined by drawing GK towards it when the table is at G, and HK towards it when the table is at H.

WITH THE THEODOLITE. Place the instrument at the first station G, and turn it till the needle points to the fleur-de-lis; take the bearing of the station H, and measure GH. Then take the angles HGC, CGD, DGE, &c., and lastly BGH. Remove the instrument to the second station H, and bring the needle to the fleur-de-lis; then the station G ought to bear upon the point opposite to that upon which H bore from G. If it does, take first the angle GHF, then FHA, AHB, &c., and lastly EHG. The sum of the angles taken at each station ought to be exactly 360°.

Every thing else which is to be put in the plan must be surveyed in the same way, by taking at G the angle between GH and the line from G to it, and the same at H. All these observations must be recorded in the field-book.

When the whole cannot be seen from two stations, more staons must be chosen. The lines between the stations must be measured, and the angles observed as before. But care must be taken to determine the position of each of the lines joining the stations.

In this manner, not only may fields be surveyed without even entering them, but a map may be made of the principal parts of an estate, or even of a county, and the chief places of a town, or any part of a river or coast, may likewise be surveyed by taking two such stations.

1. Required the plan and the area of a field from the following

FIELD-BOOK.

Angles at G.	Angles at H.	Remarks.
C 22° 0′	F 20° 0′	GH bears S. 67° 30′ W.
D 86 30	A 72 0	1038 links.
E 146 30	B 145 0	Corner of a house at K.
F 232 30	C 243 0	Angles {at G 50° / at H 323°.
A 313 30	D 317 0	
B 348 30	E 344 0	
H 360 0	G 360 0	

In this field-book, the angles at G are marked as taken with the theodolite when placed at that station. The sights, when at the beginning of the degrees, were directed to the station H, and the instrument fixed there. Then the moveable index was turned to C, and cut off 22° for the angle HGC, which, in the field-book, is marked C, the other two letters being found at the top; then it was turned to D, and cut off 86° 30′ for the angle HGD; and the difference of these two is the angle CGD. It was then turned to E, and cut off 146° 30′ for the angle HGE; and so on all the way round. In the same way the angles were taken at H, both for determining the corners of the field and for finding the corner of the house at K.

In calculating the areas of fields surveyed from more than one station, it is necessary to calculate, by trigonometry, the length of all the lines drawn from one of the stations to the angles; and for this purpose we have, in every triangle of which GH is a side, all the angles and this side to find the other side; after which the area is found as in the preceding problem. Here the distances from G are GA 1123·3, GB 893·1, GC 1409·73, GD 917·43, GE 951·44, and GF 660·74 links; from which the areas of the triangles AGB, BGC, CGD, DGE, EGF, and FGA, are to be calculated.

Ans. 2703648·8 sq. links = 27 ac. 5 per. 25 yds. 3·167 ft.

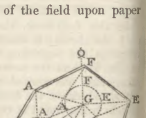
2. Required the plan and the area of a field from the following

FIELD-BOOK.

Angles at P.		Angles at R.		Remarks.
F	3°	A	6°	PR bears S. 22° 30′ E. 1827 links.
E	28	H	24	
D	49	G	64	
C	65	F	186	
B	132	E	228	
A	197	D	271	
H	247	C	319	
G	320	B	342	
R	360	P	360	

Ans. 10037324·8 sq. links = Area, 100 ac. 1 ro. 19 per. 21 yds. 3·08128 ft.

Prob. XXVI. To draw the plan of the field upon paper from the field-book.



Draw a faint line up and down the paper to represent the meridian, the upper end the north, and the under end the south. Using the data given in Ex. 1, Prob. XXV., in this line take a convenient point G for the first station. On the south side of G make an angle of 67° 30′ towards the left hand, which will give the position of GH; and take 1038 from any convenient scale, and lay that extent from G to H, to get the station H. The best protractor for laying down the angles is a circular one, divided into 360°. Place the centre at G, and the beginning of the degrees on GH. Make a mark at 22°, and at it write a faint C; make another mark at 86° 30′, and there write a faint D, and so on all the way round; and draw faint lines from G through the marks. Next place the centre of the protractor at H, and the beginning of the degrees on GH; and at 20° make a mark, and write F; at 72° make a mark, and write A, and so on; and draw lines from H through the marks. The lines from G and H, through the points where the same letter is written, must be drawn out till they meet, and their intersection is at the angle to which that letter belongs. Thus GA and HA will meet in the angle A, GB and HB will meet in the angle B, &c. After this join AB, BC, &c. for the boundaries of the field.

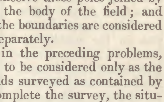
If the protractor be a semicircle, then, after laying down the angles less than 180°, the protractor must be laid on the other side of GH, and 180° taken from each of the remaining angles before they are laid down.

PROB. XXVII. To survey fields with crooked boundaries.

The boundaries of fields are seldom straight lines, and therefore surveyors generally erect poles near the corners of the ground to be surveyed, and conceive these poles joined by straight lines. This constitutes the body of the field; and the parts between these lines and the boundaries are considered as offsets, and their areas found separately.

The points, therefore, which, in the preceding problems, were called angles or corners, are to be considered only as the places of these poles, and the fields surveyed as contained by the lines joining them; and to complete the survey, the situation and distance of the boundaries from these lines must be found.

1. Let EIMP be a field to be surveyed. Poles are erected at A, B, C, D, near corners of the field, and the space ABCD is surveyed as before. The rest of the field is obtained by taking offsets from the lines AB, BC, CD, DA, and adding the spaces which are without these lines, and taking away the spaces within them.



In surveying a single field, an outline of it may be sketched upon paper, on which the dimensions may be written down as they are found. But in surveys of large estates, counties, &c. a field-book must be used, for registering all observations and dimensions. The field-book generally consists of three columns: the middle one contains the distance measured along the main lines AB, BC, &c.; and the other two are for the offsets, according as they are on the right or left of the main line. For this purpose it is best to begin at the bottom of the field-book, and to write upwards, that the offsets on the right side of the main line may be placed in the right-hand column, and the offsets on the left side in the left-hand column. Thus, in measuring from A to B, the offset A*a*, which measures 106 links, is on the left hand of AB, at the beginning of the line; therefore write 0 in the middle column, at the bottom, and opposite to it, in the left-hand column, write 106. Then measuring along AB, the point *f* is to be found, upon which the perpendicular falls from F: this is 284 links from A, and *f* F is 200 links; therefore write 284 in the middle column, and 200 opposite to it in the left-hand column. Again, at 442 links from A, the line AB crosses the boundary-line FG; therefore write 442 in the middle column, and in the adjacent columns draw straight lines in the direction of the straight line FG nearly, for the exact position of it is not required at this stage of the survey. At 530 the perpendicular from G meets AB, and G*g* is 108; place therefore 530 in the middle column, and 108 opposite to it in the right-hand column.

FIELD-BOOK.

Left offsets.	Main lines.	Right offsets.
AC, S. 60° 25′ E. 1896.		
	844	Including offset to cor.
86	746	Close to A.
152	688	
	594	
	462	200
D	64	90
	1410	D Γ
	1362	92
	924	196
	744	
146	600	
C 48	0	
⋗ 108		C Γ
104	912	
264	508	
84	152	
B 70	0	
⋗ 128		B Γ
94	1672	
172	1166	
	752	
	530	108
	442	
200	284	
A 106	0	
To left.		To right.

Proceed in this way to B, where, besides the offset, BI is measured, and placed in the left-hand column, with the mark ⋗ to show that it is not perpendicular. At the same place in the right-hand column is placed the mark Γ, to show that now the surveyor turns to the right hand. This finishes the survey along the line AB, and a line is drawn across the book to separate it from the next line. Proceed in the same way from B to C, from C to D, and from D to A.

The position of any one of the lines, as AC, being found with the compass, it will determine the position of the whole. But in using the compass, the variation should be allowed; and great care ought to be taken lest the needle be attracted by some metallic substance in its neighbourhood.

Ans. 1462335·12 sq. links ≐ 14 ac. 2 ro. 19 per. 22·27 yds.

(2.) FIELD-BOOK.

Left offsets.	Main lines.	Right offsets.
Diagonal AC, bears N. 28° W. 760 links.		
0	660	
30	450	
D 0	400	
0	490	D ⌈
10	400	
40	300	
55	200	
C 20	50	
	635	0 C ⌈
	500	25
	400	30
	300	
50	200	
B 40	100	
0	395	B ⌈
20	350	
35	300	
45	250	
50	200	
30	100	
A 15	50	

Ans. 3·1764515 acres.

(3.) FIELD-BOOK.

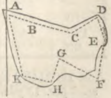
Left offsets.	Main lines.	Right offsets.
Diagonal AC, bears S. 56° E. 1560 links.		
	1350	
0	1200	
40	900	
20	750	
60	550	
85	400	
70	350	
D 35	200	
0	800	D ⌈
34	700	
	500	
	350	80
C	200	60
B	1100	C ⌈
0	912	B ⌈
40	800	
	750	
	680	50
	600	
90	450	
A 50	340	

Ans. 10·816122 acres.

Lay down the plans of the following properties from the field-book for the three examples, and calculate their contents.

Fig. 1.

Fig. 2.



(4.) (Fig. 1.)

Diagonals.
BD 1100
BE 720
BF 1080
AF 1000
AB bears N. $37\frac{1}{4}$° E.

20	510	A ┌
200	360	
20	0	
20	612	G ┌
156	320	
30	0	
30	600	F ┌
70	256	
20	0	
20	480	E ┐
28	220	
	114	
	0	30
D ┌	920	36
	826	78
	560	340
	356	90
	281	
120	180	
30	0	
25	900	C ┌
40	728	
120	560	
57	256	
20	0	
20	1040	B ┌
56	980	
	826	
	673	56
	522	
210	443	
120	156	
20	0	A

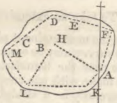
Ans. 1503446·3 sq. links = 15 ac. 5 per. 15 yds. 4·95828 ft.

(5.) (Fig. 2.)

Diagonals.
CE 620 | GB 850
CF 1000 | GK 710
CG 610 | BK 940
AK bears S. 11° E.

20	1150	A ┌
25	680	
35	420	
50	0	
60	580	K ┌
90	500	
150	300	
100	0	
89	470	H ┌
130	260	
200	0	
400	800	G ┐
380	630	
220	480	
36	230	
	153	
	110	25
	0	40
F ┌	760	50
	640	78
	520	115
	380	85
	200	40
	86	
30	0	
30	420	E ┌
35	320	
30	100	
20	0	
25	500	D ┌
89	360	
72	150	
30	0	
40	730	C ┐
150	540	
110	210	
30	0	
20	450	B ┐
70	250	
30	0	A

Ans. 1839891·2 sq. links = 18 ac. 1 ro. 23 per. 24·984 yds.



(6.)

	2180	A	
15	626	15	S. 59° E.
	426	H	
20	0	10	
20	1610	10 B Γ	
20	1590		N. 29° E.
	0	L	
To houses.			
A Γ	2050	15	
	1969		S. 13° W.
180	1000		
9	0		
61	1380	F Γ	
120	600		S. 77° E.
20	0		
20	750	E Γ	
24	500		S. 85° E.
10	0		
10	1400	D Γ	
500	1000		
400	700		N. 51° E.
300	400		
	25		
	0	20	
15	655	C Γ	
10	0		N. 45° E.
10	1450	M Γ	
350	600		N. 31° W.
20	0		
20	2280	L Γ	
220	1400		N. 85° W.
10	0		
10	640	K Γ	
100	400		N. 36° W.
20	0	A	

Ans. 89·259682 acres.

PROB. XXVIII. To take an extensive survey.

Choose for stations the most eminent places, from which the principal parts of the survey may be seen. Particularly choose such eminences as lie near the boundaries. Take the angles which these stations make with one another with great accuracy, and measure carefully in a straight line the distances from station to station, marking the places where the lines pass ditches, roads, rivulets, &c., and take offsets to near objects, leaving in the ground a mark at every place where you marked the distance in the field-book, distinguishing these marks by letters or figures, that they may not be mistaken for one another. In this way you will obtain the situation of the principal parts. Then take other stations within these, and measure the distances as before. And thus divide and subdivide the survey, till you come to single fields, which may be measured by some of the preceding methods.

The longer the distance is between the stations, if accurately measured, the more correct will the work be; but this cannot be ascertained by a single measurement, without using various methods of determining it. At the same time, an error in these primary distances affects the whole survey; and therefore every care ought to be taken to prevent it.

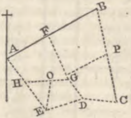
After the principal parts of the survey are laid down accurately, so as to have the whole divided into small compartments, these may be filled up by the plane-table, one by one.

In laying down the plan, proceed in the same way, first laying down the principal distances and the boundaries, and then the interior parts as they are surveyed; and, in filling up the particular departments, care must be taken to lay down the boundaries of parishes, estates, farms, &c. and to point out the particular situations of towns, villages, churches, gentlemen's seats, towers, farm-steads, also rivers, lakes, ponds, woods, plantations, rocks, precipices, and all the eminences, mines, pits, quarries, and in general every thing which can contribute to give a proper understanding of the nature of the survey. All these must be neatly sketched and properly coloured, and the names of the places printed in them.

1. I took two stations near a road, of which B lay from A, N. 61° E. 1850 links; and from A took the bearings of the eminences C, S. 70° E., D, S. 62° E., and E, S. 36° E., and at B took their bearings C, S. 14° E., D, S. 6½° W., and E, S. 26° W. Required their distances from the stations, and their bearings and distances from one another.

Ans. BC 1684·139, AE 1201·789, CD 596·638, and DE 753·3655 links.

Having drawn the plan of the observations in Example 1, is required to lay down on it, and to calculate the properties ntained in the field-book of the following examples.



(2.)

Diagonal. FH 935		
35	560	A
100	320	
88	180	
20	0	
20	695	H Γ
60	513	
	313	
O	300	4
	0	5
	870	G Γ
105	450	
50	0	
4	900	F Γ
98	734	
150	540	
122	330	
40	0	A
At the road.		

Ans. 7·286775 acres.

(3.

Diagonal. PF 1065.		
G	945	5
	878	80
	805	P Γ
44	366	
10	0	
10	950	B Γ
28	825	
90	740	
60	580	
30	430	
30	400	
78	260	
20	0	F
At the road.		

Ans. 7·3035896 acres.

(4.)

Diagonal. PD 945.		
	540	G
	360	58
	260	80
	0	20
70	597	D Γ
98	350	
	0	
	879	C Γ
203	621	
170	421	
	0	P

Ans. 6·503221 acres.

(5.)

Diagonal. EG 670		
4	564	O Γ
70	372	
130	248	
65	100	
12	0	
12	753	E Γ
90	613	
160	518	
170	416	
150	298	
40	0	D

Ans. 4·071447 acres.

The distances not mentioned in these two examples are to be taken from the preceding ones.

Prob. XXIX. To find the contents of a survey.

The areas of single fields, bounded by straight lines, may be found from the lines measured in the field, by the first twelve problems of Mensuration of Surfaces.

To calculate Offsets. The most accurate method is to compute them separately, as triangles and trapezoids, by Prob. IV. and VII. of Mensuration of Surfaces.

Method 2. If the distances between the perpendiculars be nearly equal. To half the sum of the perpendiculars at the extremities of the base add all the rest, and multiply the sum by the base, and divide the product by the number of divisions in the base made by these perpendiculars.

Common Method. Divide the sum of the perpendiculars by the number of them for a mean perpendicular, by which multiply the base.

If the boundary be a curve-line, and the distances between the perpendiculars equal, the area may be calculated by Rule II. Prob. XXV. of Mensuration of Surfaces.

The fourth Example in Prob. XXII. wrought by the first ethod.

50×110	$=$	5500	for the triangle AKa
$170 \times (110+135)$	$=$	41650	. . . . trapezoid KabH
$75 \times (135+ 85)$	$=$	16500	. . . . trapezoid HbdG
$85 \times (85+275)$	$=$	30600	. . . . trapezoid GdfF
$275 \times (275+185)$	$=$	126500	. . . . trapezoid FfiE
70×185	$=$	12950	. . . . triangle EiD
145×75	$=$	10875	. . . . triangle ABc
$330 \times (75+150)$	$=$	74250	. . . . trapezoid BceC
250×150	$=$	37500	. . . . triangle CeD
		2)356325	

Ans. 178162·5 the whole area.

By the second Method.

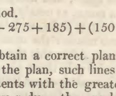
Ans. $725 \times (\frac{1}{6} \times (110 + 135 + 85 + 275 + 185) + (150 +$
$) \times \frac{1}{3}) = 149833\frac{1}{3}$ area.

By the third Method.

Ans. $725 \times (\frac{1}{5} \times (110 + 135 + 85 + 275 + 185) + (150 +$
$) \times \frac{1}{2}) = 196112 \cdot 5$ area.

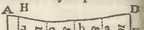
Some surveyors endeavour first to obtain a correct plan of e land, and then they measure, on the plan, such lines as ll enable them to calculate its contents with the greatest pedition; and for this purpose they reduce the crooked undaries to straight lines. Sometimes this is done by etching a hair through the crooked part, so that the small rts cut off by the hair may be equal to the parts taken in, as arly as the eye can judge; and though this can be done very ely by an experienced surveyor, it should never be trusted when it is possible to have the whole measured in the field.

Others reduce the crooked parts to a triangle, by Prob. XXIV. of PRACTICAL GEOMETRY, which can be done by e parallel ruler without drawing lines. Thus, suppose BCDEFG to be the space which is be reduced to a triangle. Lay the rallel ruler from A to C, and move ill it pass through B, and mark the nt 1 in which it cuts AG, or its exsion. Lay the ruler through 1 and and move it till it pass through



and mark 2 where it cuts AG. Again lay the ruler from hrough E, and move it till it pass through D, and mark 3 ere it cuts AG, and so on; then join 4 and F, and the angle F4G is equal to the given space. For B1 is parallel AC; therefore if C1 were drawn, the triangle AC1 = CB. Now, when the ruler passes through A and C, it takes in the triangle ACB; and when it is moved to B1, it cuts off the triangle AC1. In like manner the triangle 1DE which is taken in, is equal to the triangle 1DC cut off; and so of the rest.

Another method of calculation practised by surveyors is the following, which, though it depend upon judgment, will be found to come very near the truth, and is very expeditious.



Let ABCD be the plan of a survey, and DC a straight boundary. Draw EF perpendicular to DC, and on it lay a chain, from E to a, from a to b, from b to c, &c.; and draw parallels to CD through a, b, c, &c. and they will divide the plan into spaces, each a chain in breadth. Measure in a line parallel to DC, half-way between E and a. This is supposed to give the mean length of the first space, and therefore is to be measured where the length is a mean, as nearly as the eye can judge. It is here supposed to be 109 links, and is written so in the first space. In the same manner the mean lengths are taken in all the other divisions. After this these lengths are to be added together, and require only three places to be cut off to give the area in acres. The small space ABGH remaining beyond the last parallel, which is only 39 links in breadth, may be found by multiplying 39 by its mean length, judged of as before. Or offsets upon GH may be taken from A and B, and thus a mean breadth may be obtained, to be multiplied by GH, or the mean length. Suppose the offsets at A and B to be 44 and 31, and suppose the mean length to be 96 links; then $96 \times 39 = \cdot03744$ of an acre. Or the mean offset is 37·5, which multiplied by GH, suppose 100, gives ·03750 of an acre for the content of the part ABGH; and this, added to ·393, the sum of the mean lengths of the other pieces, gives ·4305 of an acre, or 1 rood 28·88 perches, for the whole area.

Prob. XXX. To measure and plot hilly ground.

Rule. The surface is measured as in level ground; but we must lay down on the plan only the area of the base on which the hill stands. Now the length of the base or plotting-line is found by this proportion, radius : cos. of the angle of acclivity : : the surface-line : the base-line.

1. A line of 1200 links is measured up a hill whose angle of acclivity is 12° 15′, what is the length of the base-line?

Cos. 12° 15′ — rad.	= 9·989997
Surface-line, 1200 log.	= 3·079181
Base line, 1172·7 log.	= 3·069178

2. A line of 1764 links is measured up a hill whose angle f acclivity is 17° 20′, what is the length of the base-line?

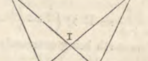
Ans. 1683·4 links.

3. The angle of acclivity of a hill is on the east side 25° 0′ and on the west side 20° 45′; a line from its base to the ummit is on the east side 5000 links, and on the west side 500 links, what is the length of the base-line?

Ans. 9654·16 links.

Prob. XXXI. To deduce from angles measured out of the tation, but near it, the true angles at the station.

When the centre of the instrument cannot be placed in the ertical line occupied by the axis of a signal, the angles oberved must undergo a reduction according to circumstances.



Let C be the centre of the tation P, the place of the intrument, or the vertex of the bserved angle APB, to find the ngle ACB.

Supposing that APB = P, PC = p; CP = d; AC = L, and BC = R are known.

Since the exterior angle of a triangle is equal to the sum f the two interior and opposite angles; then the angle AIB = P + IBP, and AIB is also equal to C + CAP; hence P + BP = C + CAP, and by transposition C − P = IBP − CAP; but the triangles CBP, CAP give

$$\text{Sin. CBP} = \text{sin. IBP} = \frac{\text{CP}}{\text{BC}} \text{sin. BPC} = \frac{d \text{ sin. } (\text{P} + p)}{\text{R}}.$$

$\text{Sin. CAP} = \frac{\text{CP}}{\text{AC}} \text{sin. APC} = \frac{d \text{ sin. } p}{\text{L}}$; and as the angles CBP, CAP are by hypothesis always very small, their ines may be substituted for their arcs; whence $\text{C} - \text{P} = \frac{d \text{ sin. } (\text{P} + p)}{\text{R}} - \frac{d \text{ sin. } p}{\text{L}}$.

When the reduction is required in seconds the equation becomes $\text{C} - \text{P} = \frac{d}{\text{sin. } 1''} \times \left\{\frac{\text{sin. } (\text{P} + p)}{\text{R}} - \frac{\text{sin. } p}{\text{L}}\right\}$.

Note 1. In using this formula, the signs of p and of $(\text{P} + p)$ must be carefully attended to: thus the first term of the correction will be positive if the angle $(\text{P} + p)$ is between 0° and 180°, and negative if that angle exceed 180°; and the contrary will obtain in these circumstances with regard to the second term, which answers to the angle of direction p.

NOTE 2. When the signal is either a circular or polygonal tower, the method of obtaining the exact angle will suffer a slight variation, which will easily be understood by any one acquainted with the rudiments of Geometry.

PROB. XXXII. When a base-line is measured at an elevated level, to find its length when reduced to the level of the sea.

Let $r =$ the mean radius of the earth, or the distance from the surface to the sea-level, $h =$ the height above the level of the sea, at which the base is measured, $B =$ the measured base and $b =$ that to which it must be reduced at the level of the sea; then since B and b are portions of similar and concentric circles to the radii $r+h$ and r, it is obvious that $r+h : r :: B : b$ or $b = B \times \frac{r}{r+h}$; hence $B - b = B - \frac{rB}{r+h} = \frac{Bh}{r+h} = B \times \left(\frac{h}{r} - \frac{h^2}{r^2} + \frac{h^3}{r^3} - \text{\&c.}\right)$; but the radius of the earth being extremely great in proportion to h, we may, for all practical purposes, assume $B - b = B \times \frac{h}{r}$.

RULE. Add the logarithm of the measured base in feet to the log. of its height above the sea also in feet, and the constant log. 2·680110, the sum will be the log. of the correction in feet, which is always subtractive.

NOTE. In order to arrive at the most accurate results in the practice of surveying, the following rules, which are demonstrated in the third volume of Hutton's Course, should be carefully attended to.

I. When only one side of a triangle is to be determined, the measured base should be as nearly equal to the side sought as possible.

II. When two sides of a triangle are to be determined, the triangle should be nearly equilateral.

III. When two sides are to be determined, and the base cannot be equal to either of the sides, it should be taken as long as possible, the two angles at the base should be equal, and not less than 23°.

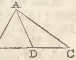
OF DIVIDING LAND.

PROB. XXXIII. To divide a triangular field ABC in any proportion, by a straight line drawn from the vertex A, to the opposite side BC.

RULE. Divide the base BC in the required proportion, by Prob. IX. PRACTICAL GEOMETRY, and draw a line from the vertex to the point found in the base.*

1. Divide the triangle BAC, of which the base BC is 950 links, in the ratio of 9 to 7, by a line drawn from the vertex A.

Ans. $16 : 7 :: 950 : 415\frac{5}{8}$ to be laid from B to D; then AD is the dividing line.



2. Divide the triangle ABC, of which the sides are AB 386, BC 428, and AC 533 feet, in the ratio of 8 to 5, by a line drawn from the angle B. Ans. AD 328, and DC 205 feet.

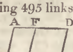
3. Divide the triangle ABC, of which AC is 374, and AB 473 links, and the angle BAC 54°, in the ratio of 5 to 6, by a line drawn from C. Ans. AD 215, and DB 258 links.

PROB. XXXIV. To cut off any portion from a parallelogram by a straight line parallel to one of the sides, having the other side given.

RULE. As the whole content is to the portion to be cut off, so is the length of the given side to the point through which the line of division must be drawn.

1. It is required to cut off 3 acres from a field ABCD of 10 acres, by a line parallel to AB, the side BC being 495 links.

Ans. $10 : 3 :: 495 : 148\frac{1}{2}$ to be laid from B to E, and from A to F; then EF is the dividing line.



2. Divide the parallelogram ABCD, of which AB is 236, and BC 574 yards, and the angle ABC 76°, in the ratio of 3 to 4, by a line parallel to AB. Ans. BE 246, and EC 328 yards.

3. Divide the rectangle ABCD, of which AB is 472, and BC 675 feet, in the ratio of 7 to 8, by a line parallel to AB. Ans. BE 315, and EC 360 feet.

PROB. XXXV. To cut off any portion from a triangular field ABC, by a straight line drawn from any point D, in the side BC.

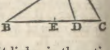
RULE. Find by Prob. XXXIII. the point E in the base, to which a line drawn from the vertex would divide the field in the given ratio. Then if E falls between B and D, the point

* It is manifest that the two triangles into which the field is divided have the same altitude; they are therefore as their bases (El. Geom. XVI. Cor.)

F will be in BA, otherwise it will be in AC, and is found by this proportion BD : BE :: BA : BF, or CD : CE :: CA : CE.*

1. It is required to cut off 2 acres from the triangle ABC of 6 acres, by a line drawn from D, 230 links from B; the line BC being 466 links, and BA 420.

Ans. $6 : 2 :: 466 : 155\frac{1}{3}$ = BE, and $230 : 155\frac{1}{3}$ :: BA 420 : BF $283\frac{15}{23}$; then DF is the dividing line.



If E had fallen between D and C, then AC must have been divided.

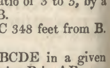
2. Divide the triangle ABC, of which the sides are AB 451, BC 528, and AC 364 links, in the ratio of 7 to 9, by a line drawn from D in BC, 363 links from B.

Ans. BF in AB 287 links.

3. Divide the triangle ABC, of which AB is 464, and BC 580 feet, and the angle ABC 64°, in the ratio of 3 to 5, by a line drawn from E in AB, 290 feet from B.

Ans. BF in BC 348 feet from B.

Prob. XXXVI. To divide any field ABCDE in a given ratio, by a straight line drawn from the point P in AB, one of its sides.



Reduce the field to the triangle AFG, by Prob. XXXIV. of Practical Geometry, having its base in the side CD, which the dividing line will cut. Divide the triangle AFG in the given ratio by the line AH, by Prob. XXX. Draw AK parallel to PH, and join PK: it will be the dividing line.†

Note 1. If the point K fall in CG, the field must be reduced to a triangle whch has its base in BC, or a triangle equal to PCK must be made by a line drawn from P to BC.

Divide the quadrilateral ABCD, of which the sides are AB 255, BC 284, CD 313, and AD 472 yards, and the angle ABC 57°, in the ratio of 6 to 7, by a line drawn from P in AD, 118 yards from A. Ans. BH 294, and BK 190·06 yds.

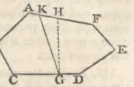
---

* Join DA, and draw EF parallel to it, the triangles BAD, BFE, are evidently similar; wherefore BD : BE :: BA : BF.

† Draw the parallels HP, KA, and join HA, intersecting PK in the point O. Since the triangles KAP, KAH have the same base KA, and lie between the same parallels KA, HP, they are therefore equal (El. Geom. 15, Cor.) Now, if from these equals we take away the common part KOA, there will remain POA = KOH, whence the line KP divides the figure in the proportion required.

NOTE 2. As the method of dividing the field geometrically by parallels is much easier than the arithmetical, it is best to do it in that way very accurately, and then to measure the result by the scale.

PROB. XXXVII. From a given field ABCDEF, to cut off any quantity by a straight line drawn from the point G in the side CD.



Draw GH as near to the position of the line required as you can conjecture, and calculate the area of the space ABCGH. Then, if this differs from the quantity to be set off, divide the difference by $\frac{1}{2}$ GH, the guess-line, the quotient is the length of a perpendicular, to be set off on either side of GH according as the quantity is too great or too small.

It is required to set off from the point G, in the side of the field ABCDEF, 8 acres towards BC.

Draw GH by guess, and measure the space GHABC, which suppose $= 9{\cdot}0496$ acres, or $1{\cdot}0496$ acres too much; now if GH $= 728$ links, then $10496 \div 364 = 288$, the perpendicular to be set off from H towards BC, and the line GK drawn through this point is the dividing line required.

When a quantity of land, such as a common, is to be divided among several proprietors in certain proportions, the quantity to be assigned to each will be as the value of his claim, divided by the quality or value of the ground allotted to him. This may be done by adding into two sums the contents and the values: then, by distributive proportion or fellowship, compute the value of each person's share; and from the quality of the ground where his share is to be determined, find what quantity will amount to the value of his share, and lay it off by the last problem.

Suppose it were required to divide 780 acres among three proprietors, whose estates are £1000, £3000, and £4000 a-year, and the values of the land in which their shares are to lie are 5s., 8s., and 10s. the acre respectively.

The claims being as 1, 3, and 4, and the qualities as 5, 8, and 10, the quantities assigned to them must be as $\frac{1}{5}$, $\frac{3}{8}$, and $\frac{4}{10}$, or as 8, 15, and 16, and their shares 160, 300, and 320 acres.

PROB. XXXVIII. To transfer, and to enlarge or diminish, a plan.

After the first plan is completed, it will be necessary to draw out a fair one upon vellum or paper.

There are various ways of doing this.

I. If the fields be generally bounded by straight lines, lay the plan upon the clean paper, keeping it firm by weights, and prick through all the corners of the plan, and then connect the points on the clean paper.

II. Lay a piece of paper covered with black-lead dust between the papers, with the powdered side towards the clean paper, and with a blunt needle trace all the lines on the rough plan with such pressure that the impression may reach the clean paper; after which they are to be traced with ink upon the clean paper.

III. Divide the rough plan into small squares, and divide the paper to which it is to be transferred into as many squares; then copy the parts of the plan found in each square into the corresponding square of the other plan. In this manner the plan may be enlarged or diminished in any proportion, by making the squares in that proportion.

IV. There are several instruments useful for transferring, enlarging, and diminishing plans, as the proportional compasses, the pentagraph, and the eidograph.

A plan may be enlarged or diminished in any proportion on the first paper, by Prob. XXXVII. of PRACTICAL GEOMETRY, and afterwards transferred to the clean paper by any of the preceding methods.

After the plan is copied upon the clean paper, write such names, remarks, or explanations as are reckoned to be necessary; draw a meridian line with a fleur-de-lis pointing to the north, and in a convenient corner lay down a scale for measuring the parts of the plan. The title of the plan must be placed in a conspicuous part, and properly ornamented. After which, every part must be coloured or illuminated in the way that appears most natural. Rivers, woods, hills, hedges, houses, roads, &c. must all be distinguished by proper representations. But these things require to be learned by practice.

GAUGING.

Gauging is the method of taking the dimensions of any vessel, and of finding the quantity of liquor in it.

In Mensuration the dimensions are taken on the outside, but in Gauging they are taken on the inside of the vessel.

The dimensions of vessels are taken in inches, and therefore the content may be found in inches by the Rules for the Mensuration of Solids; after which they may be reduced to gallons, bushels, or pounds, by dividing by the following

TABLE.

Cubic inches.	
277·274	1 imperial gallon.
2218·192	1 imperial bushel.
2273·461	1 imperial bushel ground malt.
25·67	1 pound green soft soap.
25·56	1 pound white soft soap.
27·14	1 pound cold hard soap.
30·28	1 pound tallow, gross.
34·8	1 pound starch.
8·46	1 pound green glass.
9·178	1 pound plate glass.
10·516	1 pound broad glass.

OLD MEASURES.

231	1 gallon of wine, spirits, oil, &c.
282	1 gallon of beer or ale.
268·8	1 corn gallon.
2150·42	1 malt bushel.
2204	1 bushel ground malt.
104·2034	1 Scotch pint.
2214·322	1 firlot wheat, pease, rye, and salt.
3230·305	1 firlot barley, oats, and malt.
271·25	1 Irish gallon.
8680	1 Irish barrel.

Gauging is generally performed by the sliding-rule.

DESCRIPTION OF THE SLIDING-RULE.

This rule is 1 foot long, 1·1 inch broad, and ·8 inch thick, and each of its four sides is furnished with a slider.

Upon the first side are four lines, all constructed in the same way, that is, each is divided into 1000 parts, and the numbers are placed at their logarithms. The two on the slider are marked B. or Num. The upper line on the rule is marked A., and the under one M. D. or Malt Depth. This last is inverted, and the point 2218·192 is placed at the right end, so that 10 on this line is opposite to 2218·192 or IM. B. on the line A.

Upon the opposite side of the rule, the lines on the slider are the same with those on the first side. The line on the rule is constructed in the same way, only the distance between 1 and 10 is twice as long. One-half of this line, or from 1 to 3·2, is placed above the slider, and the other half below it. Some rules have a line on which the distance between 1 and 10 is one-third of the distance on this line. On the inside of the slider are the gauge-points for imperial gallons, for imperial malt bushels, and also the multipliers and divisors for square and round vessels.

On the other sides or edges of the rule, the sliders contain lines the same with those of the other sliders; and on the rule are lines for ullaging, on the one edge for ullaging a lying cask, and on the other for a standing cask. These lines are constructed experimentally thus:—Take a cask containing 100 gallons, and fill it with water. Draw off one gallon, and measure the depth of the remaining water; then set the length, or the bung-diameter, according as the cask is standing or lying, on the slider, opposite to 100 on the rule, and opposite to the wet inches on the slider mark 99 on the rule. Draw off another gallon, measure the wet inches, and opposite to them on the slider mark 98 on the rule; and proceed in the same manner till the line is all marked. The inside of the slider, on the edge marked C, contains a line of inches and lines for reducing the first and second varieties of casks to cylinders.

There are several brasses or notches marked on the lines. Thus, on the first side, a brass with IM. B. is marked at 2218·192 for imperial bushels, and another with IM. G. for imperial gallons at 277·274. On the second side are marked on the rule the gauge-points, IM. G. for imperial gallons at 18·789, M. S. or malt bushels in square vessels at 47·097, and M. R. or malt bushels in round vessels at 53·144.

Prob. I. To multiply by the sliding-rule.

Turn up the first side, and set 1 on the slider B opposite to the multiplier on the line A; then against the multiplicand on the slider is the product on A.

1. Multiply 15 by 8. Set 1 on B to 8 on A; then opposite to 15 on B will be 120 on A, the product.

Note. The 1 at the left end of A may be read 1, or 10, or 100; and the rest of the numbers must be read accordingly, the 2 either 2, or 20, or 200, &c. Also, in reading the multiplicand on the slider, the 1 may be read 10 or 100; but then the product must be increased 10 or 100 times.

2. Multiply 250 by 56. Set 1 on B to 56 on A; then against 250 on B is 14000 on A.

3. Multiply 7·23 by 8·5............................Ans. 61·455.

4. ....82·5 by ·73.................................60·225.

5. .... ·94 by 7·4.................................. 6·956.

Prob. II. To divide by the sliding-rule.

Place the divisor on the slider B opposite to the dividend on A; then against 1 on B is the quotient on A.

1. Divide 480 by 15. Set 15 on B to 480 on A; then against 1 on B is the quotient 32 on A.

2. Divide 8142 by 59..............................Ans. 138.

3. ....8·75 by 3·25....................................2·69.

4. ....6·08 by 7·42.................................... ·819.

5. ....19·7 by 3·5......................................5·63.

Prob. III. To work a proportion by the sliding-rule.

Place the first term on the slider B opposite to the second or third on A; then against the other term on B is the answer on A.

1. If 40 yards of cloth cost £24, what will 15 cost?

Ans. Set 40 on B to 15 on A; then against 24 on B is £9 on A, the answer.

2. How many yards of cloth at 18s. may be given for 60 lbs. of tea at 7s.? Ans. $23\frac{1}{3}$ yards.

3. If 16 men do a piece of work in 48 days, in what time will 24 men do it? Ans. 32 days.

4. What number of men must be employed to perform in 84 days, a piece of work which 108 men perform in 133 days? Ans. 171 men.

5. If £15·6 pay 16 labourers for 18 days, how many, at the same rate, will £35·1 pay for 24 days? Ans. 27 labourers.

6. If 36 yards of cloth, 7 quarters wide, cost £25·2, what will 120 yards of the same quality, 5 quarters wide, cost?

Ans. £60.

Prob. IV. To extract the square root by the sliding-rule.

Take the second side of the rule. Place 1 on the slider C opposite to 1 on the rule D, then if the given number consist of 1, 3, 5, 7, &c. figures, the root is opposite to it on the line above; but if it consist of 2, 4, 6, &c. figures, the root is opposite on the line below it, on the rule.

1. Required the square root of 81. Set 1 on C to 1 on D; then opposite to 81 on C, is 9 on the line below on D.

2. Required the square root of 625. Set 1 on C to 1 on D; then against 625 on C, is 25 on D on the line above.

3. Required the square root of 1681................Ans. 41.
4. .... of 24649....................157.
5. .... of 5·0625....................2·25.
6. .... of 30·25......................5·5.

Prob. V. To find a mean proportional between two numbers.

Set the less on C to the less on D; then against the greater on C is the mean proportional on D.

1. Required a mean proportional between 18 and 72.

Set 18 on C to 18 on D; then against 72 on C is 36 on D, which is the mean proportional required.

2. Required a mean proportional between 2448 and 17.
Ans. 204.
3. Required a mean proportional between 128 and 1152.
Ans. 384.
4. Required a mean proportional between 30·25 and 272·25.
Ans. 90·75.
5. Required a mean proportional between 1248 and 78.
Ans. 312.
6. Required a mean proportional between 205·5 and 137.
Ans. 167·79.

Prob. VI. To find a number, which shall have to a given one the same ratio which the squares of two given numbers have to one another.

Set the first term of the ratio on D to the given number on C; then opposite to the other term of the ratio on D stands the answer on C.

1. Required the number which shall be to 36, as the square of 4 to that of 3.

Set 3 on D to 36 on C; then against 4 on D will be 64 on C, the answer.

2. What number is to 120, as the square of 3 to that of 2? Ans. 270.

3. Increase the number 240 in the ratio of the square of 4 to that of 5. Ans. 375.

4. Diminish the number 392 in the ratio of the square of 7 to that of 6. Ans. 288.

5. Find the number to which 196 shall have the same ratio as the square of 7 to that of 9. Ans. 324.

Prob. VII. To find a number which shall be to a given one as the square roots of two given numbers.

Set the first term on C to the given number on D; then against the other term on C stands the answer on D.

1. To what number will 3 have the same ratio as the square root of 108 to that of 48?

Set 3 on D to 108 on C; then against 48 on C is 2 on D, the answer.

2. To what number will 2 be as the square root of 120 to that of 270? Ans. 3.

3. Required the number to which 256 shall be as the square root of 16 to that of 9. Ans. 192.

4. Increase the number 433 in the ratio of the square root of 3 to that of 5. Ans. 559.

5. Diminish the number 1414 in the ratio of the square root of 8 to that of 7. Ans. 1323.

Prob. VIII. Of multipliers, divisors, and gauge-points.

Instead of first finding the content of a vessel in inches, and afterwards reducing it to the measure of capacity required, which must often be done both by multiplying and dividing by known numbers, gaugers find the content in the measure required by means of a single multiplier or divisor.

Gauge-points are numbers made use of in working by the sliding-rule. The operation is performed similar to that in Prob. VI.; and for that purpose the square root of the divisor is used as the first term, and is called the gauge-point.

TABLE I.—MULTIPLIERS, DIVISORS, AND GAUGE-POINTS, FOR CYLINDRICAL VESSELS.

Measures.	Multipliers.	Divisors.	Gauge-Points.
For inches,…………	·7853982	1·2732	1·1284
Imperial gallons,……	·0028326	353·0362	18·7893
Imperial bushels,……	·0003541	2824·2903	53·1441
Green soft soap, lbs…	·0305959	32·6841	5·7170
White soft soap, do…	·0307276	32·5440	5·7047
Cold hard soap, do…	·0289388	34·5557	5·8784
Tallow, gross, do……	·0259379	38·5537	6·2092
Starch, do…………	·0225689	44·3087	6·6565
Green glass do………	·0928367	10·7716	3·2820
Plate glass, do………	·0855740	11·6858	3·4184
Broad glass, do…….	·074686C	13·3894	3·6592
Old Wine gallons,…	·0034000	294·1183	17·1499
Old Ale gallons,……	·0027851	359·0535	18·9487
Old Corn gallons,……	·0029219	342·2468	18·4999
Old Malt bushels,….	·0003652	2738·0000	52·3259
Old Scotch pints,…..	·0075372	132·6759	11·5185
Old Wheat firlots,…	·0003547	2819·3623	53·0977
Old Barley firlots,…	·0002431	4112·9526	64·1323

TABLE II.—MULTIPLIERS, DIVISORS, AND GAUGE-POINTS, FOR CONICAL VESSELS.

Measures.	Multipliers.	Divisors.	Gauge-Points.
For inches,…………	·2617994	3·8197	1·5944
Imperial gallons,……	·0009442	1059·1086	32·5441
Imperial bushels,……	·0001180	8472·8708	92·0490
Soft soap, pounds,….	·0101986	98·0522	9·9021
White soft soap, do…	·0102425	97·6320	9·8810
Hard soap, do………	·0096463	103·6671	10·1817
Tallow, do…………	·0086460	115·6611	10·7547
Starch, do…………	·0075230	132·9261	11·5295
Green glass, do……..	·0309456	32·3148	5·6846
Plate glass, do………	·0285247	35·0575	5·9208
Broad glass, do……..	·0248953	40·1683	6·3379
Old Wine gallons,…	·0011333	882·3549	29·7045
Old Ale gallons,……	·0009284	1077·1605	32·8201
Old Malt bushels,….	·0001217	8214·0000	90·6306
Old Scotch pints,…..	·0025124	398·0277	19·9506
Old Wheat firlots,…	·0001182	8458·0870	91·9680
Old Barley firlots,…	·0000810	12338·8578	111·0812

Table III.—Multipliers, Divisors, and Gauge-Points, for Prismatic Vessels.

Measures.	Multipliers.	Divisors.	Gauge-Points.
SQUARE.			
Imperial gallons,.......	·0036065	277·274	16·6516
Imperial bushels,.......	·0004508	2218·190	47·0977
Hard soap, pounds,...	·0368460	27·140	5·2096
Tallow, do.............	·0330251	30·280	5·5027
Starch, do.............	·0287356	34·800	5·8992
Green glass, do.......	·1182033	8·460	2·9086
PENTAGONAL.			
Imperial gallons,.......	·0062050	161·1610	12·6950
Imperial bushels,.......	·0007756	1289·2884	35·9067
Hard soap, pounds,....	·0633927	15·7747	3·9717
Tallow, do.............	·0568190	17·5998	4·1952
Starch, do.............	·0494390	20·2269	4·4974
Green glass, do........	·2033661	4·9172	2·2175
HEXAGONAL.			
Imperial gallons,.......	·0093700	106·7228	10·3307
Imperial bushels,.......	·0011726	853·7824	29·2196
Hard soap, pounds,...	·0957287	10·4462	3·2321
Tallow, do.............	·0858018	11·6548	3·4139
Starch, do.............	·0746574	13·3945	3·6599
Green glass, do........	·3071012	3·2563	1·8045
HEPTAGONAL.			
Imperial gallons,.......	·0131059	76·3918	8·7351
Imperial bushels,.......	·0016382	610·4142	24·7066
Hard soap, pounds,...	·1338951	7·4685	2·7329
Tallow, do.............	·1200103	8·3326	2·8866
Starch, do.............	·1044228	9·5765	3·0946
Green glass, do........	·4295405	2·3281	1·5285
OCTAGONAL.			
Imperial gallons,.......	·0174139	57·4253	7·5780
Imperial bushels,.......	·0021767	459·4027	21·4337
Hard soap, pounds,....	·1779081	5·6209	2·3708
Tallow, do.............	·1594592	6·2712	2·5042
Starch, do.............	·1387479	7·2073	2·6846
Green glass, do........	·5707359	1·7521	1·3237

CONSTRUCTION OF THE PRECEDING TABLES.

In Table I. the *multipliers* are found by dividing ·7853982 (the area of a circle whose diameter is 1) by the number of cubic inches in 1 gallon, 1 bushel, &c.; the *divisors* are the *reciprocals* of the multipliers, and the *gauge-points** are the square roots of the divisors.

In finding the contents of conical vessels, we multiply by one-third of the length; hence the *multipliers* in Table II. are one-third of those in Table I.; the *divisors* are three times those in Table I., and the *gauge-points* are the square roots of the divisors. In using the numbers in this Table, the whole length must be taken.

The *multipliers* in Table III. are found by dividing the multipliers in the Table of Polygons, page 183, by the number of cubic inches in 1 gallon, 1 bushel, &c.; the *divisors* are the *reciprocals* of the multipliers, and the *gauge-points* are the square roots of the divisors.

PROB. IX. To gauge areas one inch deep.

I. When one side is given, set the gauge-point on D to 1 on C; and against the given side on D is the answer on C.

1. Suppose the side of a square to be 77 inches. Required its content, at 1 inch deep, in old wine and ale gallons.

Here the multipliers are ·003546 and ·004329, the divisors are 282 and 231, and the gauge-points 16·7929 for ale, and 15·1987 for wine gallons.

77	282)5929	231)5929
77	21·025 ale gal.	25·667 wine gal.
5929 content in inches.		5929
·003546		·004329
21·024 ale gallons.		25·667 wine gal.

By the Gauge-Points.

Set the gauge-point for ale gallons, 16·7929 on D, to 1 on C; then against 77 on D will be found 21 ale gallons on C.

Set the gauge-point for wine gallons, 15·1987 on D, to 1 on C; and against 77 on D is 25·7 on C, the wine gallons.

2. Required the content, in imperial gallons, of a square vessel at 1 inch deep, the side 98 inches. Ans. 34·6368 gals.

* The gauge-point for circular or polygonal vessels is the diameter of a circle, or the side of a polygon, of which the content is 1 gallon, 1 bushel, &c. when the depth is 1 inch.

3. Required the content of a regular pentagon 1 inch deep, in hard soap and starch, the side 53 inches.

Ans. 178·07 lbs. hard soap, 138·874 lbs. starch.

4. Required the content of a regular octagon 1 inch deep, in tallow, the side 83 inches. Ans. 1098·51443 lbs.

II. When two dimensions are given, it is necessary, in working by the gauge-points, to find a mean proportional between the two factors, and to work with it by the preceding rule.

1. Required the content, at 1 inch deep, of a rectangular vessel, of which the length is 100½ inches, and its breadth 20 inches, in imperial bushels and pounds of hard soap.

100·5	2010
20	·0368
2010 inches.	74·06 lbs. hard soap.
·0004508	
·906108 of an imperial bushel.	

By the Sliding-Rule.

Set 100½ on B to 1 on MD; then against 20 on A is ·906 of an imperial malt bushel on B.

Set 27·14 on A to 20 on B; then against 100·5 on A is 74·1 lbs. on B.

By the Gauge-Points.

First find a mean proportional between the breadth and length.

Set 20 on C to 20 on D; then against 100½ on C is 44·83 on D, the mean proportional.

Set the gauge-point 47·098 on D to 1 on C; and opposite to 44·83 on D is ·906 of an imperial malt bushel on C.

Set the gauge-point 5·21 on D to 1 on C; and against 44·83 on D is 74·1 lbs. on C.

2. Required the content, at 1 inch deep, of a parallelogram, in tallow and hard soap, the sides being 96 and 48, and the perpendicular upon the former 36 inches.

Ans. 114·1348 lbs. tallow, 127·34 lbs. hard soap.

3. Required the content, at 1 inch deep, in starch and green glass, of a triangular vessel, the base being 118 inches, and the perpendicular upon it 72 inches, and one of the angles 59°. Ans. 122·0688 lbs. starch, 502·1276184 lbs. green glass.

4. Required the content, in imperial gallons and bushels, at 1 inch deep, of a vessel in the form of a trapezoid, the parallel sides 68 and 142, and their perpendicular distance 76 inches.

Ans. 28·77987 gallons, or 3·5975 bushels.

5. Required the content, in pounds of starch, of a trapezium, of which the diagonal is 78, and the perpendiculars upon it 23 and 15½ inches. Ans. 43·1465 lbs.

Here the multiplier is ·0287356, the divisor 34·8, and the gauge-point 5·899. $23 + 15·5 = 38·5$.

Set 34·8 on A to $39 = \frac{1}{2}$ of 78 on B; and against 38·5 on A will be 43·146 lbs. on B.

6. Required the content, in old wine and ale gallons, of a regular octagon, of which the side is 42 inches.

Here the multipliers are ·0209023 and ·0171221, the divisors are 47·8417 and 58·4041, and the gauge-points 6·9168 for wine, and 7·6423 for ale gallons.

$42 \times 42 = 1764$		47·84)1764(36·87 wine gal.
1764	1764	58·404)1764(30·203 ale gal.
·017122	·020902	
30·203208 ale gal.	36·871128 wine gal.	

By the Gauge-Points.

Set 7·64 on D to 1 on C; then against 42 on D is 30·2 ale gallons on C.

Set 6·92 on D to 1 on C; and at 42 on D is 36·9 wine gallons on C.

7. Required the content, in imperial bushels, of a regular hexagon, of which the side is 138 inches.

Ans. 22·3309944 bushels.

III. If the inches in any of the gauge-points be laid on a rule, and this distance be divided into 100 equal parts, the dimensions may be taken with that rule, and then the content may be found without using the multipliers or divisors. Thus, if 7·64 inches be divided into 100 equal parts, the side of the octagon in Ex. 6, measured by this rule, would be 5·496 ale gallons; and this, multiplied by itself, would give 30·206 ale gallons for the content.

1. Required the content, in imperial gallons, of a circle, of which the diameter is 40 inches.

Ans. $1600 \times ·0028326 = 4·52316$ imperial gallons.

By the Gauge-Points.

Set 18·8 on D to 1 on C; then against 40 on D is 4·53 gallons on C.

If 18·8 inches be divided into 100 equal parts, the diameter measured by this scale would be 2·13, which, multiplied by itself, gives 4·5369 imperial gallons for the content.

2. Required the content, in imperial gallons, of a sector of a circle, of which the radius is 42 inches, and the arc 118 inches. Ans. 8·936907 imperial gallons.

3. Required the content, in hard soap, of a trapeze, the diagonal being 32 inches, and the perpendiculars upon it from the angles 18 and 14 inches. Ans. 18·865152 lbs.

4. Required the content of a quadrant, at 1 inch deep, in plate glass, the radius being 16 inches. Ans. 21·906944 lbs.

IV. Some preparation is often necessary before the question can be wrought by the sliding-rule, as in the following examples:

1. Required the content, in imperial gallons, of a segment of a circle, the diameter 50, and the versed sine 10 inches.

Ans. $10·000 \div 50 = ·200$ the tabular versed sine, opposite to which is ·111823 the tabular area; and $·111823 \times 50^2 \times ·0036065 = 1·00823$ imperial gallon.

Set 16·65 on D to ·1118 on C; and at 50 on D is 1·01 imperial gallon on C.

2. Required the content, at 1 inch deep, in tallow, of a triangular vessel, of which the sides are 36, 24, and 20 inches.

Ans. 7·472655.

Here the half sum is 40, and the remainders are 20, 16, and 4. A mean proportional between 20 and 40 is 28·284, and between 16 and 4 is 8. Then,

Set 30·28 on A to 28·284 on B; and against 8 on A is 7·47 lbs. tallow on B.

3. What is the content at 1 inch deep, in imperial gallons, of an ellipse, of which the axes are 72 and 50 inches?

Ans. 10·17936 imperial gallons.

Prob. X. To gauge solids.

When the depth is greater than one inch, set the gauge-point to the depth instead of 1.

1. Required the content, in imperial gallons, of a rectangular prism, of which the length is 81, the breadth 26, and the depth 25 inches.

Ans. $81 \times 26 \times 25 \times ·0036065 = 189·882$ imp. gallons.

By the Gauge-Points.

Set 25 on C to 25 on D; and at 81 on C is 45 on D, a mean proportional between 25 and 81. Then,

Set 16·65 on D to 26 on C; and at 45 on D is 190 imperial gallons on C.

2. Required the content, in imperial gallons and bushels, of an octagonal prism, of which the depth is 80 inches, and each side of the base 63 inches.

Ans. $63^2 \times 80 \times ·0174139 = 5529·262$ imperial gallons = 591·158 bushels.

By the Sliding-Rule.

Set 7·578 on D to 80 on C; and at 63 on D is 5529·3 imperial gallons on C.

Set 21·434 on D to 80 on C; and at 63 on D is 691·16 imperial bushels on C.

3. Required the content, in imperial gallons, of a cylindrical vessel, the depth 40 inches, and the diameter of the base 27 inches. Ans. 82·599 imperial gallons.

4. Required the content, in imperial gallons, of the frustum of a square pyramid, the depth 24 inches, each side of the lower base 26, and of the higher 34 inches.

Ans. $34 + 26 = 60$, and $(60^2 - 34 \times 26) \times 24 \times \cdot 0036065 \div 3 = 78 \cdot 364$ imperial gallons.

First set 26 on C to 26 on D; and at 34 on C is 29·72 on D, the mean proportional between 26 and 34. Then,

Set 28·84 on D to 24 on C; and at 60·0 on D is 104·2 on C.
.... 28·84 .... 24 .... 29·7 .... —25·8 ...

78·4 im. gal.

Note. These, with most other questions, may be worked more easily thus: Find the squares or products of the diameters or sides at the top and bottom, and of the double of those in the middle: the sixth part of the sum of these multiplied by the proper multiplier in Table I. or III. will give the content. (See Prob. XII. Mensuration of Solids.)

Table IV.—Gauge-points to be used when the middle area is taken.

Measures.	For Squares.	For Circles.
Imperial gallons,	40·7878	46·024
Imperial bushels,	115·3653	130·176
Soft soap, pounds,	12·4105	14·004
White soft soap, do.	12·3839	13·974
Hard soap, do.	12·7609	14·399
Tallow, do.	13·4789	15·209
Starch, do.	14·4499	16·305
Green glass, do.	7·1246	8·039
Plate glass, do.	7·4208	8·373
Broad glass, do	7·9433	8·963
Old Wine gallons,	37·2290	42·008
Old Ale gallons,	41·1339	46·415
Old Corn gallons,	40·1597	45·315
Old Malt bushels,	113·5892	128·172
Old Scotch pints,	25·0044	28·214
Old Wheat firlots,	115·2646	130·062
Old Barley firlots,	139·2187	157·091

The gauge-points for squares in the preceding Table are found by taking the square roots of six times the divisors in Table III., and for circles by taking the square roots of six times the divisors in Table I.

By the Sliding-Rule.

Set the gauge-point on D to the length on C; then opposite to the sides or diameters at the ends, and to twice that in the middle on D, will be found three numbers on C; and these three, added together, will give the content.

To work the last question by this rule. $\frac{1}{6}(26^2+34^2+60^2) \times 24 \times \cdot 0036065 = 78\cdot 362$ imperial gallons.

By the Sliding-Rule.

Set 40·79 on D to 24 on C; then at 60 on D is 51·6 on C.
.... 40·79 ....24.................34..........16·6 ...
.... 40·79 ....24.................26.......... 9·65 ...
77·85 imp. gal.

5. Required the content, in imperial gallons, of a frustum of a rectangular pyramid, the depth of the frustum 100 inches, the sides of the upper base 18 and 8 inches, and the sides of the lower base 27 and 12 inches.

Ans. 82·2282 imperial gallons.

6. Required the content, in imperial gallons, of the frustum of a cone, the depth of the frustum 100 inches, and the diameters of the bases 18 and 12 inches.

Ans. 64·58328 imperial gallons.

7. If the axis of a globe be 100 inches, how many imperial gallons will it contain?

In a sphere, the square of twice the middle diameter is three times the square of the axis.

Ans. $\frac{1}{6}(10000 + 30000 + 0) \times 100 \times \cdot 0028326 = 1888.4$ imperial gallons.

Set 46·02 on D to 100 on C; then at 200 on D is 1888·7 imperial gallons on C.

8. Required the content, in imperial gallons, of a bowl or segment of a sphere, the depth 15 inches, the diameter of the base 60 inches, and the middle diameter 45 inches.

Ans. $\frac{1}{6}(60^2+90^2+0) \times 15 \times \cdot 0028326 = 82\cdot 85355$ imperial gallons.

Set 46·02 on D to 15 on C; then at 60 on D is 25·5 on C.
.... 46·02 ....15..................90 ....57·37 ...
82·87 imp. gal.

Or by Prob. XV., Case 2, of Mensuration of Solids.

Set 32·544 on D to 15 on C; then at 15 on D is 3·18 on C.
.... 32·544............15..................30.........12·75 ...
.... 32·544............15..................30.........12·75 ...
.... 32·544............15..................30.........12·75 ...
41·43

And $41·43 \times 2 = 82·86$ imp. gal.

9. Required the content, in imperial bushels, of a hexagonal prism, of which the depth is 96 inches, and each side of the base 18 inches. Ans. 36·47255 imperial bushels.

10. Required the content, in imperial gallons, of a cylindrical vessel, of which the depth is 84 inches, and the diameter of the base 63 inches. Ans. 944·37751 imperial gallons.

11. Required the content, in pounds of hard soap, of a frustum of a pentagonal pyramid, the depth 60 inches, and the sides of the bases 18 and 6 inches. Ans. 593·355672 lbs.

12. Required the content of the frustum of a cone, in imperial gallons, the depth being 50 inches, and the diameters of the bases 24 and 30 inches. Ans. 103·67316 gallons.

PROB. XI. To gauge malt.

RULE. Take the depths at a great number of places, particularly where the malt is deepest, and where it is ebbest. Add all these depths, and divide the sum by the number of them for a mean depth. Find the content at one inch deep, as before, and multiply it by the mean depth.

The barley must remain covered with water in the *cistern* for at least 40 hours; it is then removed into a frame called a *couch*, where it remains for 26 hours; after which it is reckoned a *floor*, and continues to be so till it is ready for the *kiln*.

The malt must be several times gauged during these processes, and the duty charged upon the best gauge, or the largest quantity in bushels of grain, after making the legal deductions. When gauged in the *cistern*, or the *couch*, one-fifth is allowed by law for the swell or increase, and when gauged in the *floor*, before it has been 72 hours out of the cistern, one-third is allowed for the growth; but after 72 hours, one-half is the quantity allowed by law. To obtain, therefore, the net measure when gauged in the *cistern* or *couch*, multiply it by ·8, and, when gauged in the *floor*, multiply by $\frac{2}{3}$ if it has not been out of the cistern 72 hours, otherwise multiply by ·5. When the measure of the dry barley is given, multiply it by 1·2, to find what it should be in the couch, and that again by 1·6, to find what it should be in the floor, after it has been 72 hours out of the cistern.

1. Required the content of a rectangular floor of malt, of which the length is 72 inches, the breadth 48, and the depth, taken at five different places, 4·7, 5·4, 5·6, 4·9, and 4·4 inches.
Ans. The sum of the depths, 25, divided by 5, gives 5 the mean; then $72 \times 48 \times 5 \times \cdot0004508 = 7\cdot7898$ imp. bushels.

By the Sliding-Rule.

Set the length 72 on B to the breadth 48 on MD; then against the depth 5 on A is 7·79 imperial bushels on B.

2. Suppose the length 270, the breadth 56·2, and the mean depth 5·2 inches. Required the quantity of malt.
Ans. $270 \times 56\cdot2 \times 5\cdot2 \times \cdot0004508 = 35\cdot570284$ imp. bush.

3. Let the length be 140, the breadth 72, and the mean depth 18·2 inches. Required the quantity.
Ans. 82·7019648 imperial bushels.

4. Let the length be 1250, the breadth 360, and the mean depth 9 inches. Required the quantity.
Ans. 1825·74 imperial bushels.

5. How many imperial bushels of malt are in an octagonal cistern, the length of the side being 10 feet, and the depth in eight different places 10·2, 9·6, 9·1, 9·8, 10·5, 10·7, 10·3, and 10·4 inches? Ans. 315·795636 imperial bushels.

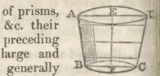
6. There is an oval cistern of malt, of which the diameters are 72 and 48, and the depth 5 inches. Required its content.
Ans. 6·118848 imperial bushels.

7. What should be the couch and floor measure of 13·8 imperial bushels of dry barley?
Ans. $13\cdot8 \times 1\cdot2 = 16\cdot56$ the couch measure.
$16\cdot56 \times 1\cdot6 = 26\cdot496$ the floor measure.

8. Suppose a floor to measure 100·8 imperial bushels, what should have been the couch measure? Ans. 63 bushels.

9. Suppose a couch to measure 56 bushels, what should have been the floor measure, and the quantity of dry barley?
Ans. 89·6 the floor measure, and $46\frac{2}{3}$ bushels dry barley.

Prob. XII. To gauge open vessels.



These vessels being in the form of prisms, cylinders, frustums, cylindroids, &c. their contents may be found by the preceding rules. But as they are often large and fixed vessels, their contents are generally required at every inch, or tenth part of an inch, of depth. These contents must therefore be found and placed in a table, so that, by taking the depth of the liquor, the content may be known at once from the table.

When the vessels are prisms or cylinders, find the content at one inch deep; and this doubled, tripled, &c. will give the contents at two, three, &c. inches. If the dimensions at the top and bottom be unequal, divide the difference of corresponding sides or diameters at the bases by the depth, to get the difference at one inch deep; and this difference, added to the bottom diameter if it be less than that at the top, or subtracted from it if greater, will give the side or diameter at one inch deep; and the same difference, added to the side or diameter at one inch deep, or subtracted from it, will give it at two inches deep, and so on.

Having found the dimensions, find the content of each part; and, by adding them, the contents at all the depths will be found.

Generally the dimensions are found only in the middle of every six inches, and the content, being found from these dimensions for one inch deep, is added to itself six times, to get the contents for each of these six inches of depth.

1. Suppose an elliptical vessel to be 6 inches perpendicular depth, the axes at the top 65 and 60, and those at the bottom 110 and 100 inches, all taken parallel to the horizon, the vessel inclining so that it requires 15 gallons to reach to the upper part of the bottom where the axes were taken.

The difference of the two greater axes is 45, which, divided by 6, gives 7·5 inches, the difference for every inch of depth; and, in the same manner, the difference of the lesser axes for every inch of depth is $6\frac{2}{3}$ inches: consequently, at 1 inch from the bottom, the axes will be 72·5 and 66·7 inches; at 2 inches, 80 and 73·3 inches, and so on. These are placed in the second and third columns of the table; and the particular contents being found and added together regularly, both from the top and the bottom, are placed in the fourth and fifth columns.

In such vessels there is a place marked on the edge of the vessel for the dipping-place; and it is here supposed, that, at the dipping-place, the wet inches are 2, when the 15 gallons are in the vessel to cover the bottom, and also that there is 1 inch dry at the top when the vessel is full.

TABLE.

Dry Inches.	Length.	Breadth.	Content from Top.	Content from Bottom.	Wet Inches.
1	65·0	60	0·00000	136·51854	8
2	72·5	66⅔	12·34542	124·17312	7
3	80·0	73⅓	27·47622	109·04232	6
4	87·5	80	45·67567	90·84287	5
5	95·0	86⅔	67·22704	69·29150	4
6	102·5	93⅓	92·41357	44·10497	3
7	110·0	100	136·51854	15·00000	2

Suppose the wet inches at the dipping-place to be 5; then against 5 wet inches in the column titled *Content from Bottom*, is found 90·84287 imperial gallons for the quantity of liquor in the vessel.

2. The depth of a circular mash-tun is 60, the top-diameter 48, and the bottom-diameter 36 inches, and supposing the content of the *drip or fall* to be 20 imperial gallons. Required the content of each 10 inches from the top, and also the whole content.

Ans. First 10 inches from the top 62·57213, whole content 321·7852 imperial gallons.

3. Suppose the depth of a circular tun to be 80 inches, the top-diameter 50, and the bottom-diameter 30. Required the content of the tun, and also of every 10 inches from the bottom, allowing 10 gallons for the *drip or fall*.

Ans. Whole content 380·00836, content of first 10 inches from the bottom 37·66211 imperial gallons.

Coolers, &c. are very wide and ebb, and their bottoms uneven; therefore the depths must be taken at various parts, and their sum divided by the number of them, to get a mean depth. Tables are constructed for such vessels, exhibiting the content at every tenth part of an inch in depth. They are made and used in the same way as the last table.

It often happens that the depth taken at the dipping-place differs from the mean depth for which the table was calculated. The difference must be marked on the vessel and in the table, with the sign — when the depth at the dipping-place is greater than the mean depth, or with the sign + if it be less; and this difference must be subtracted or added to get the mean depth, before using the table.

Suppose the mean depth to be 4·89, and that at the dipping-place 5 inches; the difference, 0·11, must always be taken from the wet inches to reduce them to mean ones.

NOTE. When the wort is gauged hot, one-tenth part is deducted from the content, to find how much there will be when cold; as it has been found that 10 gallons of hot wort measure only about 9 gallons when the wort is cold.

1. The length of a cooler is 120, the breadth 84, and the depth at 10 equidistant places 4·6, 4·5, 4·7, 4·4, 4·2, 4, 3·9, 3·7, 3·5, and 3 inches. How many gallons of hot wort will it contain, and how many gallons will there be when the wort is cold?

Ans. 115·6380624 gallons hot, and 104·07426 gallons cold wort.

2. Suppose the length to be 280, the breadth 200, and the mean depth 5·1 inches. Required the content in hot, and also in cold wort.

Ans. 808·99056 gallons hot, and 728·0915 gallons cold.

PROB. XIII. To gauge a copper, still, &c.

If the greatest width be at the top, and the least at the bottom, or the contrary, take diameters perpendicular to one another at both ends, and also exactly in the middle, between the top and bottom. (By the bottom is meant the top of the crown in the bottom.) Then work by Prob. XII. of MENSURATION OF SOLIDS: That is,

To 4 times the product of the middle diameters, add the products of those at the top and of those at the bottom. Multiply the sum by the depth from the top of the vessel to the top of the crown: the product, multiplied by ·0004721, will give the imperial gallons in the content of all above the crown. Water must then be measured into the vessel, just to cover the crown; and this measure, added to that found by the rule, will give the whole content.

1. Let the depth to the top of the crown be 36 inches, the diameters at the top 116 and 115·5, at the top of the crown 111 and 110, and in the middle 114 and 113, and the liquor required to cover the crown 16·3 imperial gallons. Required the content.

Ans. $(4 \times 114 \times 113 + 116 \times 115{\cdot}5 + 111 \times 110) \times 36 \times {\cdot}0004721 = 1310{\cdot}9726$, and $1310{\cdot}9726 + 16{\cdot}3$ the content of the crown $= 1327{\cdot}2726$ imperial gallons whole content.

If the broadest part be not at the top or bottom; suppose the vessel to be divided into two or more frustums, so that the broadest part of each frustum be at one end of it, and the least breadth at the other. Find the content of each frustum separately, and add these contents, and the liquor required to cover the crown: the sum will be the whole content.

2. Suppose the depth 36 inches, and the greatest bulge 15 inches from the top; the diameters at the top 80·5 and 80·8, at the bulge 89·0 and 89·5, and in the middle between these 85·5 and 86·0; also, the diameters at the top of the crown 83·0 and 83·5, and half-way between it and the greatest bulge 86·5 and 87·0; the liquor required to cover the crown 18·5 gallons. Required the content.

Ans. 775·36434665 imperial gallons.

3. Let the depth of a still be 42·8 inches, and the height of the greatest bulge from the bottom 20·5; and let the diameters at the top be 21·0, at the bulge 47·8 and 47·3, and half-way between them 45·4 and 46·0; also, the diameters at the bottom 43·5 and 44·0, and half-way between the bottom and the bulge 47·0 inches. Required the content in imperial gallons, supposing 7 gallons to cover the crown.

Ans. 249·31137 imperial gallons.

Stills are generally measured by taking cross diameters at the middle of every six inches, and finding the content of each part as if it were a cylinder; and the top is calculated like a frustum or zone of a sphere.

4. Suppose the top of the still to be 7·3 inches, its greatest diameters 41·5 and 40·8, and its least 21·0; the body of the still 35·5 inches deep, and the cross diameters in the middle of every six inches from the top to be, first, 43·9 and 43·2; second, 47·0 and 46·2; third, 47·8 and 47·3; fourth, 47·6 and 47·4; fifth, 46·5 and 46·5; and in the middle of the undermost $5\frac{1}{2}$ inches, 45·0 and 45·2 inches. Required the content in imperial gallons, supposing 7·5 gallons to cover the crown. Ans. 248·22624492 imperial gallons.

CASK GAUGING.

The easiest way of finding the contents of casks is by the diagonal-rod.

OF THE DIAGONAL-ROD.

This rod is 4 feet long and ·4 of an inch square. It is divided into 4 equal parts by joints. The principal line on it is the diagonal line for imperial gallons, which may be made thus:

It is found by experiment, that a cask containing 144 imperial gallons has a diagonal of 40 inches: therefore 144 is placed at 40 inches; and, since the contents are as the cubes of the diagonals, $144 : 40^3 = 64000 :: 114 : 152000$, the cube root of which is 37, therefore 114 is put at 37; and in the same manner any other number of gallons may be placed upon the rod. A line of inches is also upon the same side of the rod.

Upon another side of the rod is a line marked Seg. St. for finding the ullage of a standing cask.

On a third side are tables for ullaging lying casks, viz. those of half or whole hogsheads, of 84, 108, 110, and 120 old wine gallons. The depth is taken in inches, and the ullage is given in gallons.

The fourth side contains lines for ullaging casks of known dimensions, as a half-anker, a firkin, a barrel, a hogshead, a puncheon, &c., either lying or standing. Put into the bung that end of the rod from which the divisions for the given cask are numbered, until it rests upon the opposite stave; and the division on the rod intersected by the surface of the liquor will be the ullage.

Prob. XIV. To find the content of a cask by the rod.

Put in the end covered with brass at the bung, and extend it to the opposite corner of the head, and mark the gallons and parts at the middle of the bung; then extend it to the other head of the cask, and mark the gallons and parts. Half the sum of these two, if they do not agree, will be the content.

Note. The contents on the rod are made for the most common forms of casks.

1. Suppose a cask to be 21 inches long, the bung-diameter 19, and the head-diameter 16. Required the content in ale gallons.

If the rod be extended from the bung to the opposite corner of the head, it will give 19·3 imperial gallons nearly.

Prob. XV. To find the content of a cask by the pen.

In common casks, the cube of the diagonal divided by $444\frac{4}{9}$ will give the content in imperial gallons. Therefore, to the square of half the length add the square of half the sum of the diameters, to get the square of the diagonal: this multiplied by its square root, and divided by $444\frac{4}{9}$, gives the content.

Half the sum of the diameters in last example is 17·5; therefore $17{\cdot}5^2 + 10{\cdot}5^2 = 416{\cdot}6{\cdot}5$, the square root of which is 20·4, and $416{\cdot}5 \times 20{\cdot}4 \div 444\frac{4}{9} = 19{\cdot}12$ imperial gallons the content.

OF THE VARIETIES OF CASKS.

Casks are commonly divided into four varieties, according to the degree of their curvature.

I. The middle zone of a spheroid, measured by Prob. VIII. Conic Sections; that is, $(2D^2+d^2)\times\frac{1}{3}\,an$.

II. The middle zone of a parabolic spindle, gauged by Prob. XVI. Conic Sections; that is, $(2D^2+d^2)-\frac{2}{5}(D-d)^2 \times \frac{1}{3}an$, which, when reduced, becomes $\frac{8D^2+4Dd+3d^2}{5}\times\frac{1}{3}an$.

III. Two equal frustums of a parabolic conoid, by Prob. X. Conic Sections; that is, $(D^2+d^2)\times\frac{1}{2}an$.

IV. Two equal frustums of a cone, by Prob X. Mensuration of Solids; that is, $\{(D+d)^2 - Dd\}\times\frac{1}{3}an = (D^2+Dd+d^2)\times\frac{1}{3}an$.

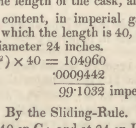
In these formulæ, D denotes the bung-diameter, d the head-diameter, a the length of the cask, and $n = \cdot7854$.

2. Required the content, in imperial gallons, of a cask of the first variety, of which the length is 40, the bung-diameter 32, and the head-diameter 24 inches.

$(2\times32^2+24^2)\times40 = 104960$
$\cdot0009442$
$\overline{99\cdot1032}$ imperial gallons.

By the Sliding-Rule.

Set 32·544 on D to 40 on C; and at 24 on D is 21·75 on C
.... 32·544 .... 40 .... 32 .... 38·67 ...
.... 32·544 .... 40 .... 32 .... 38·67 ...
$\overline{99\cdot09}$ imp. gals.



3. Suppose the cask to be of the second variety, and the dimensions the same as in the last.

$(2\times32^2+24^2-\cdot4\times8^2)\times40 = 103936$
$\cdot0009442$
$\overline{98\cdot1364}$ imperial gallons.

Set 32·544 on D to 40 on C; then at 8 on D is 2·417 on C, which, multiplied by ·4, gives ·9668 of an imperial gallon to be taken from the content found in the last example, and leaves 98·13643 imperial gallons.

4. Let the cask be of the third variety, and the dimensions as before.

Ans. $(32^2+24^2)\times 20\times \cdot0028326 = 90\cdot6432$ imp. gals.

Set 18·79 on D to 20 on C; then at 32 on D is 58·00 on C.
.... 18·79 .... 20 .... 24 .... 32·63 ...

$\overline{90\cdot63}$ imp. gal.

5. Let the cask be of the fourth variety, and the dimensions still the same.

Ans. $(56^2 - 32\times 24)\times 40\times \cdot0009442 = 89\cdot4346$ imperial gallons.

Set 24 on D to 24 on C; and at 32 on C is 27·7 on D, the mean proportional.

Set 32·544 on D to 40 on C; and at 56 on D is 118·4 on C.
.... 32·544 .... 40 .... 27·7 ... — 29·0 ...

$\overline{89\cdot4}$ imp. gal.

6. Let the length be 20, and the diameters 16 and 12 inches. Required the contents in imperial gallons, according to all the varieties.

Ans. First var. 12·3879, second var. 12·267046, third var. 11·3304, fourth var. 11·179328 imperial gallons.

7. Let the length be 40, and the diameters 32 and 26 inches. Required the content, according to all the varieties.

Ans. First var. 102·88, second var. 102·3361728, third var. 96·3084, fourth var. 95·628576 imperial gallons.

8. Let the length be 45 inches, and the diameters 36 and 30 inches. Required the content, according to all the varieties, in imperial gallons.

Ans. First var. 148·371588, second var. 147·759746, third var. 139·958766, fourth var. 139·193964 imperial gallons.

9. Let the length be 48 inches, and the diameters 40 and 32 inches. Required the content, according to all the varieties.

Ans. First var. 191·4384384, second var. 190·2782054, third var. 178·3858176, fourth var. 176·9355264 imperial gallons.

Note. The second variety comes nearer to the form of common casks than any of the others, but it does not entirely agree with them.

Prob. XVI. To gauge a cask by reducing it to a cylinder.

Rule. Divide the head by the bung diameter, and find the quotient in the column titled *Quot.* in the following table. In the column answering to the variety of the cask, on the same line with the quotient, will be found a number, which, multiplied by the difference between the bung and head diameters, and the product added to the head-diameter, will give the mean diameter, or that of a cylinder equal to the cask. Then multiply the square of the mean diameter by the length of the cask, and by ·0028326, for the content in imperial gallons.

By the Sliding-Rule.

Find the difference between the head and bung diameters on the edge of the inside of the slider, and against it, in the proper line, is the number to be added to the head, to get the diameter of the cylinder, called the mean diameter.

Then set the gauge-point on D to the length on C, and opposite the mean diameter on D is the content in imperial gallons on C.

Quot.	1st Var.	2d Var.	3d Var.	4th Var.	Quot.	1st Var.	2d Var.	3d Var.	4th Var.
·50	·732	·693	·581	·527	·76	·695	·678	·534	·511
·51	·730	·692	·579	·527	·77	·694	·677	·532	·510
·52	·729	·692	·577	·526	·78	·693	·677	·530	·510
·53	·727	·691	·575	·526	·79	·691	·676	·529	·510
·54	·726	·690	·573	·525	·80	·690	·676	·527	·509
·55	·724	·690	·571	·524	·81	·689	·675	·526	·508
·56	·723	·689	·569	·523	·82	·688	·675	·524	·508
·57	·721	·689	·567	·523	·83	·686	·674	·522	·508
·58	·720	·688	·565	·522	·84	·685	·674	·521	·507
·59	·719	·688	·563	·521	·85	·684	·673	·520	·506
·60	·717	·687	·562	·521	·86	·683	·673	·519	·506
·61	·716	·686	·559	·520	·87	·682	·672	·517	·505
·62	·714	·686	·558	·519	·88	·680	·671	·516	·505
·63	·713	·685	·556	·519	·89	·679	·671	·515	·504
·64	·712	·685	·554	·518	·90	·678	·671	·513	·504
·65	·710	·684	·552	·517	·91	·677	·670	·511	·503
·66	·709	·684	·551	·517	·92	·675	·670	·510	·503
·67	·708	·683	·549	·516	·93	·674	·669	·509	·503
·68	·706	·682	·547	·516	·94	·673	·668	·507	·502
·69	·705	·682	·545	·516	·95	·672	·668	·506	·501
·70	·703	·681	·543	·515	·96	·670	·667	·505	·500
·71	·702	·681	·541	·514	·97	·670	·667	·503	·500
·72	·701	·680	·540	·513	·98	·667	·666	·501	·500
·73	·699	·680	·539	·513	·99	·666	·666	·500	·500
·74	·698	·679	·537	·512	1·00	—	—	—	—
·75	·697	·678	·535	·512					

CONSTRUCTION OF THE PRECEDING TABLE.—It follows from the formulæ, page 321, that, when the bung-diameter is = 1, the contents of the four varieties of casks will be an multiplied by $\frac{2+d^2}{3}$; $\frac{8+4d+3d^2}{15}$; $\frac{1+d^2}{2}$; and $\frac{1+d+d^2}{3}$ respectively; but the contents must also be equal to an multiplied by the squares of the mean diameter, and consequently these mean diameters are respectively $\sqrt{\frac{2+d^2}{3}}$; $\sqrt{\frac{8+4d+3d^2}{15}}$; $\sqrt{\frac{1+d^2}{2}}$; and $\sqrt{\frac{1+d+d^2}{3}}$. If in these expressions we substitute the value of d, we will obtain the mean diameters; thus, if $d = \cdot 60$, then the mean diameters for the four varieties are ·8869; ·8748; ·8246; and ·8082; and so on for any other value of d. Now if the difference between the bung and head diameters be multiplied by x, and the product added to the head-diameter, we have $\cdot 40x + \cdot 60 = \cdot 8869$; $\cdot 40x + \cdot 60 = \cdot 8748$; $\cdot 40x + \cdot 60 = \cdot 8246$; and $\cdot 40x + \cdot 60 = \cdot 8082$; whence $x = \cdot 717$; ·687; ·562; and ·521, the multipliers opposite ·60 in the Table; and, in like manner, by taking $d = \cdot 50, \cdot 51, \cdot 52$, &c, the multipliers opposite these numbers are found.

1. Suppose the length 40, and the diameters 32 and 26 inches. Required the mean diameter and the content in imperial gallons, according to all the varieties.

$26 \div 32 = \cdot 81$, opposite to which, in the table, are ·689, ·675, ·526, ·508. Then,

$6 \times \cdot 689 + 26 = 30\cdot 134$ mean diameter, and $30\cdot 134^2 \times 40 \times \cdot 0028326 = 102\cdot 8866$ imperial gallons in the first variety.

$6 \times \cdot 675 + 26 = 30\cdot 05$ mean diameter, and $30\cdot 05^2 \times 40 \times \cdot 0028326 = 102\cdot 3138$ imperial gallons in the second variety.

$6 \times \cdot 526 + 26 = 29\cdot 156$ mean diameter, and $29\cdot 156^2 \times 40 \times \cdot 0028326 = 96\cdot 3166$ imperial gallons in the third variety.

$6 \times \cdot 508 + 26 = 29\cdot 048$ mean diameter, and $29\cdot 048^2 \times 40 \times \cdot 0028326 = 95\cdot 6044$ imperial gallons in the fourth variety.

Set 18·79 on D to 40 on C; and at 30·188 on D is 103·26 imperial gallons on C, first variety.

Set 18·79 on D to 40 on C; and at 30·05 on D is 102·31 imperial gallons on C, second variety.

Set 18·79 on D to 40 on C; and at 29·156 on D is 96·32 imperial gallons on C, third variety.

Set 18·79 on D to 40 on C; and at 29·048 on D is 95·60 imperial gallons on C, fourth variety.

2. Suppose the length 60, and the diameters 40 and 32 inches. Required the content, according to all the varieties.

Ans. First var. 239·25563, second var. 237·829367, third var. 222·91406, fourth var. 221·14491 imperial gallons.

3. Suppose the length 50, and the diameters 36 and 30 inches. Required the content, according to all the varieties.

Ans. First var. 164·84336, second var. 164·1483, third var. 155·47142, fourth var. 154·68408 imperial gallons.

4. Suppose the length 56, and the diameters 40 and 36 inches. Required the content, according to all the varieties.

Ans. First var. 237·7193292, second var. 237·37557353, third var. 229·6826837, fourth var. 229·248296 imp. gals.

Prob. XVII. To gauge a cask by the middle diameter.

Add the squares of the head, of the bung, and of twice the middle diameter: the sum, multiplied by the length, and by ·0004721, gives the content in imperial gallons.*

Note. This is the most accurate method of finding the contents of casks.

1. Let the length of the cask be 40, and the diameters 32 at the bung, 26 at the head, and 30·4 inches in the middle.

Ans. $(32^2 + 26^2 + 60{\cdot}8^2) \times 40 \times {\cdot}0004721 = 101{\cdot}91015$ imperial gallons.

Set 46·024 on D to 40 on C; and at 60·8 on D is 69·81 on C.
.... 46·024......... 40 .... 32·0......... 19·34 ...
.... 46·024......... 40 .... 26·0......... 12·76 ...
101·91 imp. gal.

2. Let the length be 42, the bung diameter 34, the head 27, and the middle diameter 32 inches. Required the content in imperial gallons. Ans. 118·592464 imperial gallons.

3. Let the length be 44, the head 30, the bung 36, and the middle diameter 33 inches. Required the content.

Ans. 136·1007648 imperial gallons.

4. Let the length be 50, the middle 36, the head 34, and the bung diameter 40 inches. Required the content.

Ans. 187·4237 imperial gallons.

Prob. XVIII. To find the content of a cask without the middle diameter.

Rule. From 12 times the head subtract 7 times the bung diameter, and multiply the remainder by twice the bung diameter, and subtract the product from the square of 5 times the sum of these diameters. Multiply the remainder by the length, and by ·00003147: the product will give the content in imperial gallons.

* This is the same rule as that given in Mensuration of Solids, Prob. XII.; the number ·0004721, being one-sixth of the multiplier for imperial gallons, Table I., page 306.

1. Let the length of the cask be 40 inches, the bung diameter 32, and the head diameter 24 inches. Required the content.

Ans. $(12 \times 24 - 7 \times 32) \times 64 = (288 - 224) \times 64 = 64 \times 64 = 4096$ and $5 \times (32 + 24) = 280$, and $280^2 - 4096 = 74304$ and $74304 \times 40 \times \cdot 00003147 = 93\cdot 534$ imp. gallons.

2. Suppose the length to be 41 inches, the bung diameter 32·2, and the head diameter 26·3 inches. Required the content. Ans. 102·8956391 imperial gallons.

3. Let the length be 45, the bung 34, and the head diameter 28 inches. Required the content.

Ans. 126·229946 imperial gallons.

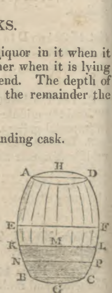
4. Let the length be 48, the bung 36, and the head diameter 30 inches. Required the content.

Ans. 152·753869 imperial gallons.

OF ULLAGING CASKS.

The Ullage of a cask is the quantity of liquor in it when it is not full. The dimensions are taken either when it is lying on its side, or when it is standing on its end. The depth of the liquor is called the Wet Inches, and the remainder the Dry Inches.

Prob. XIX. To find the ullage of a standing cask.



Rule I. Add together the squares of the diameter at the top of the liquor, of the diameter at the nearest end, and of twice the diameter half-way between these two, and multiply the sum by the length or distance from the surface of the liquor to the nearest end, and by ·0004721: the product will be the content of the less part of the cask in imperial gallons, whether full or empty.*

* To obtain the diameters at the surface of the liquor, and that in the middle between it and the nearest end, suspend a plummet from a rod laid over the centre of the head, so as just to touch the bulge of the cask; then measure the distance of the cord from the side of the cask at the surface of the liquor, and also in the middle point between it and the nearest end;—the double of these distances, including also the thickness of the staves, taken from the bung diameter, will leave the diameters required.

1. Suppose the wet inches to be 10, and the diameters 24, 27, and 29 inches. Required the ullage.

Ans. $(24^2+29^2+54^2)\times 10\times \cdot 0004721 = 20\cdot 4561$ imperial gallons.

2. Suppose the length 40, the wet inches 30, the diameters 30, 24, and 28 inches, bung diameter 32, and the middle $30\frac{1}{2}$. Required the ullage.

$$(24^2+32^2+61^2)\times 40\times \cdot 0004721 = 100\cdot 482 \text{ im. gal. cask.}$$
$$(24^2+30^2+56^2)\times 10\times \cdot 0004721 = \underline{21\cdot 773} \ldots\ldots \text{ empty.}$$
$$78\cdot 709 \ldots\ldots \text{ ullage.}$$

3. Suppose the length 28, the wet inches 12, the diameters 20, 22, and 24 inches, bung diameter 25, and the middle 23 inches. Required the ullage. Ans. 16·49706 imperial gallons.

4. Suppose the length 50, the wet inches 30, the diameters 26, 30, and 32 inches, bung diameter 36 inches, and the middle 32 inches. Required the ullage.

Ans. 93·19254 imperial gallons.

Rule II. Multiply the square of the dry or wet inches (the less of the two) by the difference between the head and bung diameters, and divide the product by the square of the length: the quotient, subtracted from the bung diameter, will give the mean diameter.

Multiply the square of the mean diameter by the wet or dry inches (the less of the two), and then by ·0028326 to get the content of the filled or empty part (the less of the two).*

5. Suppose the length 40, the bung 32, the head diameter 26, and the wet inches 10. Required the ullage.

Ans. $30^2\times 6 \div 1600 = 3\frac{3}{8}$, and $32 - 3\frac{3}{8} = 28\frac{5}{8}$, and $(28\frac{5}{8})^2\times 10\times \cdot 0028326 = 23\cdot 21006$ imperial gallons.

6. Suppose the length 60, the bung 49, the head diameter 40, and the wet inches 25. Required the ullage.

Ans. 149·43763 imperial gallons.

7. Suppose the length 48, the bung 40, the head diameter 34, and the wet inches 18. Required the ullage.

Ans. 72·298934 imperial gallons.

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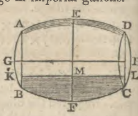
* This rule is only an approximation, and is founded on the supposition that, for a small part of a cask, the diameter in the middle may be considered as a mean diameter; and this supposition is very nearly true for all the varieties of casks. Now if L = the length of a cask of the second variety, B = the bung diameter, H = the head diameter, I = the wet or dry inches (the less of the two), and M = the diameter in the middle, then from the property of the parabola $L^2 : (L-I)^2 :: B-H : \frac{B-H}{L^2}\times(L-I)^2$, and $B-\frac{B-H}{L^2}\times(L-I)^2 = M$, which is the first part of the rule, and the second part is so evident as to require no demonstration.

PROB. XX. To ullage a lying cask.

RULE I. Divide the wet or dry inches (the less of the two) by the bung diameter, and find the quotient in the column of versed sines in the table of segments. Take out its corresponding segment, and multiply it by the content of the cask, and by $1\frac{1}{4}$: the product is the ullage in imperial gallons.*



1. Suppose the content 92 imperial gallons, the bung diameter 32, and the wet inches 8.

32)8·00(·25 the versed sine, of which the segment is ·153546, and $\cdot 153546 \times 92 \times 1\frac{1}{4} = 17\cdot 65779$ imperial gallons the ullage.

2. Let a lying cask be 40 inches long, and the diameters 32, 26, and $30\frac{1}{2}$. Required the ullage, when the dry inches are 12.

Ans. 67·946533 imperial gallons.

3. Let a lying cask be 46 inches long, and the diameters 34, 30, and 28. Required the ullage, when the wet inches are 8·5 and 23 inches respectively.

Ans. 23·09139 and 87·230138 imperial gallons.

4. Let a lying cask be 56 inches long, and the diameters 40, 34, and 36. Required the ullage, when the dry inches are 30, 18, and 12 respectively.

Ans. 40·289423, 119·970885, and 157·9166 imp. gals.

RULE II. Find a mean diameter according to the variety of the cask; from the wet inches subtract half the difference between the bung and mean diameter, divide the remainder by the mean diameter, and find the quotient in the column of versed sines in the table of segments. Take out its corresponding segment, and multiply it by the square of the mean diameter, by the length of the cask, and by ·0036065: the product will be the ullage in imperial gallons.

1. Let the length of a lying cask of the first variety be 40 inches, the bung diameter 30, the head diameter 24, and the wet inches 12. Required the ullage.

$24 \div 30 = \cdot 80$, opposite to which in the table, page 323, is ·690, and $\cdot 690 \times 6 + 24 = 28\cdot 14$ mean diameter; then $(12 - \cdot 93) \div 28\cdot 14 = \cdot 393\frac{183}{469}$ versed sine, the corresponding segment of which is ·286881, and $\cdot 286881 \times 28\cdot 14^2 \times 40 \times \cdot 0036065 = 32\cdot 77147$ imperial gallons the ullage.

* This rule is a near approximation to the true ullage, and is founded on the supposition, that the whole of the bung-circle is to the segment of it cut off by the surface of the liquor as the whole content of the cask is to the ullage.

2. Let the length of a lying cask of the first variety be 48 inches, the bung diameter 32, the head diameter 24, and the wet inches 14. Required the ullage.

Ans. 49·26027 imperial gallons.

3. Let the length of a lying cask of the second variety be 38 inches, the bung diameter 36, the head diameter 32, and the wet inches 18. Required the ullage.

Ans. 64·7422344 imperial gallons.

4. Let the length of a lying cask of the third variety be 50 inches, the bung diameter 45, the head diameter 36, and the wet inches 15. Required the ullage.

Ans. 63·7410478 imperial gallons.

Prob. XXI. To ullage a cask by the sliding-rule.

Rule. First find the whole content of the cask. Next set the length or bung diameter on the slider to 100 on the rule, and against the wet or dry inches on the slider, is a number upon Seg. St. or upon Seg. Ly. to be reserved. Then set 100 on B to this reserved number on A; and opposite to the content on B will be found the ullage on A.

1. Suppose the length of a standing cask 40 inches, the wet inches 10, and the content 92 imperial gallons. Required the ullage.

Set 40 on the slider to 100 on the rule; and at 10 on the slider is 23 on Seg. St. to be reserved.

Set 100 on B to 23 on A; and at 92 on B is 21·2 imperial gallons on A.

2. Let the bung diameter of a lying cask be 32 inches, the wet inches 8, and the content 92 gallons. Required the quantity of liquor in it. Ans. 16·4 gallons.

3. Let the length of a standing cask be 20 inches, its content 11·5 gallons, and the wet inches 5. Required the ullage.

Ans. 2·65 gallons.

4. Let the diameter of a lying cask be 34 inches, the dry inches 25, and the content 138 gallons. Required the ullage.

Ans. 26·8 gallons.

SPECIFIC GRAVITY.

THE weight of a cubic foot of a body, in proportion to that of a cubic foot of water, is called its Specific Gravity.

A cubic foot of water, at the temperature of 40° of Fahrenheit's thermometer, weighs 1000 ounces avoirdupois; and therefore the following table of specific gravities expresses in ounces the weight of a cubic foot of these bodies.

TABLE OF SPECIFIC GRAVITIES.

SOLIDS.

Platina, from 16000 to	23000	Mean of the globe, about	5210
Pure gold, hammered,	19326	Loadstone . . .	4930
Guinea of George III.	17629	Spar, heavy, . .	4430
Tungsten, . . .	17600	Jargon of Ceylon, .	4416
Mercury at 32° Fahr. .	13598	Ruby, oriental, . .	4283
Lead, . . .	11352	Garnet, precious, .	4230
Palladium, . . .	11800	. . . . common, .	3576
Rodium, . . .	11000	Topaz, . from 3536 to	4061
Pure silver, . .	10744	Sapphire, oriental, .	3994
Shilling of George III.	10534	Diamond, from 3523 to	3550
Bismuth, molten, . .	9823	Beryl, oriental, . .	3549
Copper of Japan, . .	9000	English flint glass, .	3329
. . . . . wire-drawn, .	8878	Tourmaline, . .	3155
. . . . . red, molten, .	8788	Hornblende, . .	3000
Cadmium, . . .	8694	Asbestus, . . .	2996
Molybdena, . .	8611	Limestone, . .	2950
Brass, wire-drawn, .	8544	Basalt, . . .	2860
. . . . common, . .	7824	Marble, Parian, . .	2837
Arsenic, . . .	8306	. . . . . green Campanian,	2742
Nickel, molten, . .	8279	. . . . Egyptian,	2668
. . . . forged, . .	8666	Chalk, British, . .	2784
Uranium, . . .	8109	Emerald of Peru, .	2775
Meteoric iron, hammered,	7965	Jasper, . . .	2710
Steel, ...	7833	Glass, white, . .	2892
Cobalt, molten, . .	7812	. . . . bottle, . .	2733
Bar iron, . . .	7788	. . . . green, . .	2642
Cast iron, Carron, .	7248	Pearl, oriental, . .	2684
Wootz, hammered, .	7787	Coral, . . .	2680
Pewter, . . .	7471	Slate, ...	2670
Tin, hardened, . .	7299	Granite, Cornish, .	2662
. . . pure Cornish, .	7291	. . . . . Aberdeen, .	2625
Zinc, molten, . .	7191	Rock crystal, . .	2653
Wolfram, . . .	7119	Quartz, . . .	2640
Manganese, . .	6900	Pebble, English, .	2619
Antimony, . . .	6702	Felspar, . . .	2564
Tellurium, . . .	6115	Stone, common, .	2500
Chromium, . .	5900	Porcelain, China, .	2385

Porcelain, Limoges,	2341	Sodium,	972
Obsidian,	2348	Oak, heart of,	950
Gypsum,	2280	Butter,	942
Clay,	2160	Ice,	930
Opal,	2114	Gunpowder, shaken,	922
Sulphur, native,	2033	Pumice-stone,	915
Brick,	2000	Logwood,	913
Ivory,	1917	Living men,	891
Nitre,	1900	Potassium,	865
Alabaster,	1874	Beech,	852
Gunpowder, solid,	1745	Ash,	845
Alum,	1714	Apple-tree,	793
Phosphorus,	1714	Maple,	755
Bone, dry,	1660	Citron,	726
Sand,	1500	Orange-tree,	705
Gum Arabic,	1452	Walnut,	681
Opium,	1337	Pear-tree,	661
Ebony, American,	1331	Hazel,	609
Lignumvitæ,	1327	Linden-tree,	604
Coal,	1250	Elm,	600
Pitch,	1150	Cypress,	598
Rosin,	1100	Cedar, American,	561
Amber,	1078	Fir, male,	550
Mahogany,	1063	... female,	498
Brazil-wood, red,	1031	Poplar,	383
Boxwood,	1030	Cork,	240

LIQUIDS.

Sulphuric acid,	1848	Wine of Bordeaux,	994
Boracic acid,	1830	Wine of Burgundy,	991
Nitric acid, or Aquafortis,	1500	.... red port,	990
Nitrous acid,	1452	Castor-oil,	970
Honey,	1450	Linseed-oil,	940
Water of the Dead Sea,	1240	Proof-spirit,	935
Aqua regia,	1234	Whale-oil,	923
Muriatic acid,	1170	Moselle wine,	916
Strong ale, from 1020 to	1050	Olive-oil,	915
Human blood,	1045	Muriatic ether,	874
Milk,	1030	Oil of turpentine,	870
Sea water,	1026	Brandy,	837
Tar,	1015	Alcohol, absolute,	792
Distilled water,	1000	Sulphuric ether,	739
White Champagne,	997	Air at earth's surface,	1·222

GASES.

Atmospheric air,	1·000	Muriatic acid,	1·280
Hydriodic acid,	4·300	Oxygen,	1·111
Fluosilicic acid,	3·611	Nitrous,	1·042
Chlorine,	2·500	Olefiant,	0·972
Sulphurous acid,	2·222	Nitrogen,	0·972
Cyanogen,	1·805	Ammonia,	0·590
Carbonic acid,	1·527	Hydrogen,	0·069

PROB. I. To find the magnitude of a body from its weight.

RULE. Divide the weight of the body by its specific gravity, both being in ounces: the quotient is the content in cubic feet.

1. How many cubic inches are in 1 lb. of gunpowder?

Ans. $1728 \times 16 \div 922 = 30$ inches nearly.

2. What is the content of a block of Parian marble weighing 5 cwt.? Ans. 3·158 cubic feet.

3. What is the content of a ton weight of mahogany?

Ans. 33·716 cubic feet.

4. What is the content of a block of Cornish granite which weighs 4 tons? Ans. 53·85 cubic feet.

5. What is the content of a cast iron ball which weighs 100 lb.? Ans. 381·457 cubic inches.

PROB. II. To find the weight of a body from its magnitude.

RULE. Multiply the content in feet by the specific gravity: the product is the weight in ounces.

1. What is the weight of a stone of green Campanian marble 63 feet long, and its breadth and thickness each 12 feet?

Ans. $63 \times 12 \times 12 \times 2742 = 24875424$ oz. $= 694\frac{11}{160}$ tons.

2. What is the weight of a log of beech 10 feet long, 3 broad, and $2\frac{1}{2}$ feet thick? Ans. $3993\frac{3}{4}$ lb.

3. What is the weight of a cast iron ball 2 inches in diameter? Ans. 13·177 ounces.

4. What is the weight of a log of mahogany 40 feet long, 3 broad, and $2\frac{1}{2}$ thick? Ans. 8·898 tons.

5. What is the weight of a leaden ball 6 inches in diameter?

Ans. 185·747 ounces.

PROB. III. To find the specific gravity of a body.

CASE I. When the body is heavier than water.

RULE. Weigh the body both in air and in water, and, annexing three ciphers to the weight in air, divide by the difference of the weights, to get the specific gravity.

1. Suppose a piece of Aberdeenshire granite to weigh $10\frac{1}{2}$ lb. in air, and $6\frac{1}{2}$ lb. in water. What is its specific gravity?

Ans. $10500 \div 4 = 2625$ ounces the specific gravity.

2. A piece of copper weighs 36 oz. in air, and 32 in water. What is its specific gravity? Ans. 9000.

3. Suppose a piece of gold weighs 40 lb. in air, and 37·93 lb. in water. What is its specific gravity? Ans. 19323·67.

4. Suppose a piece of platina weighs 10 lb. in air, and 9·5 lb. in water. What is its specific gravity? Ans. 20000.

CASE II. When the body is lighter than water.

RULE. Having weighed the light body in air, and a body heavier than water both in air and in water, fasten them together with a slender tie, then weigh the compound in water, and subtract it from the weight of the heavy body in water; to the remainder add the weight of the light body in air, and by the sum divide the weight of the light body in air with three ciphers annexed; the quotient is the specific gravity of the light body.

1. A piece of copper weighs 18 lbs. in air, and 16 lbs. in water; a piece of elm which weighs 15 lbs. in air is fixed to the copper; and the compound weighs 6 lbs. in water. What is the specific gravity of the elm?

Ans. $15000 \div (16 - 6 + 15) = 600$ the specific gravity of the elm.

2. A piece of copper which weighs 27 ounces in air, and 24 in water, is attached to a piece of cork which weighs 6 ounces in air, and the compound weighs 5 ounces in water. What is the specific gravity of the cork? Ans. 240.

3. A piece of lead weighs 60 lbs. in air, and 55 lbs. in water; a piece of poplar which weighs 30 lbs. in air is fixed to the lead; and the compound weighs 7 lbs. in water. What is the specific gravity of the poplar? Ans. $384\frac{8}{13}$.

4. A piece of steel weighs 140 lbs. in air, and 122 lbs. in water; a piece of fir which weighs 30 lbs. in air is fixed to the steel; and the compound weighs $97\frac{1}{2}$ lbs. in water. What is the specific gravity of the fir? Ans. $550\frac{50}{109}$.

PROB. IV. Given the specific gravity and the weight of a mass composed of two ingredients, and also the specific gravity of each ingredient; to find the quantity of each of them.

RULE. As the specific gravity of the mass, multiplied by the difference between those of the ingredients, is to the specific gravity of the heaviest ingredient, multiplied by the difference between those of the mass and the other ingredient, so is the whole weight to the weight of the highest ingredient; and that of the other may be found in the same way.

1. A composition of 112 lbs. is made of copper of Japan and tin. Required the quantity of each ingredient, the specific gravity of the mass being 8784.

Ans. $(9000 - 7291) \times 8784 : (8784 - 7291) \times 9000 :: 112 : 100{\cdot}25$ lbs. of copper.

2. A mixture of gold and silver weighed 170 lbs. and its specific gravity was 15630. Required the quantity of each metal in it. Ans. 119·673 lbs. gold, 50·327 lbs. silver.

3. A composition of 100 lbs. is made of platina and steel, and its specific gravity is 15000. Required the quantity of each ingredient. Ans. 78·54 lbs. platina, 21·64 lbs. steel.

4. A composition of silver and steel weighs 1000 lbs. and its specific gravity is 8000. Required the quantity of each ingredient. Ans. 77·046 lbs. silver, 922·954 lbs. steel.

TO FIND THE TONNAGE OF A SHIP.

The length is taken in a straight line along the rabbet of the keel, from the back of the main sternpost to a perpendicular from the fore part of the main stem, under the bowsprit, from which subtract $\frac{3}{5}$ of the breadth; the remainder is the length for tonnage. The breadth is taken at the broadest part of the ship, from the outside to the outside of the plank.

Rule. Multiply the square of the breadth by the length, and divide the product by 188; the quotient will be the tonnage.

1. Required the tonnage of a ship, of which the length is 75 feet, and the breadth 26 feet.

Ans. $26 \times 26 \times 75 \div 188 = 269\frac{32}{47}$ tons.

2. Required the tonnage of a ship, of which the length is 96 feet, and the breadth 33 feet. Ans. $556\frac{4}{47}$ tons.

3. Required the tonnage of a ship, of which the length is 100 feet, and the breadth 40 feet. Ans. $851\frac{3}{47}$ tons.

4. Required the tonnage of a ship, of which the length is 150 feet, and the breadth 60 feet. Ans. $2872\frac{16}{47}$ tons.

Note. This rule is very erroneous, and no other general rule can be given that is perfectly accurate. The best way is to find the quantity of water displaced by the ship when she is loaded; but as this must be done by means of ordinates, the operation is laborious. It is easier to load her with ballast, weighing the load as it is put on board.

The following rule is a near approximation.

1*st*, *For Ships of War*. Take the length of the gun-deck, from the rabbet of the stem to that of the sternpost; subtract $\frac{1}{24}$ of it, and the remainder is the length. Take the extreme breadth from outside to outside of the plank, add it to the length of the plank, and $\frac{1}{23}$ of the sum is the depth. Set up this height from the limber-strake, and at that height take a breadth from outside to outside of the plank, where the extreme breadth was taken, and take another breadth in the middle, between this and the limber-strake; add the extreme breadth and these two breadths together, and take $\frac{1}{3}$ of the sum for e breadth. Multiply the length, breadth, and thickness together, and divide the product by 49; the quotient is the urden in tons.

2d, For Ships of Burden. Take the length of the lower eck, from the rabbet of the stem to that of the sternpost, and om it subtract $\frac{1}{32}$ of it, for the length. Take the extreme eadth from outside to outside of the plank, add it to the ngth of the lower deck, and take $\frac{3}{55}$ of the sum for the pth. Set up this depth from the limber-strake, where the treme breadth was taken, and at this height take a breadth om outside to outside of the plank, take another breadth at of this height, and a third at $\frac{1}{3}$ of the height; add these ree breadths to the extreme breadth, and $\frac{1}{4}$ of the sum is e mean breadth. Multiply the length, breadth, and depth gether, and divide the product by $36\frac{2}{3}$ for the tonnage.

The following rules for ascertaining the tonnage of vessels e established by an Act of Parliament passed on September 9, 335, and ordered to take effect on the 1st January 1836:—

Divide the length of the upper deck between the after part the stem and the fore part of the stern-post into six equal rts, and take the following dimensions:—

Depths: At the foremost, the middle, and the aftermost those points of division, measure in feet and decimal parts a foot the depths from the under side of the upper deck the ceiling at the limber strake. In the case of a break in e upper deck, the depths are to be measured from a line retched in continuation of the deck.

Breadths: Divide each of those three depths into five equal rts, and measure the inside breadths at the following ints: *videlicet,* at one-fifth and at four-fifths from the upr deck of the foremost and aftermost depths; and at twoths and four-fifths from the upper deck of the midship pth.

Length: At half the midship depth, measure the length the vessel from the after part of the stem to the fore rt of the stern-post; then

To find the Tonnage: To twice the midship depth add e foremost and the aftermost depths for the sum of the pths; add together the upper and lower breadths at the remost division, three times the upper breadth, and the wer breadth at the midship division, and the upper and ice the lower breadth at the aftermost division for the sum the breadths; then multiply the sum of the depths by the m of the breadths, and this product by the length, and dile the final product by 3500; the quotient will give the umber of tons for register.

NOTE 1. If the vessel have a poop, or half-deck, or a break in the upper deck, measure the inside mean length, breadth, and height of such part thereof as may be included within the bulkhead, and divide the product of the three measurements by 92·4, the quotient will be the number of tons to be added to the result found by the rule.

NOTE 2. In ascertaining the tonnage of steam-vessels, the contents of the engine-room is found thus: measure the inside length of the engine-room in feet and decimal parts of a foot, from the foremost to the aftermost bulkhead, then multiply this length by the depth of the ship or vessel at the midship division, and the product by the inside breadth at the same division at two-fifths of the depth from the deck, and divide the last product by 92·4. The quotient is the tonnage due to the cubical contents of the engine-room, which must be deducted from the tonnage found by the rule for the register tonnage of the steam-vessel.

NOTE 3. In order to ascertain the tonnage of vessels when laden, measure the length on the upper deck between the after part of the stem and the fore part of the stern-post, take the inside breadth on the under side of the upper deck at the middle point of the length, and take the depth from the under side of the upper deck down the pump-well to the skin, then multiply these three dimensions together, and divide the product by 130, the quotient will be the amount of the register tonnage.

NOTE 4. In open vessels, the depths are to be measured from the upper edge of the upper strake.

TO FIND THE WEIGHT OF CATTLE.

TAKE the girt close behind the shoulder, and the length from the fore part of the shoulder-blade along the back to the bone at the tail, which is in a vertical line with the buttock, both in feet. Multiply the square of the girt by 5 times the length, and divide the product by 21; the quotient is the weight, nearly, of the four quarters, in imperial stones of 14 lbs. avoirdupois.*

NOTE. The four quarters in very fat cattle will be about $\frac{1}{20}$ more, and in very lean cattle $\frac{1}{20}$ less than the weight obtained by the rule. The four quarters are very little more than half the weight of the living animal; the skin weighs about the 18th part, and the tallow about the 12th part of the whole.

* It has been found by experiment, that the weight of a bullock divided by the product of the square of the girt behind the shoulder-blade into the length from the shoulder-blade to the buttock is $= 3\frac{1}{3}$ lbs. avoirdupois $= \frac{5}{21}$ of an imperial stone. Hence the weight of all cattle whose specific gravity is nearly alike will be obtained by the rule.

1. What is the weight of the four quarters of an ox, of which the girt is 6 feet 6 inches, and the length 5 feet 10 inches?

Ans. $6{\cdot}5^2 \times 5$ ft. 10 in. $\times 5 \div 21 = 58$ stones 10·8888 lbs.

2. What is the weight of the quarters of a sheep, of which the girt is 3 feet 1 inch, and the length 2 feet 8 inches?

Ans. 6·036 stones.

3. What is the weight of a hog which is 4 feet 6 inches in girt, and 3 feet 4 inches in length? Ans. $16\frac{1}{14}$ stones.

4. What is the weight and value of an ox measuring $6\frac{1}{2}$ feet in girt, and $5\frac{3}{4}$ feet in length, at 11s. 6d. a-stone, sinking offals? Ans. 57·8425 stones, value £33·2594.

5. What was the value of the four quarters of the Dunearn ox, which measured 9 feet $3\frac{1}{2}$ inches in girt, and 5 feet $7\frac{1}{2}$ inches in length, at 10s. 6d. a-stone? Ans. £60, 14s. $0\frac{3}{4}$d.

6. What is the weight of the four quarters of a calf measuring 3 feet in girt by $2\frac{1}{4}$ feet in length? Ans. $4\frac{8}{6}\frac{?}{?}$ stones.

TO FIND THE WEIGHT OF A STACK OF HAY.

To the height from the ground to the eaves add half the height from the eaves to the top; then multiply the sum, and the length and breadth of the stack, into one product, all of them being taken in feet. Divide the product by 27, to bring it to yards. This, multiplied by 6, will give the number of stones, if the hay be new; but if the stack has stood a considerable time, add a third to it; or if it be old hay, add a half to it.

NOTE. If the form of the stack resemble any of the figures in MENSURATION OF SOLIDS, its content in cubic yards may be found by the rules there given, and its weight found as in the rule.

1. How much hay does a new stack contain, of which the length is 25 feet, the breadth 9 feet, the height from the ground to the eaves 14 feet, and above the eaves 8 feet?

Ans. $18 \times 25 \times 9 \times 6 \div 27 = 900$ stones.

2. How much old hay in a stack 40 feet long and 16 feet broad, the height to the eaves 15 feet, and above 8 feet?

Ans. $4053\frac{1}{3}$ stones.

3. How much new hay in a stack 50 feet long and 30 feet broad, the height to the eaves 20 feet, and above 14 feet?

Ans. 9000 stones.

4. How much hay in a stack which has stood 4 weeks 60 feet long and 35 feet broad, the height to the eaves 24 feet, and above 16 feet? Ans. $19911\frac{1}{9}$ stones.

PRACTICAL GUNNERY.

I. THEOREMS relating to projectiles on horizontal planes.

Let s denote the sine; c, the cosine; and t, the tangent of the angle of elevation; S, the sine; and v, the versine of twice the angle of elevation; R, the horizontal range; T the time of flight; H, the greatest height of the projectile; $g = 32{\cdot}2$ feet; and a, the impetus, or altitude due to V, the velocity; any two of these being given the others can be found; thus,

1. $R = 2as = 4asc = \frac{SV^2}{g} = \frac{scV^2}{\frac{1}{2}g} = \frac{\frac{1}{2}gcT^2}{s} = \frac{\frac{1}{2}gT^2}{t} = \frac{4H}{t}.$

2. $V = \sqrt{2ag} = \sqrt{\frac{gR}{s}} = \sqrt{\frac{gR}{2sc}} = \frac{\frac{1}{2}gT}{s} = \frac{2}{s}\sqrt{\frac{1}{2}gH}.$

3. $T = \frac{sV}{\frac{1}{2}g} = 2s\sqrt{\frac{a}{\frac{1}{2}g}} = \sqrt{\frac{tR}{\frac{1}{2}g}} = \sqrt{\frac{sR}{\frac{1}{2}gc}} = 2\sqrt{\frac{H}{\frac{1}{2}g}}.$

4. $H = as^2 = \frac{1}{2}av = \frac{1}{4}tR = \frac{sR}{4c} = \frac{s^2V^2}{2g} = \frac{vV^2}{4g} = \frac{1}{8}gT^2.$

II. Theorems relating to projectiles on inclined planes.

Let c = cos. of direction above the horizon; C, cos. of inclination of the plane; s, sine of direction above the plane; R, the range on the oblique plane; T, the time of flight; V, the projectile velocity; H, the greatest height above the plane; a, the impetus, or altitude due to the velocity V, and $g = 32{\cdot}2$ feet; then,

1. $R = \frac{cs}{C^2}4a = \frac{2cs}{C^2g}V^2 = \frac{gc}{2s}T^2 = \frac{4c}{s}H.$

2. $H = \frac{s^2}{C^2}a = \frac{s^2V^2}{2gC^2} = \frac{sR}{4c} = \frac{g}{8}T^2.$

3. $V = \sqrt{2ag} = C\sqrt{\frac{gR}{2cs}} = \frac{gc}{2s}T = \frac{2C}{s}\sqrt{\frac{1}{2}gH}.$

4. $T = \frac{2s}{C}\sqrt{\frac{a}{\frac{1}{2}g}} = \frac{sV}{\frac{1}{2}gC} = \sqrt{\frac{sR}{\frac{1}{2}gc}} = 2\sqrt{\frac{H}{\frac{1}{2}g}}.$

III. To determine the velocity of any shot or shell.

Let $v =$ the velocity; B, the weight of the shot or shell; and C, the weight of the charge of powder; then, according to the experiments of Dr Hutton, $v = 1600\sqrt{\frac{2C}{B}} = 2263\sqrt{\frac{C}{B}}$.

From Dr Gregory's experiments, however, on better powder, $v = 1600\sqrt{\frac{3C}{B}} = 2771\sqrt{\frac{C}{B}}$.

IV. Given the range at one elevation, to find the range at any other elevation.

Rule. As the sine of twice the first elevation is to the sine of twice the second elevation, so is the given range to the range required.

V. Given the range for one charge, to find the range for any other charge; or the charge for any other range.

Rule. The ranges are directly as their charges, the elevation remaining the same; or as one range is to any other range, so is the given charge to the charge required.

Example 1. If a ball of 1 lb. has an initial velocity of 1600 feet per second when fired with a charge of 8 oz. of powder; required with what velocity each of the several kinds of shells will be projected by the full charges of powder.

BY HUTTON'S FORMULA.

$$2263\sqrt{\frac{9}{196}} = 2263 \times \frac{3}{14} = \frac{6789}{14} = 485 \text{ feet.}$$

$$2263\sqrt{\frac{4}{90}} = \frac{4526}{\sqrt{90}} = \frac{4526}{9{\cdot}486833} = 477 \text{ feet.}$$

$$2263\sqrt{\frac{2}{48}} = 2263\sqrt{\frac{1}{24}} = \frac{2263}{4{\cdot}89898} = 462 \text{ feet.}$$

$$2263\sqrt{\frac{1}{16}} = 2263 \times \tfrac{1}{4} = \frac{2263}{4} = 566 \text{ feet.}$$

$$2263\sqrt{\frac{{\cdot}5}{8}} = 2263\sqrt{\frac{1}{16}} = \frac{2263}{4} = 566 \text{ feet.}$$

BY GREGORY'S FORMULA.

$$2771\sqrt{\frac{9}{196}} = 2771 \times \frac{3}{14} = \frac{8313}{14} = 594 \text{ feet.}$$

$$2771\sqrt{\frac{4}{90}} = \frac{5542}{\sqrt{90}} = \frac{5542}{9{\cdot}486833} = 584 \text{ feet.}$$

$$2771\sqrt{\frac{2}{48}} = 2771\sqrt{\frac{1}{24}} = \frac{2771}{4{\cdot}89898} = 568 \text{ feet.}$$

$$2771\sqrt{\tfrac{1}{16}} = 2771 \times \tfrac{1}{4} = \frac{2771}{4} = 693 \text{ feet.}$$

$$2771\sqrt{\tfrac{\cdot 5}{8}} = 2771\sqrt{\tfrac{1}{16}} = \frac{2771}{4} = 693 \text{ feet.}$$

Diam. of Shell.	Weight of Shell.	Charge of Powder.	Velocity in Feet per Second, according to the Formula	
Inches.	Pounds.	Pounds.	Of Hutton.	Of Gregory.
13	196	9	485	594
10	90	4	477	584
8	48	2	462	568
$5\frac{1}{2}$	16	1	566	693
$4\frac{2}{5}$	8	$0\frac{1}{2}$	566	693

2. If a shell range 1500 yds. when discharged at an elevation of 45°, how far will it range when the piece is elevated 32° 30′, the charge of powder being the same? As the sine of twice 45° or 90° is = the radius, we have twice the sine of 32° 30′, = ·906308 × 1500 = 1359 yds. the range.

3. At an elevation of 45° the range of a ball was 4000 feet, at what angle must the ball be fired to strike an object at 3000 feet? Ans. 24° 36′, or 65° 24′ equally distant from 45°.

4. With what impetus, velocity, and charge of powder must a 13-inch shell be fired at an elevation of 28° 30′ to strike an object at the distance of 2250 feet?

Ans. 1341·5 impetus, 293·92 velocity, 2·2358 lbs. of powder.

5. A shell being found to range 3500 ft. when discharged at an elevation of 25° 12′, how far, with the same charge of powder, will it range with an elevation of 34° 36′?

Ans. 4237 feet.

6. A shell with a charge of 9 lbs. of powder will range 4000 feet, how much powder will throw it 3700, the elevation in both cases being 35°? Ans. $8\frac{14}{4\,0}$ lbs. of powder.

7. Required the time of flight for any given range, the elevation being 45°?

Ans. $\frac{1}{4}$ of the square root of the range in feet.

8. In what time will a shell range 3000 at an angle of 30°?

Ans. In 10·36 seconds.

9. How far will a shell range on a plane which ascends 10° 30′, and also on another which descends 6° 30′, the impetus being 4000 feet, and the elevation of the mortar 35°?

Ans. Range on the ascent, 5621·86 feet.
Range on the descent, 8840·7 feet.

10. How much powder will throw a 10-inch shell 5621·86 feet on a plane which ascends 10° 30′, the elevation of the mortar being 35°? Ans. 3·021 lbs. of powder.

11. At what elevation, with a charge of 3·021 lbs. of powder, must a 10-inch mortar be discharged to range 8840·7 feet on a plane which descends 6° 30′? Ans. 35° 42′, or 47° 48′.

12. In what time will a 10-inch shell strike a plane which rises 10° 30′ when the mortar is elevated 45°, and discharged with an impetus of 2500 feet? Ans. 14·15 seconds nearly.

13. How far will a shell range on a plane which ascends 8° 15′, and also on another which descends 8° 15′, the impetus being 3000 feet, and the elevation of the mortar 32° 30′?

Ans. 4244 feet on the ascent, and 6745 feet on the descent.

14. How much powder will throw a 13-inch shell 4244 feet on an inclined plane which ascends 8° 15′, the elevation of the mortar being 32° 30′? Ans. 7·3765 lbs., or nearly 7 lbs. 6 oz.

15. At what elevation must a 13-inch mortar be discharged to range 6745 feet on a plane which descends 8° 15′, the charge of powder being $7\frac{3}{8}$ lbs.? Ans. 32° $41\frac{1}{2}$′.

16. In what time will a 13-inch shell strike a plane which rises 8° 30′, when the mortar is elevated 45°, and discharged with an impetus of 2304 feet? Ans. $14\frac{2}{5}$ seconds.

17. If a shell with a charge of 9 lbs. of powder range 4000 feet, what charge will be sufficient to throw it 3000 feet, the elevation in both cases being 45°? Ans. $6\frac{3}{4}$ lbs. of powder.

18. With what impetus, velocity, and charge of powder must a 13-inch shell be fired, at an elevation of 32° 12′, to strike an object at the distance of 3250 feet?

Ans. 1802 impetus, 340 velocity, and 4 lb. $7\frac{1}{2}$ oz. charge of powder.

WEIGHT AND DIMENSIONS OF BALLS AND SHELLS.

An iron ball 4 inches in diameter weighs 9 lbs. nearly; and a leaden ball $4\frac{1}{4}$ inches in diameter weighs about 17 lbs. Also, a pound of gunpowder measures about 30 cubic inches. And similar solids are to one another as the cubes of their diameters, or like sides.

PROB. I. Given the diameter of an iron ball, to find its weight, and conversely.

RULE. As the cube of 4 is to the cube of the diameter, so is 9 to the weight in pounds; or, divide the cube of the diameter by $7\frac{1}{9}$; the quotient will be the weight in pounds.

Multiply the weight by $7\frac{1}{9}$; the cube root of the product is the diameter.

1. What is the weight of an iron ball, of which the diameter is $3\frac{1}{2}$ inches? Ans. $3{\cdot}5^3 \div 7\frac{1}{9} = 6{\cdot}0293$ lbs.

2. What is the diameter of an iron ball which weighs 24 lbs.? Ans. $\sqrt[3]{24 \times 7\frac{1}{9}} = \sqrt[3]{170{\cdot}\dot{6}} = 5{\cdot}547$ inches the diameter.

3. What is the weight of an iron ball, of which the diameter is 4·6 inches? Ans. 13·688 lbs.

4. What is the diameter of an iron ball which weighs 36 lbs.? Ans. 6·349 inches.

5. What is the weight of an iron ball, of which the diameter is 5·5 inches? Ans. 23·3965 lbs.

6. What is the diameter of an iron ball which weighs 48 lbs.? Ans. 6·988 inches.

Prob. II. Given the diameter of a leaden ball, to find its weight, and the converse.

Rule. As the cube of $4\frac{1}{4}$ is to the cube of the diameter, so is 17 to the weight in pounds; or, divide the cube of the diameter by $4\frac{1}{2}$: the quotient will be the weight in pounds nearly.

Multiply the weight by $4\frac{1}{2}$; the cube root of the product will be the diameter in inches nearly.

1. What is the weight of a leaden ball, of which the diameter is 4·25 inches? Ans. $4{\cdot}25^3 \div 4\frac{1}{2} = 17{\cdot}059$ lbs.

2. What is the diameter of a leaden ball which weighs 36 lbs.? Ans. 5·45 inches.

3. What is the weight of a leaden ball, of which the diameter is 4·6 inches? Ans. 21·63 lbs.

4. What is the diameter of a leaden ball which weighs 48 lbs.? Ans. 6 inches.

Prob. III. To find the weight of an iron shell.

Rule. Take the difference between the cubes of the external and internal diameters, and divide it by $7\frac{1}{9}$; the quotient will be the weight in pounds.

1. What is the weight of a 13-inch shell, the inner diameter being 9 inches? Ans. $(13^3 - 9^3) \div 7\frac{1}{9} = 206{\cdot}4375$ lbs.

2. What is the weight of a shell, of which the diameters are 11·1 and 8 inches? Ans. 120·323 lbs.

3. What is the weight of a 16-inch shell, the inner diameter being $11\frac{1}{2}$ inches? Ans. 362·127 lbs.

4. What is the weight of a shell whose diameters are 15·4 and 11·2 inches? Ans. 316·0316 lbs.

PROB. IV. To find how much powder will fill a case.

RULE. Find the content in inches, and divide it by 30; the quotient will be the weight in pounds.

1. How much powder will fill a cubical box, of which the side is 18 inches? Ans. $18^3 \div 30 = 194{\cdot}4$ lbs.

2. How much powder will be contained in a cylinder which is 1 foot in length, and the diameter of its base 4 inches? Ans. 5·02656 lbs.

3. How much powder will a chest hold, which is 15 inches long, 13 inches broad, and 5 inches deep? Ans. 32·5 lbs.

4. What is the side of a cubical box which will hold 12 lbs. of powder? Ans. 7·113 inches.

5. What is the side of a cubical box which will hold 24·3 lbs. of powder? Ans. 9 inches.

PROB. V. To find how much powder will fill a shell.

RULE. Divide the cube of the internal diameter in inches by 57·3;* the quotient will be the weight in pounds.

Multiply the weight by 57·3; the cube root of the product will be the diameter.

1. How much powder will a shell of 9 inches internal diameter hold? Ans. $729 \div 57{\cdot}3 = 12{\cdot}7225$ lbs.

2. Required the diameter of a shell which will hold 9 lbs. of powder. Ans. 8·02 inches.

3. How much powder will fill a shell, of which the inner diameter is $11\frac{1}{2}$ inches? Ans. 26·5423 lbs.

4. Required the diameter of a shell which will hold 15 lbs. of powder. Ans. 9·51 inches.

PILING OF BALLS AND SHELLS.

BALLS AND SHELLS are piled up in horizontal courses, upon a base of the form of an equilateral triangle, or of a square, or of a rectangle. The number of balls in a row diminishes, till, in the two first forms, it ends in a single ball, and in the last in a single row. The number of horizontal courses in a triangular or square pile is equal to the number of balls in a side of the under course. The number in the top row of a rectangular pile is one more than the difference between the length

* The bulk of 1 lb. of gunpowder being about 30 inches, it is manifest that if d = the internal diameter of the shell, it will hold $d^3 \times {\cdot}5236 \div 30 = d^3 \div 57{\cdot}3$ lbs. of powder very nearly.

and breadth of the bottom row, and the number of courses is equal to the number of balls in the breadth of the bottom course.

Prob. I. To find the number of balls in a triangular pile.

Rule. Multiply the number of balls in a side of the bottom row by that number increased by 1, and again by that number increased by 2; the product, divided by 6, will be the number of balls in the pile.*

1. Required the number of balls in a triangular pile, of which each side of the base contains 30 balls. Ans. 4960 balls.

2. Required the number of balls in a triangular pile, each side of the base containing 64 balls. Ans. 45760 balls.

3. Required the number of balls in a triangular pile, each side of the base containing 80 balls. Ans. 88560 balls.

Prob. II. To find the number of balls in a square pile.

Rule. To twice the number of balls in a side of the bottom row add 1, and multiply the sum by the number in that row, and by that number increased by 1; the product, divided by 6, will give the number of balls in the pile.†

1. Let the side of the bottom row of a square pile contain 20 balls. How many balls are in the pile? Ans. 2870 balls.

2. Let the side of the bottom row of a square pile contain 80 balls. How many balls are in the pile? Ans. 173880 balls.

3. Let each side of the bottom row of a square pile contain 50 balls. How many balls are in the pile? Ans. 42925 balls.

Prob. III. To find the number of balls in a rectangular pile.

Rule. From 3 times the number in the length of the bottom row, increased by 1, subtract the number in the breadth,

* The triangular numbers, 1, 3, 6, 10, 15, 21, &c., are the number of balls in the different courses from the top of a triangular pile; it is therefore manifest, that the number of balls in a triangular pile is equal to the sum of as many terms of this series as there are courses, or as there are balls in one side of the bottom row. Now, if $n =$ the number of courses, then $\frac{n(n+1)(n+2)}{1.2.3} =$ the sum of the series, or the number of balls in the pile. (Algebra, Prob. IV., page 87.)

† Here the square numbers 1, 4, 9, 16, 25, &c., are the number of balls in the different courses from the top of a square pile; consequently the sum of as many terms of this series as there are courses will give the number of balls in the pile, and if $n =$ the number of courses, then $\frac{n(n+1)(2n+1)}{1.2.3}$ is the sum of the series, or the number of balls in the pile. (Algebra, Prob. IV. page 88.)

and multiply the remainder by the breadth, and by the breadth increased by 1; the product, divided by 6, will give the number of balls in the pile.*

1. Suppose the number of balls in the length of a rectangular pile to be 59, and in the breadth 20. What is the number in the pile? Ans. 11060 balls.

2. Suppose the length contains 80, and the breadth 60. How many balls are in the pile? Ans. 110410 balls.

3. Suppose the length contains 100, and the breadth 75. How many balls are in the pile? Ans. 214700 balls.

PROB. IV. To find the number of balls in an incomplete pile.

RULE. From the number of balls in the complete pile subtract the number in the pile that is wanting, both computed as before; the remainder is the number in the incomplete pile.

1. Required the number of balls in a rectangular pile of 15 courses, the numbers in the bottom row being 60 and 25.

Ans. 14590 balls.

2. Required the number of balls in a triangular pile of 15 courses, when each side of the base contains 60.

Ans. 11605 balls.

3. Required the number of balls in a square pile of 20 courses, each side of the base containing 160.

Ans. 453670 balls.

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* Since the number of balls in the length and breadth of the several courses of a rectangular pile decreases each by unity from the bottom to the top, it is evident, that the number of balls in the whole pile is equal to the sum of as many terms of a series of products as there are balls in the breadth. The factors of the first term are the number of balls in the length and breadth, and those of each succeeding term are diminished by unity till the series terminates. Now, if the factors of the first term are represented by p and q, then the sum of such series is $= \frac{3pq^2 + 3pq - q^3 + q}{6}$ (ALGEBRA, Prob. IV., Ex. 9, page 88), and this expression is equivalent to $\frac{(3p + 1 - q) \times d \times (d + 1)}{6}$, which is the rule.

THE WORKS OF ARTIFICERS.

ARTIFICERS take the dimensions of their work with a measuring-line, divided into feet and inches, or by the carpenter's rule, or by a yard divided into inches and parts.

The work is generally computed by duodecimal multiplication, in which the inch is supposed to be divided into 12 parts, and each part into 12 seconds, &c.

RULE. Multiply each denomination of the multiplicand by the feet of the multiplier, and place the product under that denomination of the multiplicand from which it arises, carrying at 12. Then multiply by the inches of the multiplier, and set each product a denomination farther towards the right hand. Next multiply by the parts, if any, and set the products a place still farther to the right. Then add the products.

1. Multiply 9 f. 4 in. by 3 f. 8 in.

3	8		
28	0		
6	2	8	
34	2	8	product.

2.	Multiply	98	3		by 5	6	........Ans.	540	4	6			
3.	....	148	3		by 8	9	....	1297	2	3			
4.	....	87	6	8	by 11	10	....	1036	0	10	8		
5.	....	63	4	6	by 8	9	6.........	557	2	0	9		
6.	....	55	8	7	by 72	6	3.........	4040	6	2	7	9	
7.	....	105	3	4	by 27	9	6.........	2925	10	1	8		
8.	....	208	7	9	by 12	5	4.........	2596	5	9	4		
9.	....	365	11	8	by 13	6	3.........	4948	2	11	11		
10.	....	185	10	9	by 15	9	8.........	2938	2	2	11		

NOTE. The feet in the product are square feet, 9 of which make a square yard, and 36 square yards make a rood of building. The inches in the product are 12th parts of a square foot, or each of them is 12 square inches, and the parts are square inches. The lower denominations are commonly expressed in fractions of a square inch: thus, 8 seconds are $\frac{2}{3}$ of a square inch, 9 seconds are $\frac{3}{4}$, and 7 seconds 6 thirds are $\frac{5}{8}$.

OF THE CARPENTER'S SLIDING RULE.

The works of artificers, as well as the quantity of timber, are often computed by the sliding-rule.

This rule consists of two pieces, each a foot long, fastened together with a folding joint, with a slider in one of the pieces.

The edge of each piece of the rule is divided into 10 equal parts, and each part is subdivided into 10 equal parts; so that by it the dimensions may be taken in feet and decimals.

One of the faces is divided into inches, and 8th or 10th parts; and on the same face are several plane and diagonal scales, the diagonal being divided into 12 parts.

On the other face, the piece which has the slider contains four lines, two on the slider marked B and C, and two on the rule; one under the slider marked A, and the other above it marked D. The three lines A, B, and C, are of the same length, and divided in the same way: the divisions on D are double of those on the other lines. These divisions are all logarithmical; that is, if the distance between the first 1 and the other 1 be divided into 1000 equal parts, the distance between 1 and 2 is 301 parts, which is the logarithm of 2, and the distance between 1 and 3 is 477, the logarithm of 3, &c.

The first 1 may be read 1, or 10, or 100, and all the rest are valued according to it. If it be read 1, the second 1 is 10, and the third 1 is 100, and then the first 2 is read 2, and the second 2 is 20; but if the first 1 be called 10, the second 1 is 100, and the third 1000, and then the first 2 is 20, and the second 2 is 200. And all the other divisions and subdivisions are valued in the same way.

On the same face of the rule, there is on the other piece of it a table of the value of a load, or of 50 cubic feet of timber, at all prices, from 6d. to 2s. each foot.

Prob. I. To multiply numbers by the rule.

Set 1 on B opposite to the multiplier on A; then opposite to the multiplicand on B will be the product on A.

1. Multiply 16 by 6..............Ans. 96.
2. .... 23 by 14...................322.
3. .... 27 by 23...................621.
4. .... 68 by 46...................3128.

Prob. II. To divide numbers.

Set the divisor on B to 1 on A; then against the dividend on B will be found the quotient on A.

1. Divide 96 by 24..................Ans. 4.
2. .... 576 by 48........................12.
3. .... 156 by 23........................6·8.
4. .... 988 by 76........................13.

Prob. III. To work a proportion.

Set the first term on B to the second on A; then against the third on B will stand the fourth on A.

1. Required the fourth proportional to 12, 28, and 114. Ans. 266.
2. Required the third proportional to 18 and 54. Ans. 162.
3. If 14 men build 4 roods, how many will in the same time build 28 roods? Ans. 98 men.
4. If 42 men perform a piece of work in 108 days, in what time will 72 do it? Ans. 63 days.

Note. This, with the two preceding rules, depends upon this principle: In a proportion, the difference between the logarithms of the first and second terms is equal to the difference of the logarithms of the third and fourth; and 1 is to the multiplier or divisor, as the multiplicand or quotient is to the product or dividend.

Prob. IV. To extract the square root.

Set 1 on C to 1 on D; then against the given number on C is its square root on D.

Note. The 1 on C must be read 1, or 100, or 1000; and the 1 on D must be read 1, or 10, or 100 accordingly.

1. Required the square root of 576..............Ans. 24.
2. .... of 196....................14.
3. .... of 4096....................64.
4. .... of 9216....................96.

Prob. V. To find a mean proportional between two numbers.

Set the less on C to the same number on D; then against the greater number on C will stand the mean proportional on D.

1. Required the mean proportional between 4 and 36. Ans. 12.
2. .... 144 and 576. 288.
3. .... 513 and 57. 171.
4. .... 128 and 32. 64.

TO MEASURE TIMBER.

Prob. I. To find the area of a board.

Rule. Multiply the length by the mean breadth.

Note. When the board tapers regularly, half the sum of the breadths at the ends is the mean breadth.

By the Sliding-Rule.

Set the breadth in inches on A to 12 on B; then against the length in feet on B will be the content on A, in square feet and decimals.

1. Required the content of a board 12 feet 6 inches long, and 1 foot 3 inches broad. Ans. 15 feet 7 inches 6 parts.

2. Required the content of a board 13 feet 4 inches long, and 1 foot 8 inches broad. Ans. 22 feet 2 inches 8 parts.

3. Required the content of a board 11 feet 10 inches long, and 11 inches broad. Ans. 10 feet 10 inches 2 parts.

4. Required the content of a board 16 feet 9 inches long, and 2 feet 2 inches broad. Ans. 36 feet 3 inches 6 parts.

5. Required the content of a board 14 feet 11 inches long, and 9 inches broad. Ans. 11 feet 2 inches 3 parts.

6. Required the content of a board 10 feet 10 inches long, and 8 inches broad. Ans. 7 feet 2 inches 8 parts.

Prob. II. To find the content of squared or four-sided timber.

Rule. Multiply the mean breadth by the mean thickness: the product, multiplied by the length, will give the content.*

Note. If the tree tapers regularly from the one end to the other, take the mean breadth and thickness in the middle, or take half the sum of the dimensions at the two ends for the mean dimensions. If it does not taper regularly, take several dimensions at equal intervals, and divide their sum by the number of them for the mean dimensions; or divide the tree into several convenient lengths, find the content of each piece separately, and their sum will be the whole content.

By the Sliding-Rule.

Find a mean proportional between the breadth and thickness. Then set the length on C to 12 on D; and against the

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* Sometimes $\frac{1}{4}$ of the circumference of the tree is used as the side of the mean square, and this multiplied by itself and by the length is accounted the solid content. This method is, however, very erroneous, always giving the content too great; and supposes the quarter-girt to be an arithmetical instead of a geometrical mean proportional between the breadth and thickness. The content of the log, in the second example, by this method would be 64 feet [illegible] inches $0\frac{3}{4}$ part, or 2 feet 3 inches $6\frac{3}{4}$ parts too great.

mean proportional on D in inches will be the solid content in feet on C.

NOTE. If the mean proportional be in feet, use 1 instead of 12 on D.

1. Required the content of a log, the length 24 feet 6 inches, mean breadth 1 foot 1 inch, and mean thickness 1 foot 1 inch.
Ans. 28 feet 9 inches $\frac{1}{2}$ part.

2. Required the content of a log, the length 27 feet, mean breadth 1 foot 10 inches, and mean thickness 1 foot 3 inches.
Ans. 61 feet 10 inches 6 parts.

3. Required the content of a log, the length 18 feet 6 inches, mean breadth 1 foot $4\frac{1}{2}$ inches, and mean thickness 1 foot 2 inches.
Ans. 29 feet 8 inches $1\frac{1}{2}$ part.

4. Required the content of a log, the length 20 feet 6 inches. mean breadth 1 foot $2\frac{1}{2}$ inches, and mean thickness 1 foot $2\frac{1}{2}$ inches.
Ans. 29 feet 11 inches $2\frac{1}{8}$ parts.

5. Required the content of a log, the length 30 feet 8 inches, mean breadth 2 feet 1 inch, and mean thickness 2 feet 2 inches.
Ans. 138 feet 5 inches $1\frac{1}{3}$ part.

6. Required the content of a log, the length 40 feet 7 inches, mean breadth 2 feet 3 inches, and mean thickness 1 foot 9 inches.
Ans. 159 feet 9 inches $6\frac{3}{4}$ parts.

PROB. III. To find the content of round timber.

COMMON RULE. Take one-fourth of the mean girt, and square it, and multiply it by the length for the content.

By the Sliding-Rule.

Set the length in feet on C to 12 on D; then against the quarter-girt in inches on D will be the content in feet on C.

NOTE 1. In order to reduce the tree to such a circumference as it would have without its bark, $\frac{1}{4}$ of an inch should be deducted from the quarter-girt when the thickness of the bark is $\frac{1}{2}$ of an inch; but when the bark is thicker, which is rarely the case, a little more than $\frac{1}{4}$ of an inch must be allowed. No rough timber under 6 inches diameter is accounted measurable.

NOTE 2. The common rule gives the content too small, by 3 feet on every 11 feet of content; yet it is universally used in practice, being originally introduced in order to compensate the purchaser of round timber for the waste occasioned by squaring it.*

* Let l = the length, and c = the mean circumference; then the rule is $\left(\frac{c}{4}\right) \times l = \cdot0625c^2l$; but the content of the cylinder is $= \cdot0795775c^2l$; hence the content, as given by the rule, is to the true cylindrical content, as $\cdot0625 : \cdot0795775$, or as $11 : 14$ nearly.

RULE II. Take one-fifth of the girt, and square it, and multiply by twice the length for the true content nearly.*

By the Sliding-Rule.

Set twice the length on C to 12 on D; then against $\frac{1}{5}$ of the girt on D will be the content in feet on C.

1. Required the content of a piece of round timber $9\frac{1}{2}$ feet long, and its mean girt 14 feet.

Ans. 116 feet $4\frac{1}{2}$ inches by the common rule; or, true content 148 feet 11·52 inch by Rule II.

2. Required the content of a tree 24 feet long, and its girts at the ends 14 and 2 feet.

Ans. 96 feet by the common rule; the true content is 122·88 feet.

3. How much timber in a tree 18 feet long, and its mean girt 5 feet 8 inches?

Ans. Common rule 36 feet $1\frac{1}{2}$ inch; true content 46 feet 2 inches 10·56 parts.

4. How much timber in a tree 32 feet long, its girts in the middle of every 8 feet being 64, 56, 52, and 46 inches?

Ans. 41 feet $10\frac{1}{6}$ inches by the common rule; true content 53 feet 6 inches 9·28 parts.

5. Required the content of a tree 30 feet long, the girts in the middle of every 10 feet being 50·4, 54·8, and 60·8 inches.

Ans. 40 feet 1 inch 2·9 parts by the common rule; true content 51 feet 3 inches 11·872 parts.

6. Required the content of a tree 55 feet long, the girts in the middle of every 11 feet being 72, 56, 42, 35, and 25 inches.

Ans. 56 feet 11 inches $8\frac{5}{8}$ parts by the common rule; true content 72 feet 11 inches 1·92 part.

7. Required the content of a tree 50 feet long, its mean girt being 7 feet.

Ans. 153 feet $1\frac{1}{2}$ inch by the common rule; true content 196 feet.

8. Required the content of a tree 48 feet long, the girts at its ends being 60 and 18 inches.

Ans. 31 feet $8\frac{1}{4}$ inches by the common rule; true content 40 feet 6·72 inches.

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* The solidity of the cylinder is $\cdot0795775c^2l$, and that by the rule is $\left(\frac{c}{5}\right) \times 2l = \frac{c^2}{25} \times 2l = \frac{2}{25}c^2l = \cdot08c^2l$; hence the content, as given by the rule, is to the true content as ·08 : ·0795775, or as 1 : ·99472; hence the rule gives 1 foot too much in 190 feet of content.

9. Required the content of a tree 45 feet long, the mean girt being 74 inches.

Ans. 106 feet $11\frac{7}{18}$ inches by the common rule; true content 136 feet 10·8 inches.

10. Required the content of a tree $17\frac{1}{4}$ feet long, the girts in five different places being 9·43, 7·92, 6·15, 4·74, and 3·16 feet.

Ans. 42·5195 feet by the common rule; true content 54·424992 feet.

Prob. IV. To calculate the value of roods, yards, feet, inches, &c. at any number of shillings and pence per rood.

Rule. Bring the roods, yards, feet, inches, &c. into a state of duodecimals; thus,

Write down in the same line with the roods one-third of the yards, and if there is any remainder, for each unit of it add 9 feet to the given feet, inches, &c., then annex four-ninths of this sum to complete the multiplicand, and multiply it by the shillings and pence, as in duodecimal multiplication, The highest or left-hand number of the product is shillings, the second pence, and the third twelfth-parts of a penny, &c.*

Note. Instead of taking $\frac{4}{9}$ of the feet and inches, add $\frac{1}{3}$ to them, and then take $\frac{1}{3}$ of the sum.

Find the value of 12 roods 15 yards 4 feet 3 inches, at 25s. per rood.

Here we write down 12 on the left, and next to it the third part of 15, or 5, and as there is no remainder, we next write down $\frac{4}{9}$ of 4 feet 3 inches, or 1 : 10 : 8, and the complete multiplicand is

	12	5	1	10	8 × 25 = 5 × 5
					5
	62	1	9	5	4
					5
2,0)31,0	8	11	2	8	
£15	10	9 very nearly.			

Find the value of 15 roods 16 yards 7 feet 3 inches, at 31s. 6d. per rood.

* The editor is indebted for this simple and useful rule to Mr Duff, surveyor, Edinburgh.

Here the prepared multiplicand is

15	5	7	2	8	to be multiplied by $31\frac{1}{2} = 3 \times 10 + 1\frac{1}{2}$.
				3	
46	4	9	8	0	
				10	
464	0	0	8	0	
15	5	7	2	8	
7	8	9	7	4	
487	2	5	6	0	= £24, 7s. $2\frac{1}{2}$d. very nearly.

1. Find the value of 6 roods 5 yards 8 feet 3 inches, at 21s. 4d. per rood. Ans. £6, 11s. $4\frac{1}{2}$d.

2. Find the value of 35 roods 27 yards 4 feet 8 inches, at 27s. $9\frac{1}{2}$d. per rood. Ans. £49, 13s. $11\frac{1}{2}$d.

3. Find the value of 56 roods 17 yards 5 feet 4 inches, at 15s. $10\frac{1}{2}$d. per rood. Ans. £44, 16s. $9\frac{1}{11}$d.

4. Find the value of 47 roods 19 yards 7 feet 6 inches, at 34s. $8\frac{3}{4}$d. per rood. Ans. £82, 11s. $4\frac{3}{4}$d.

MASON WORK.*

RUBBLE WORK is measured in three different ways.

I. When the tradesman furnishes all materials.

Find the depth of the foundation at several places, and take the mean height from the foundation to the top of the side-walls. Take the length of the side-walls on the outside, and the breadth of the gables or cross-walls on the inside of the building.

Gable-ends are measured by multiplying the height from the level of the side-walls to the bottom of the chimney-stacks by half the sum of the breadths at the top of the side-walls and at the bottom of the chimney-stack; and the chimney-stack is measured by multiplying half the girt by the height from the bottom of the stack to the top of the cope.

Chimney-flues are measured by the lineal foot, from the top of the stack to the bottom of the jambs.

Dormer-windows on side-walls are measured by adding the thickness of one haunch to the length of the square part, and

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* The rules for the Mensuration of Artificers' Works, with the various allowances, have been furnished by an eminent surveyor in Edinburgh, and cannot fail to be of great advantage to the students for whom this section is intended. The allowances apply principally to Scotland; but the rules for taking the dimensions are also applicable both to England and Ireland.

multiplying it by the height from the level of the side-walls to the bottom of the angle; and the angular part and stack are measured the same way as a gable-end and chimney-stack.

All projections, whether external or internal, if they do not exceed 2 feet, are found by adding one return to the length, and multiplying the sum by the height and thickness, and reducing it to the standard of the wall.

An allowance for workmanship of 1 foot by 9 inches, multiplied by the length, is made for every levelling for joists and belts in rubble walls; and when walls are 2 feet thick or more, 1 foot by the thickness of the wall, multiplied by the length, is made for levelling the tops of side-walls, skews, and chimney-stacks; and when under 2 feet thick, the thickness by its half, and by the length, is allowed; but no allowance is made for belts on ashlar fronts.

An allowance of 9 inches square by the length is made for levelling for bond timbers and ragulates for roofs in the chimney-heads only; 1 foot by 9 inches is allowed for ragulates left for stairs; and 1 foot by 6 inches for thin walls. These allowances must all be reduced to the standard of the walls in which they are made, and rated as workmanship only.

The daylight or net opening of all apertures is to be deducted.

Rough stones more than 3 feet in length, placed as safes over voids, are to be taken by number, according to their different lengths.

Arches over cellars, &c. are taken by the net average girt, and by the length and by the depth of the arch-stones for the thickness, and are double measure; and arches having been included in the general dimensions are to be again taken by their height, thickness, and length, and reduced to the standard of the wall.

All upright circular walls are double measure, if not above 2 feet thick, and if above that thickness single measure, and a 2 feet wall *extra* added; and walls circular on one side only are allowed 1 foot thick round the circular part as double measure, and reduced to the standard of the wall, besides the solid content of the straight part.*

The rubble of stair-steps and platts is taken by their length without the wall, and by their breadth and thickness, and in all cases reduced to 1 foot thick.

Rubble is allowed for pavement laid on lime, and in no case is the thickness reckoned less than 4 inches.

* The double measure for circular walls is understood to be far too great an allowance, except when the circle is of small diameter and the work well executed.

In measuring separated pillars, when the face or front of the pillar does not exceed 5 feet in length, they are taken by their net height and length, and an allowance of 2 feet square by the height is made for carrying up the scontion. But this allowance applies only to pillars at and above 2 feet thick: all below that have the net thickness added to the length.

II. When the tradesman furnishes workmanship only.

The dimensions are taken over both side-walls and gables, and no deduction is made for apertures.

III. When the tradesman furnishes workmanship, lime, and sand.

The outside-walls are measured by including the thickness of one side-wall, and one-half of the apertures is deducted.

NOTE. Rubble walls, in all the three cases, at and below 18 inches thick, are to be reduced to 1 foot, and all above 18 inches reduced to 2 feet thick, and measured by the rood of 36 square yards.

On doors and windows where there is no hewn work, an allowance is made of 1 foot square by the length, in name of hammer-dressed scontions.

HEWN WORK.

Hewn Work in Rubble Walls. The rybats of doors and windows are measured by girting from the bottom of the check outward, including the backset, if any. Sills and lintels are taken for the length over the face of the rybats, including the projection of one end, if projected; and the girt is taken as in the rybats.

Hewn corners are taken by the height for the length, and by the mean girt for the breadth.

Skews are taken by the length and by the girt, and chimney-copes by the extreme length all round, and for the breadth by girting from the open of the flue down to the chimney-back.

When the whole front of a building is of hewn or polished work, it is taken by the extreme length and height of the different species of work, including the sides of projections, if any; but no allowance is made for the internal corners of such projections. All apertures are deducted; but the breasts and checks of rybats, together with the under bed and checks of the lintel, and upper bed of the sill, including their rests, are measured and added.

When architrave rybats are placed in a hewn front, the deductions are taken over these; and such moulded architrave rybats are measured by the height, and by girting from the bottom of the check outwards to the face of the plain ashlar.

The lintels are girted in the same manner, and the length is taken round the ends.

Moulded architrave rybats of main doors, or otherwise, are taken in the same manner, and the whole reported as moulded work, excepting when plain ashlar stones are placed in the reveals, between the outband rybats and the checks; in which case these must be deducted, and added to the plain ashlar.

The Hewn Work of Arches is measured by finding the mean height of the arch stones, and for the length by laying the line round the middle of the face of the arch. The soffit and check are taken for the length round the check, and for the breadth by girting from the bottom of the check outward to the face of the arch. Both face and soffit are reckoned double measure. Arches in upright circular walls are allowed three measures.

When pannels are sunk on ashlar work, after they are included in the surface, the sunk part, and that round the edges, are taken over again; but if a moulding is round it the whole is taken as moulded work, and not included in the plain surface.

All hewn work cut circular for skews is allowed 6 inches by the length for cutting.

Rustic work, whether square or champhered, is first measured superficially, and the checks or champhers are measured over again. Giblat checks, in like manner, are measured over again, after having been included in the face on scontions.

Pilasters, when they are raised out of the solid stones, and built in courses along with the ashlar, are girted in along with the ashlar, and the sunk part and edges are taken over again. If the pilasters are fluted, they are measured over again as moulded work, girting into the flutes and over the fillets. The cabled part, if any, is measured in the same way, and allowed double measure. The bases and capitals are girted as mouldings.

NOTE. The measuring over again of pilasters ought to be only for workmanship. But a better way is to measure single as mouldings.

Columns, of which the shafts are diminished with a curve or swell, are allowed double measure and a half; and if the neck-moulding is wrought on the shaft, they are allowed three measures. When the shafts are diminished straight, without a swell, double measure is only allowed, and a half more if the neck-moulding is wrought on the shaft. The fluted and cabled parts of columns are measured the same as in pilasters, after they are taken for plain work, as above. The bases and capitals are girted as other mouldings, with the usual allow-nce; and the number and size of carved capitals must be given.

Cornices are taken for their length at the extremities of their greatest projection, and for their breadth by girting their mouldings; and so much of the superficies of the upper bed as is without the wall is added.

Block and dentil cornices, after being measured in the same manner, have the backs and soffits of the recesses, together with the ends of the blocks and dentils, added.

Dentils are, however, generally measured lineally and not as surfaces.

The steps of hanging stairs, whether moulded or plain, are girted at their mean breadth, including both joints, and for their length what is seen, including 6 inches of rests in the wall. The soffit and ends of wheel steps are taken over again, so far as is without the walls; and the ends of both square and wheel steps are taken at their extreme breadth and depth.

The joints of platts, if joggled, are also taken.

The skirting of hanging stairs is taken by the extreme length and breadth of the stones, including the upper edge.

The steps and platts of newel stairs are taken by girting at the mean breadth, allowing 1 inch of overlap on each step, and for the length by what is seen, allowing 6 inches on each end for rests. The newels are girted round, including the backset. The tails are taken as scribbled work, and the soffits of steps according to the kind of work upon them.

Pyramids or obelisks, if they are built in courses, are girted for the length at the bottom of each course, and between the joints for the breadth. When they are made of one stone, they are girted for the length at the bottom, and for the breadth by the sloping height; and if they are polygonal figures, the peends or angles are generally allowed about 3 inches broad on each angle over and above the net girts.

In measuring curb-stones, besides the upper bed, 6 inches are allowed on the edge, of the same work with the upper bed.

Hewn work of every kind, as well as coursed, hammer-dressed, or scribbled work, is measured by the superficial foot.

NOTE. In measuring rough-casting, the whitewashing on the faces and breasts of rybats, belts, chimney-copes, &c. is taken as rough-casting.

1. How much rubble work of the standard thickness of 2 feet is in a house of 3 stories, 60 feet long and 30 broad within walls, the height 30 feet from the foundation to the top of the side-walls, 12 more to the foot of the chimney-stacks, which are 7 feet high, 10 broad, and 3 thick, the skews are

14 feet 6 inches long, the side-walls $2\frac{1}{2}$ feet thick, and the end-walls 3 feet thick, with two doors, each 7 feet by 4 feet, 22 windows, each 5 feet by 3 feet, and 12 windows, each 4 feet by $2\frac{3}{4}$ feet?

A side-wall,	$66\times30\times1\frac{1}{4}$	$=2475$ sq. feet.
An end-wall,	$30\times30\times1\frac{1}{2}$	$=1350$
A gable-end,	$\frac{1}{2}(35+10)=22\frac{1}{2}\times12\times1\frac{1}{2}$	$=\ 405$
A chimney-stack,	$10+3=13\times7\times1\frac{1}{2}$	$=\ 136\frac{1}{2}$
		$4366\frac{1}{2}$
		2
		8733
Levelling side-walls,	$60\ \times2\times2\frac{1}{2}$	$=\ 300$
Ditto for joists,	$60\ \times\frac{3}{4}\times2$	$=\ 90$
Ditto 4 skews,	$14\frac{1}{2}\times4\times3$	$=\ 174$
Ditto chimney-tops,	$10\ \times2\times3$	$=\ 60$
		9357

2 doors,	$7\ \times4\times\ 2$	$=\ 56$	
22 windows,	$5\ \times3\times22$	$=330$	
12 windows,	$2\frac{3}{4}\times4\times12$	$=132$	
		518	
Add $\frac{1}{4}$ for thickness,		$129\frac{1}{2}$	
			$647\frac{1}{2}$

Content 26 roods 31 yards $6\frac{1}{2}$ feet, $=\ 8709\frac{1}{2}$ sq. feet.

2. How much hewn work in 22 window-lintels and sills, each 4 feet by $1\frac{2}{3}$ foot; 12 lintels and sills, each $4\frac{1}{2}$ feet by $1\frac{1}{3}$ foot; two door-lintels and sills, each 5 feet by $1\frac{2}{3}$ foot; 22 pairs of rybats of 5 feet, 17 rybats of 4 feet, and 2 ditto of 7 feet, all of them 14 inches broad; skews 64 feet by 14 inches, coping of chimney-stacks 44 feet by 20 inches, and coping of the roof 60 feet by 16 inches?

Ans. $687\frac{1}{3}$ square feet.

3. What should be charged for the workmanship of a house of 2 stories, 36 feet long and 24 feet broad within the walls, the side-walls 2 feet thick and 24 feet high, measured from the foundation; the gables 3 feet thick and 16 feet higher than the side-walls to the bottom of the chimney-stacks, which are 7 feet broad, 3 deep, and 8 high; the skews are 19 feet 3 inches in length, and there are 110 feet of flues; the rubble work, reduced to the standard, is at £2 per rood, and the flues at 4d. per foot? Ans. £36, 5s. 11d.

4. A house of 3 stories is 45 feet long and 28 feet broad within walls, and the height from the foundation to the top of the side-walls is 30 feet; the gables rise 18 feet above the side-walls to the bottom of the chimney-stacks, which are 8 feet wide, 3 deep, and 10 high; the skews are 21 feet 11

inches long; the side-walls are $2\frac{1}{2}$ feet, and the gables are 3 feet thick; there are 2 doors in the sides, each $7\frac{1}{2}$ feet by 4 feet; 12 windows in the sides, and 6 in the ends, each 6 feet by $3\frac{1}{2}$ feet. Required the expense of the materials and workmanship of the rubble work, at £10, 6s. 8d. per rood, allowing £2, 14s. per rood for levelling the side-walls.

Ans. £234, 15s. 1d.

5. A house is 41 feet long, $20\frac{1}{2}$ feet broad within the walls, and 18 feet 9 inches high from the foundation to the top of the side-walls, which are 2 feet thick; the gables are $2\frac{1}{2}$ feet thick, and rise 8 feet 6 inches above the side-walls to the bottom of the chimney-stacks, which are 4 feet wide, $2\frac{1}{2}$ feet thick, and 5 feet 1 inch high. The broached hewn work consists of 4 skews, each 11 feet 6 inches by 1 foot 7 inches; corners, each 18 feet 9 inches by $2\frac{1}{2}$ feet; and 2 chimney-stacks, the girt of each 13 feet, and the height 5 feet 3 inches. The droved hewn work consists of the rybats and lintels of 6 windows, each 13 feet 11 inches by 15 inches; 6 sills of ditto, each 3 feet 11 inches by 19 inches; the rybats and lintels of one window, 9 feet 3 inches by 15 inches; sill of ditto, $3\frac{1}{4}$ feet by 19 inches; the rybats and lintel of a door, $19\frac{1}{4}$ feet by 15 inches; sill of ditto, $4\frac{1}{4}$ feet by 19 inches; 3 pairs of jambs, each 6 feet by 2 feet; the lintels of ditto, each 4 feet inches by 15 inches; 3 inner hearths, each 3 feet 1 inch by 3 inches; 3 outer hearths, each 3 feet 8 inches by 20 inches; kitchen jambs, 8 feet 8 inches by 2 feet 3 inches; lintel of ditto, 5 feet 8 inches by 15 inches; the hearth, 4 feet by 21 inches; and also $106\frac{1}{2}$ feet of flues. Required the content of the rubble work, and of the hewn work, and also the expense of the workmanship; the rubble work being at £3 per rood, the broached hewn work at 5d. per foot, the droved hewn work at 8d. per foot, and the flues at 6d. per foot.

Ans. 11 roods 1 yard 2 feet $4\frac{1}{4}$ inches rubble work; 396 feet 10 inches broached hewn work; 307 feet $5\frac{1}{4}$ inches droved hewn work. Expense of the whole £51, 14s. 5d.

BRICK WORK.

Brick Work is measured by the square yard, and reported brick on edge or brick on bed, 9 inches or 14 inches thick; and all above that is reduced to 14 inches as the standard.

Brick walls are measured the same way as stone walls, and the net daylight of all apertures deducted.

Upright circular walls and arches are allowed measure and half; and arches over apertures in upright walls are taken over again. Groin-arches are double measure; and 18 inches by the length and thickness allowed on the groin for cutting.

The tops of niches and spherical arches, whether of brick or stone, are allowed three measures.

When the skews on brick gables are feathered on edge or feathered and tongued, they must be taken over again; and in all cases $4\frac{1}{2}$ inches by the thickness of the skew above the thackgate is allowed for cutting. Chimney-stacks are taken by the height and by the breadth, adding the thickness of one haunch, if it does not exceed 18 inches; and in all above that thickness one-half of the haunch is added.

1. The sides of a brick vault are 18 feet long, 5 feet high, and 2 bricks thick; the girt of the arch 10 feet, and 1 brick thick; the end walls 8 feet long, 7 feet high, and $1\frac{1}{2}$ brick thick; the door 5 feet by $2\frac{1}{2}$ feet. How much does the vault measure at standard thickness?

Ans. 56·6516 square yards.

2. How many square yards of standard brick work in a wall 75 feet long, 15 feet 9 inches high, and 3 bricks thick?

Ans. 262 yards $4\frac{1}{2}$ feet.

3. A garden is 160 feet broad, and contains an imperial acre. Required the expense of enclosing it with a brick wall 10 feet 6 inches high, and $2\frac{1}{2}$ bricks thick, at 5s. $7\frac{1}{2}$d. per square yard of standard thickness, deducting 2 doors, each 6 feet 9 inches by 4 feet, and a gateway 11 feet wide.

Ans. £468, 0s. $11\frac{1}{4}$d.

CARPENTERS' AND JOINERS' WORK.

Common rough joisting is measured by adding to what is in sight the rests or hold of the joisting in the walls; and when that cannot be ascertained, an allowance not exceeding one foot on each end is made, and the content is estimated in square yards, stating the size and distance.

Framed joisting is measured in the same way for the scantling or bridged joists, and the surface-measure includes the beams. But beams and transoms are measured by the cubic foot. When joists are laid on the tops of walls, and the ends of couples joined to them, or when beam-fitted, and the wall plates fixed down to them,—in both cases they are taken as joisting.

Trussed and dressed beams are measured by the cubic foot, and the oak in trussed beams is reported lineal, stating the se. Dwangs put between joists are classed with rough tim-er, such as safe-lintels, &c.

Deafening-boards are measured superficially; and when e composition is laid the hearth-places are deducted, and e boxing for the hearths stands as an equivalent for the floor.

Flooring is measured superficially, and reported according its quality, as deal, or batten flooring, &c. No deduction made for hearths where there is strong boxing under them t measured separately; but when that is the case, the arths are deducted. When floors are cut to any angle or cle at or exceeding 45°, an allowance of 6 inches by the gth is made for cutting. Hearth-borders are taken by the eal foot.

Framed and bound roofing is measured by taking all the ncipal timbers that are connected with the main couples, the cubic foot, and also the extra size of diagonals, when ey are above 9 inches by 3 inches, and reported as cubic med timber.

The surface of a square roof is measured by taking twice depth from ridge to eave, and by the length from skew skew.

A pavilion or hipped roof is taken by adding the length breadth at the eaves to the length at the ridge, or to the gth and breadth of the platform, and by the depth from ge to eave. The platform is taken as flooring and joisting. pavilion square roof, finishing in a point at the top, is taken girting one end and one side at the eave for the length, the depth, as before.

A conical or turnpike roof is measured by taking the cir-nference at the eaves, and by the slant height.

Eighteen inches by the length are allowed for each hip and ey. All openings for dormer-windows, skylights, and mney-stacks are deducted, except when the opening is at under 2 feet square; and when such deductions are made, ches for the width of it are allowed at top and bottom, bridling.

The contents of the wall-plates, including the sleepers built r fixing them, are added to the surface-measure of the unless when the wall-plates are above 2 inches thick; in h case they are taken as rough cubic timber.

The putting on of the iron-work of framed roofs ought to ncluded in the price; and, if furnished by the tradesman, ged by weight.

When there are two baulks in a common roof, the upper are included, and the under ones taken as joists, men-ng their size and distance.

Roofing and tile-lath are also measured superficially; and when sarking is put on slate eaves, it is measured as sarking only. Roofs upon circular walls are allowed double measure, and all domes three measures.

Roofs put upon polygons, when the scantlings are curved, are allowed double measure.

Battens for ridges and hips, whether square or rounded, and filleting for skews, are measured by the lineal foot, specifying the size.

Framing for brick partitions, if the standards are placed at regular distances, is measured superficially by the yard, as brick on edge, brick on bed, stating the distance of the standards; and when dressed door-standards are placed in such partitions, the apertures are deducted over the dressed standards.

When there are only a few detached standards in partitions, these are calculated to 3 inches square, and reported as rough standards; and the warpings, in that case, are to be reduced to 4 inches broad, and their thickness stated.

Standards for lath partitions are measured by the square yard, stating their size and distance, and deducting the doors over the door-standards, where dressed ones are placed. Run-trees at top and bottom (if any) must be reduced to 3 inches square, or 4 inches broad, stating the thickness.

Wall-battens for lath are measured by the superficial yard, and, if fixed with plugs, or iron holdfasts, are reported as such.

All standards set circular are allowed measure and half. Bond-timbers are taken by the lineal foot, stating the general size.

Lathing is measured by the square yard, and, when on circular walls, allowed measure and half. All arches and coves are allowed the same.

Domes and tops of niches are double measure, unless otherwise specified. An allowance of 6 inches by the length is made for cutting round circles and angles; and all apertures are deducted.

Dressed door-standards in brick or lath partitions are measured by their actual height, and, along with the lintels, reduced to 3 inches square.

All dressed posts or standards below 6 inches square are reduced to 3 inches square, and those at 6 inches square and above are reduced into cubic feet.

Dressed deal door-breasts or standards, not exceeding 8 inches broad and $1\frac{1}{4}$ inch thick, are reduced to 4 inches broad, and reported according to the thickness. All above 8 inches broad are reported by the superficial foot, as an rticle by itself; and all above $1\frac{1}{4}$ inch thick are reduced to 3 aches square.

Grounds are measured by the lineal foot, specifying whether hick or thin, or if checked or grooved.

Sash-windows are allowed 2 inches more than the daylight r the height, and 3 inches for each side-facing more than the aylight for the breadth, when they are not more than 3 feet ride: all above that are allowed 1 inch on each side of the acing for every foot in width.

Windows with circular tops are allowed double measure for ne circular part. Convex or concave windows are double easure; and, if made to fit an arch on the top, the arched art is taken at its extreme height and breadth, and allowed hree measures. Flat segment-topped windows are allowed inches for cutting; and, when the panes are square, are aken as windows without glass. Cupola lights with curved bs, or astragals, are allowed three measures; but when raight, only two. Common skylight hatch-windows are aken by the net surface.

The sash-part of doors is measured by adding as much of ne belt-rail to the height as the breadth of the stiles, and the emaining part is taken as bound work.

Chinese sash-lights are allowed double measure when the anes are of various figures, or circular; and if in a circular oor, three measures are allowed; but only single measure hen the panes are all one figure and one size.

Bound doors are measured by adding as many inches to the eight as there are pannels in the height, and by the net readth; and when the thickness is at or above $1\frac{3}{4}$ inch, ouble measure is allowed; below that thickness, when dressed n both sides, measure and half; but when dressed only on ne side, no more than single measure.

Bound window-shutters are measured in the same way: if at, two thicknesses are added to the length; and if checked r backfolds, the girt of one checked edge is added to the et breadth of both shutters.

Bound flush-and-bead shutters are measured by the square ot, specifying the thickness. Plain deal backfolds have the readth of the cross heads added to the height, and are reorted by the yard; and, if not more than 6 inches broad, ney are reduced to 4 inches broad.

All circular bound work is allowed double measure.

Bound flush-and-bead doors, having two leaves, are measured like shutters with backfolds; but in shop-doors the sash deducted from the bead-and-flush, and the sash and shutters aken by themselves.

Torus mouldings on bound work are taken by the lineal foot.

Plain deal-backed work, double deal doors, &c., are taken by the square foot; and if beaded on the joints, it should be specified, as well as the thickness.

Common plain deal, if dressed on both sides, is allowed measure and half, and reported by the square yard, stating the thickness; and whenever beads are put on the joints, half an inch is allowed in the measure for each bead.

Bound dado-lining is allowed, on the length, one inch for every external corner for nailing, and an inch of cover for every architrave; and on the height, besides an inch for the pannels, 3 inches more than the net measure between the base and surbase, and no notice taken of the stile-ends.

Plain dado and window linings, when done in a superior manner, are reported as such by the yard, stating the thickness, and the bars behind are included in the price.

Shelving in general is taken by the yard, stating the thickness. When cut circular, the net area is taken, and an allowance of 3 inches on each edge for cutting; and when circular on one edge, an allowance of 6 inches is made for cutting. When shelves are wrought on both edges, they are allowed measure and half; and grooves for shelves are reported by the lineal foot.

Plain deal work, dovetailed, is measured by the superficial foot, stating the thickness and quality; and all broad plain deal work, of whatever description, above 1½ inch thick, is taken by the square foot, and the thickness stated.

Mouldings are taken by their greatest length, and for their breadth by girting over the mouldings, allowing an inch more than is seen on base-mouldings for a rest on the plinth, and another allowance of one foot for every mitre more than four, on base and surbase in one room.

The blocks on which architraves are set are included in the height of the architraves, and then taken over again as skirting, along with the base-plinth.

Cornices of doors and chimney-pieces are taken at their greatest projection for the length, and by girting the moulding for the breadth; the upper bed being taken as moulded work as far back as the projection, the remaining part to be of plain deal, if there be any.

The frieze-board is taken as plain deal, by the square foot, including what is behind the cornice; but when the frieze is under 6 inches broad, with an astragal at the bottom, the whole is taken as mouldings.

All mouldings, except small single ones, are estimated by the superficial foot; dentils, Doric bells, &c. by the lineal foot.

The shafts of plain pilasters are taken by their extreme height and breadth, and estimated by the square foot, stating the thickness, and both edges are girted on the face. Fluted pilasters are taken in the same way, girting over the fillets and into the flutes; and if the edges or returns are fluted, they are also girted in; but if they are planted returns, not fluted, they are taken as plain work, when above 2 inches broad. Cabled or reeded pilasters are taken as such, and the thickness in all cases stated. The bases and capitals of both plain and fluted pilasters are taken by themselves, as the mouldings of the pilasters.

Solid columns are taken by their height and greatest diameter; and when their mouldings are turned out of the solid, the diameter is taken at the base. The shafts of built columns are taken superficially by the whole height, and by the girt of the greatest diameter, and allowed two measures. When columns are fluted and reeded, they are taken as such; and if reeds are planted in, they are taken lineally. The bases and capitals are measured as mouldings, and the circular part only allowed double measure.

Facings, skirtings, base-plinths, and door-stops, under 8 inches broad, are reduced to 4 inches broad; and all above 8 inches broad are taken as plain linings.

The stanchel part of railing is taken by the yard, stating the size of the stanchels and the distance between them; the posts and rails are reduced to 4 inches broad. The posts of the rail are included in the surface-measure.

The Chinese part of railing is measured by the square yard, as such, the posts reduced to 3 inches square, and the rails to 4 inches broad, stating the thickness.

The square steps of timber stairs are taken by their length and by girting over the step and breast, allowing an inch of cover to each. The wheel steps are taken at their extreme length, and by girting at the mean breadth, allowing 3 inches to each step for cutting. Spring-boards and brackets are taken by the square foot, specifying their thickness.

Stair hand-rails are taken by the lineal foot, stating the quality. Circular parts are double measure, twist and circle three measures, and the measure taken round the scroll.

1. What is the value of a sash-window which measures 6 feet 10 inches by 3 feet 8 inches, at 2s. per square foot?

Ans. £2, 10s. $1\frac{1}{3}$d.

2. How many square yards of roofing and sarking are in a house 60 feet long from skew to skew, and each side of the

roof 22 feet, allowing 9 inches for the breadth of the wall-plate; and what is the value of it, at 9s. 6d. per square yard?

Two sides 45 feet 6 inches.
Length 60 0

9)2730(303⅓ square yards at 9s. 6d. = £144, 1s. 8d.

3. How many yards of flooring in a house of three stories, 56 feet by 28 feet within the walls, deducting the vacancy for the stair, 13 feet by 8 feet; and what is the value, at 5s. 6d. per square yard? Ans. £134, 4s.

4. How much wainscoting in a room 25 feet by 18 feet, and 14 feet 3 inches high when girt over the mouldings, allowing a door 7 feet 2 inches by 3 feet 4 inches, 2 windows with shutters, each 5 feet 8 inches by 3 feet 6 inches, and a chimney 6 feet 4 inches by 5 feet 6 inches; the doors and shutters being charged work and half-work?

Ans. 135 yards 7 feet 5½ inches.

5. A partition is 173 feet 10 inches in length, and 10 feet 7 inches in height. How many squares are in it?

Ans. 18·397361 squares.

6. How many yards of flooring and joisting in a house of 3 floors, 48 feet by 27 within walls, allowing 9 inches for the rests of the joists, and deducting from each floor the vacancy for the stair, 12 feet by 8 feet 3 inches; and what is the expense of the materials and workmanship, the joisting and flooring at 7s. 6d. per yard, and the naked joisting at 3s. 6d. per yard? Ans. £153, 16s. 6d.

PLASTER WORK.

Plain plaster work is measured by the square yard, stating the number of coats and the quality of the finishings.

Upright circular walls, soffits of arches, coves, &c. are allowed double measure. Domes and tops of niches are allowed three measures. When new and old plaster are joined, an allowance is made of one foot for splicing; and when mouldings are put on plain plaster, to form pannels, the whole wall is taken as plain plaster, and the mouldings are taken again by the lineal foot.

Where stiles are raised, the general superficies of the wall is measured as pannelled plaster. The stiles and mouldings are taken by the lineal foot, stating the breadth.

These rules apply to ceilings as well as walls, and to mouldings, whether plain or enriched.

All circular mouldings on domes are double measure. Pannelled soffits of arches, and pannelled scontions of stair-windows are taken by girting over the mouldings both ways; and if at or above 12 inches broad, they are estimated by the square foot; but if under 12 inches, by the lineal foot, stating the breadth.

Architraves of arches are taken as other mouldings.

Plain cornices, at or above 12 inches in girt, are taken by the square foot, and all under that by the lineal foot.

Enriched cornices are measured in the same way, stating the number and nature of the enrichments; and for all mitres in a room, &c. more than four, one foot is allowed for each, whether external or internal.

Plain and enriched entablatures are measured by the square foot, by girting from the ceiling down to the plain plaster of the walls; and the number and quality of the enrichments are stated.

Entablatures on the bottom of coves are measured on the upper bed, as far as the mould goes back, and down to the plain plaster.

If the ornaments and mouldings on a ceiling do not exceed 12 inches in their distance from each other, the whole ceiling is taken by the superficial foot, as an ornamented one; but when their distance exceeds 12 inches, the mouldings and margins are taken in the same way as pannelled plaster.

Centre ornaments above 3 feet diameter are taken by the square foot, and all at or under that by the piece, stating the size.

Heads, trusses, and other detached ornaments, are reported by the number and size.

Plaster beads are taken as plain mouldings, and relieved corner beads by the lineal foot, as double cut.

1. How much plastering on a partition 7 feet 8 inches long and 10 feet 3 inches high, deducting a door 6 feet 3 inches by 2 feet 10 inches; and what will it cost, at 5d. per yard?

10 feet 3 inches.			6 feet 3 inches.	
7	8		2	10
78	7 wall.		17	8½ door.
17	8½ door.			
9)60	10½ content.			

6 yards 6 feet 10½ inches content, at 5d. is 2s. 9¾d.

2. How many square yards of plastering on the walls and ceiling of a room 30 feet long, 25 broad, and 12 high, deducting 3 windows, each 8 feet 2 inches by 5 feet, 2 doors, each

7 feet by 3 feet 6 inches, and a fireplace 4 feet 6 inches by 4 feet 10 inches, the sides of the windows behind the shutters being plastered, and measuring 8 feet 2 inches by 15 inches; and what will it cost, at 6¼d. per square yard?

Ans. 215 yards 3 feet, cost £5, 12s. 1¾d. ⅓.

SLATERS' WORK.

SQUARE roofs are girted for their deepness from the top of the ridge downwards, allowing 9 inches for the double eaves, and for the length between the skews, and 6 inches more for cover.

Chimney-stacks, and all apertures above 4 square feet of daylight, are deducted, allowing the double eaves above such openings, and also 9 inches for cutting along each side; but no deductions are made at or under 4 square feet.

Stormont and roof windows are measured according to the form of the different parts, and 9 inches by the length allowed for every cutting on peends, flanks, and skews.

Close flanks made waterproof without lead are allowed double of a common flank for cutting.

Circular work and dome roofs are double measure. Ridge stones are reported by the lineal foot.

Tile roofs are measured in the same way as slate roofs, but no allowance for double eaves, unless when slate eaves are put on, in which case 6 inches more than what is seen is allowed on the slating for cover.

The pointing of slate or tile roofs is measured as before stated, but no allowance for cutting or for eaves. The deepness of the plaster is to be added to the length of the roof.

Slate and tile roofs are estimated by the rood of 36 square yards.

1. How much slating is in a roof 46 feet long, and 18 feet from the coping to the eaves? Ans. 5 roods 11 yards 6 feet.

2. Required the content of a tile roof 42 feet 7 inches long, and 16 feet 10 inches from the ridge to the eaves; and what does it amount to, at £3, 15s. per rood?

Ans. 4 roods 15 yds. 2 ft. 7 in. 8 pts. cost £16, 11s. 10¼d.

3. Required the expense of a slate roof measuring 48 feet 6 inches in length, and 24 feet from ridge to eaves, breadth of the wall-plate 9 inches, reckoning the roofing and sarking at 7s. per square yard, and the slating, including slates, at £5, 8s. per rood. Ans. £133, 7s. 6d.

PAINTERS' WORK.

PLAIN painting is measured wherever the brush touches, and estimated by the square yard, stating the colour and quality, whether oil or size, and the number of coats.

Party-coloured work is measured first as plain work, and then the stiles and mouldings are taken and estimated by the lineal foot, according to the number of different colours; and this rule applies to skifting and mouldings of a room, when different colours form the general body of the work.

An allowance of 6 inches for each enrichment in cornices is added to the girt, when enriched cornices are picked in; and if at or above one foot of girt, they are taken by the superficial foot; all under that girt by the lineal foot. In both cases, the number of enrichments are to be stated, besides being included along with the plain work with which they may class.

Ornamented ceilings are measured in the same way as plaster work.

Mock mouldings in passages, staircases, &c. are reported by the lineal foot. Outsides of windows are allowed one-fourth more than the net daylight.

Stanchel-railing, at or under 6 inches in the open, is allowed double measure; above 6 and under 9 inches, measure and half; from 9 to 12 inches, one and one-fourth; and all above that, single measure. Stanchels put into windows are taken by including one of the side spaces between the stanchel and the rybats.

Ornamented railing on stairs is allowed double measure, and figures of every description are reported by number.

1. How much painting on a wall 14 feet by $9\frac{1}{2}$ feet, deducting the chimney, 4 feet 6 inches by 3 feet 10 inches; and what does it come to, at 10d. per square yard?

Ans. Content 12 yards $7\frac{3}{4}$ feet, value 10s. $8\frac{1}{2}$d.

2. A room is 20 feet long, 14 feet 6 inches broad, and 10 feet 4 inches high. How much painting is in it, deducting a fireplace 4 feet 4 inches by 4 feet, and 2 windows, each 6 feet by 3 feet 2 inches? Ans. 73 yards $0\frac{2}{3}$ foot.

3. Required the expense of painting a room 28 feet long and 20 broad, the girt of the wainscoting or *dado-work* round the bottom of the room 2 feet 10 inches by 84 feet; the height from the wainscoting to the ceiling 7 feet 10 inches; 3 windows, each 7 feet 10 inches by 4 feet 9 inches; 2 doors, and 2 presses, each 7 feet 6 inches by 4 feet; and a fireplace 4 feet 9 inches by 5 feet. The wood work is painted in oil,

the window-shutters and doors on both sides, at 9d. per square yard; the walls with size at 3d., and the ceiling is white-washed at $1\frac{1}{2}$d. per yard. Ans. £4, 1s. $11\frac{3}{4}$d. $\frac{2}{3}$.

GLAZIERS' WORK.

GLASS is measured by the superficial foot, stating the quality. Every pane is measured at the extreme points, including the back-check of the astragal.

1. A window is 5 feet 4 inches by 3 feet 2 inches of day-light. What does the glazing amount to at 14d. per square foot? Ans. Content $16\frac{8}{9}$ feet, value 19s. $8\frac{1}{2}$d.

2. An oval window is 4 feet 3 inches by 2 feet 5 inches. Required the expense of glazing it, at 1s. 3d. per square foot. Ans. Content $10\frac{13}{48}$ feet, value 12s. 10d.

3. Required the expense of glazing the windows of a house of three stories, at 1s. 4d. per square foot, the common breadth of the windows being 3 feet 10 inches; the height of the lower tier 7 feet 8 inches, of the second 6 feet 10 inches; and of the highest 5 feet 3 inches; 4 windows in each tier.

Ans. £20, 3s. $9\frac{1}{3}$d.

PLUMBERS' WORK.

PLUMBERS' WORK is generally done by the pound or hundred-weight; but the laying down of lead is done by the day.

Sheet-lead used in roofing, &c. weighs from 7 to 12 lb. per square foot. Leaden pipes of $\frac{3}{4}$ inch bore weigh 10 lb.; of 1 inch bore, 12 lb.; of $1\frac{1}{4}$ inch bore, 16 lb.; of $1\frac{1}{2}$ inch bore, 18 lb.; of $1\frac{3}{4}$ inch bore, 21 lb.; and of 2 inches bore, 24 lb. per yard, in length.

1. Required the expense of a leaden pipe of $1\frac{1}{4}$ inch bore, and 72 feet long, at $3\frac{1}{4}$d. per lb. Ans. £5, 4s.

2. Required the expense of lining a water-cistern 2 feet 10 inches long, 2 feet 6 inches deep, and 2 feet broad, with sheet-lead of 10 lb. to the square foot, at £1, 18s. 9d. per cwt.

Ans. £5, 3s. $2\frac{1}{2}$d. $\frac{13}{98}$.

3. The platform on the roof of a square building measures 40 feet square, and is covered with lead of 9 lb. to the square foot; the hips are each 16 feet 6 inches long, and covered to the breadth of 18 inches with lead of 10 lb. to the square foot; the water-pipe is of 1 inch bore and 48 feet long, and the soil-pipe is of 2 inches bore and 30 feet long; the water-cistern is 3 feet 6 inches long, 2 feet 6 inches deep, and 3 feet

wide, and lined with lead of 11 lb. to the square foot. Required the expense of the whole, the sheet-lead being rated at £1, 11s. 6d. per cwt., and the pipes at 4¾d. per lb.

Ans. £231, 12s. 5½d. ½.

PAVIERS' WORK.

CAUSEWAYING is measured by the rood or yard, stating whether rubble or coursed work. One foot by the length is added as an allowance for every channel, and 6 inches by the length for cutting on coursed work, and for warpings.

Hewn pavement is measured by the square foot, stating the quality; and, if grooved pavement, the grooves are added to the surface measure.

The hollow part of gutters cut in pavement is taken over again; and sinks are taken two times, after being included in the surface-measure.

1. A court-yard is 50 feet long by 40 feet 6 inches broad. What will the paving of it amount to, at 3s. 7½d. per square yard? Ans. £40, 15s. 7½d.

2. What will be the expense of paving a square court, the length of the side being 150 feet? The outside, to the breadth of 10 feet, is paved with Arbroath pavement at 3s. per square yard, and the rest is done with common pavement at 1s. 9d. per yard. Ans. £257, 12s. 9¼d.

3. A hexagonal space, the outside of which to the breadth of 12 feet, in a line from the corner to the centre, is to be paved with Arbroath pavement at 2s. 10½d. per yard; the remainder, deducting a circular garden in the centre, of 300 feet diameter, is to be done with common pavement at 1s. 8¾d. per yard. Required the amount of the expense, supposing the length of the side 250 feet. Ans. £977, 14s. 1d.

OF VAULTS.

VAULTS are formed by arches springing from opposite walls, and meeting in a line at the top.

PROB. I. To find the surface of a vault.

RULE. Apply a line close to the arch, from one side to the other, to get the girt, and multiply it by the length of the vault to get the surface; and this, multiplied by the thickness of the arch, will give the solid content of the arch.

1. Required the surface of a vault 106 feet long, and the girt of the arch $42\frac{2}{3}$ feet. Ans. $499{\cdot}3\dot{7}$ yards.

2. Required the surface of a vault 56 feet long, the girt of the arch 36 feet 4 inches; and also the solidity of the arch, its thickness being 2 feet.

Ans. $226\frac{2}{27}$ yards surface, 150 yards $19\frac{1}{3}$ feet solidity.

3. Required the surface of a vaulted roof, the length being 125 feet, and the girt 36 feet. Ans. 500 square yards surface.

Prob. II. To find the concavity of a vault.

Rule. Find the area of one of its ends according to its form, whether circular, elliptical, or Gothic, and multiply it by the length of the vault.

1. Required the content of a semicircular vault, the span being 30 feet, and the length 150 feet.

Ans. 53014·5 cubic feet.

2. Required the content of an oval vault, the span being 30 feet, the height 12, and the length 60 feet.

Ans. 16964·64 cubic feet.

3. Required the vacuity of a Gothic vault 20 feet long, the span 50 feet, the chord of each of the arches 50 feet, and the versed sine of the arch 15 feet. Ans. 43024·215 cubic feet.

OF GROINS.

Groins are formed by the intersection of vaults with one another.

Prob. I. To find the surface of a groin.

Rule I. Divide the area of the base by 7, and add the quotient to the dividend: the sum will be the area.

Note. This rule is correct only when the groin is a semicircle.

1. Required the surface of a groin raised upon a square, of which each side is 14 feet. Ans. 224 square feet.

2. Required the surface of a groin raised upon a rectangular base, of which the sides are 14 and 18 feet.

Ans. 288 square feet.

3. Required the surface of a circular groin-arch raised on a square base, each side 20 feet. Ans. $457\frac{1}{7}$ square feet.

Rule II. Multiply the square of the base by 1·1416 for the surface of the groin-arch, and add to the product twice the product of the diameter of the four semicircular spaces between the piers into their breadth, and into 3·1416 for the whole surface required.

1. Required the surface of a groin-arch, 30 feet square, having 4 semicircular spaces between the piers, each 30 feet in diameter and 18 inches broad.

$30 \times 30 \times 1{\cdot}1416 = 1027{\cdot}44$ square feet, the groin, and $30 \times 2 \times 1{\cdot}5 \times 3{\cdot}1416 = 282{\cdot}744$ feet, surface of semicircular spaces; then $1027{\cdot}44 + 282{\cdot}744 = 1310{\cdot}184 = 1310{\cdot}184$ square feet, the whole surface.

2. Required the surface of a groin-arch, 33 feet square, having 4 semicircular spaces between the piers, each 33 feet in diameter and 20 inches broad. Ans. 1588·7784 sq. feet.

Prob. II. To find the solidity of the masonry in a semicircular groin-arch.

Rule I. Multiply the product of the length and breadth by the height from the springing of the arch to the top; from this product subtract the square of the inside measure, multiplied by the height within, and by $\cdot 9041\dot{3}$; subtract also the four semicircles between the piers, and the last remainder will be the solid content of the arch when made up level to the crown of the arch.

1. There is a semicircular groin-arch, 15 feet high, and the opening within, *ei*, 30 feet; the arch A*e*, or *i*B, is 18 inches thick, and the square of the arch over the piers ABCD is 33 feet in the side. Required the cubic content of the arch.

First $33^2 \times 16{\cdot}5 = 17968{\cdot}5$, and $30^2 \times 15 \times \cdot 9041\dot{3} = 12205{\cdot}8$; also $30^2 \times \cdot 7854 \times 2 \times 1{\cdot}5 = 2120{\cdot}58$; then $17968{\cdot}5 - (12205{\cdot}8 + 2120{\cdot}58) = 17968{\cdot}5 - 14326{\cdot}38 = 3642{\cdot}12$ cubic feet, the solidity.

Note. As the arch is generally of a better description of materials than the making up of the corners, it is therefore necessary sometimes to find the arch separately, which may be done by the following rule.

Rule II. Find the cubic content of a hollow cylinder, whose length and diameter over all is the same as that of the arch, and whose inside diameter is the same as the inside of the arch. Square the diameter over all, and multiply the product by the half of the same diameter; from this product subtract the square of the inside diameter, multiplied by its half, and subtract $\frac{2}{3}$ of the remainder from the content of the cylinder: the remainder is the content of the groin-arch, including the four spaces between the piers.

Taking the last example, we have $33^3 \times \cdot 7854 - 33 \times 30^2 \times \cdot 7854 = 4898{\cdot}5398$ the content of the cylinder; then $33^2 \times 16{\cdot}5 - 30^2 \times 15 = 4468{\cdot}5$, $\frac{2}{3}$ of which is 2979; hence $4898{\cdot}5398 - 2979 = 1919{\cdot}5398 =$ cubic feet, content of the arch, which, taken from the content found by Rule I. or $3642 \cdot 12 - 1919 \cdot 5398 = 1722 \cdot 5802$ cubic feet, the filling up of the corners.

2. There is a semicircular groin-arch, 25 feet opening within, and $12\frac{1}{2}$ feet high, the arch is 18 inches thick, and the square of the arch over the piers is 28 feet. Required the cubic content of the arch. Ans. 1387·6 cubic feet.

Prob. III. To find the vacuity of a groin.

Rule. Multiply the area of the base by the height, and from the product subtract $\frac{1}{10}$ of it: the remainder will be the solidity.

Note 1. Instead of subtracting $\frac{1}{10}$ of the product, it may be multiplied by ·9, or by ·904.

Note 2. This rule is correct only when the groin is a semicircle.

1. Required the vacuity of a circular groin upon a square base, of which the side is 14 feet, and its height 7 feet.

Ans. $14^2 \times 7 - 14^2 \times \cdot 9 = 1234 \cdot 8$ cubic feet.

2. Required the vacuity formed by an elliptical groin, the side of its square base being 28 feet, and its height 9 feet.

Ans. 6350·4 cubic feet.

3. Required the vacuity of an elliptical groin upon a rectangular base 20 feet by 30, and the height 12 feet.

Ans. 6480 cubic feet.

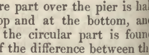
OF BRIDGES.

Prob. To measure the spandril walls of a bridge when they are thicker at the bottom than the top.

Rule. Find the areas of each of the pieces separately, multiply each by its mean thickness, and the sum of the products is the content.

The mean thickness of the centre part over the pier is half the sum of the thickness at the top and at the bottom, and the mean or average thickness of the circular part is found by adding to the top thickness $\frac{1}{4}$ of the difference between the thickness at the bottom and that at the top.

1. The wall ABCD over the pier of a bridge is 30 feet high and 10 feet broad, the thickness at the bottom is 2 feet, and that at the top 1 foot 6 inches; the two quarter circles EAC, and BFD are the same thickness, and each 30 feet by 30 feet. Required the content of the masonry in cubic feet.



First $\frac{1}{2}(2 + 1\cdot5) = 1\cdot75$ mean thickness of the wall over the pier, and $1\cdot5 + \frac{1}{4}(2 - 1\cdot5) = 1\cdot5 + \cdot125 = 1\cdot625$ foot mean thickness of the quarter circles; then $30 \times 10 \times 1\cdot75 = 525$ cubic feet over the pier, and $(30^2 - 30^2 \times \cdot7854) \times 2 = 193\cdot14 \times 2 = 386\cdot28$ the area of both sides of the circular part, and $386\cdot28 \times 1\cdot625 = 627\cdot705$; then $525 + 627\cdot705 = 1152\cdot705$ cubic feet the content.

2. The wall over the pier of a bridge is 40 feet high and 12 feet broad, the thickness at the bottom 2 feet 6 inches, and at the top 1 foot 9 inches; the two quarter circles are the same thickness, and each 40 feet by 40 feet. Required the content of the masonry in cubic feet.

Ans. 2350·52 cubic feet.

OF DOMES.

A DOME is formed by arches springing from a circular or polygonal base, and meeting in a point at the top.

PROB. I. To find the surface of a spherical dome.

RULE I. Multiply twice the area of the base by the height; and the product, divided by the radius of the base, will give the surface.

NOTE. This rule is accurate only when the dome is a semicircle.

RULE II. Multiply the circumference of the great circle of the dome (or that of which double the radius is the diameter) by the perpendicular height, the product is the curve surface.

1. The chord line or diameter at the base of a dome is 80 feet, and the perpendicular height or versed sine is 30 feet. Required the curve surface.

First $\left(\frac{80}{2}\right)^2 \div 30 = 40^2 \div 30 = 1600 \div 30 = 53\cdot\dot{3}$ and $53\cdot\dot{3} + 30 = 83\cdot\dot{3}$ feet, the diameter of the great circle of the dome, and $83\cdot\dot{3} \times 3\cdot1416 = 261\cdot8$ feet, the circumference; then $261\cdot8 \times 30 = 7854$ feet, the surface.

The area by Rule I. is only 7539·84 feet, or 314·16 feet too little.

2. Required the surface of a spherical dome upon a hexagonal base, of which the side is 10 feet.

NOTE. The radius of the base being equal to the height, twice the area of the base is the surface = 519·61524 square feet.

3. Required the expense of painting a spherical dome upon an octagonal base, of which the side is 20 feet, at 8d. per square yard.

Ans. £14, 6s. $1\frac{1}{2}$d.

PROB. II. To find the surface of a spherical dome which has a circular opening for a lantern or cupola at the top.

RULE. Divide the product of half the sum of the chord and the diameter of the top opening multiplied by half their difference by the perpendicular height, and add the quotient to the perpendicular height. To the square of half this sum, add the square of half the diameter of the top opening, and take the square root of the sum for the radius of the great circle of the dome. Multiply the circumference of the great circle by the perpendicular height for the curve surface of the dome.

1. A spherical dome whose chord line at the bottom is 88 feet, has a circular opening at the top, the diameter of which is 30 feet; the perpendicular height from the chord line to the diameter of the top opening is 33·42 feet. Required the curve surface. Ans. 1069 yards 5 feet $2\frac{1}{2}$ inches.

2. A dome at the bottom is 102 feet in diameter, the lantern opening is 40 feet in diameter, and the perpendicular height is 38 feet. Required the curve surface.

Ans. 1378 yards 4 feet $9\frac{1}{2}$ inches.

3. There is a spherical dome, covered with lead, whose chord line at the bottom, across the base, is 68 feet; at the top is a circular opening for a skylight 20 feet in diameter; the perpendicular height from the top opening to the base chord is 20 feet; there are 28 battens or rolls for the lead, each making an additional 6 inches: how many square feet of lead is on the dome, including the 28 rolls, what is its weight at $7\frac{1}{4}$ lbs. per square foot, and what is the cost at 1s. 9d. per foot?

Ans. 5195·182 square feet, which weighs 347 cwt. 3 qrs. 15 lb. 13 oz. 13 drams, and costs £454, 11s. $6\frac{3}{4}$d.

4. There is a staircase 13 feet 6 inches square, on which is a spherical dome whose diameter from angle to angle of the staircase is 18 feet, the perpendicular height from the base to the opening of the circular skylight is 6 feet 9 inches; there are four semicircular spaces to be deducted, whose chord line is 12 feet, the depth is 27 inches, or half the difference between the diameter of the dome and the square side of the staircase. Required the superficial content of the four spherical angle spaces left. Ans. 14 square yards 1 foot $2\frac{3}{4}$ inches.

PROB. III. To find the surface of an elliptical dome.

RULE. Divide the difference between the squares of the diameters by the square of the less diameter, if for an oblate;

but by the square of the greater, if for an oblong spheroid, and call the quotient x.

Add $\frac{1}{6}$ of x to unity, if for an oblate; but subtract it from unity if for an oblong spheroid, and take $\frac{4}{9}$ of this sum or difference, which call y.

From this sum or difference, subtract $\frac{1}{40}$ of x, multiply the remainder by $\frac{8}{27}$, and *retain* the product. Add $\frac{1}{3}$ of x to unity if for the oblate; but subtract it from unity if for the oblong spheroid; from the square root of the sum or difference take the number represented by y, and subtract the remainder from the product which was *retained*. Multiply this remainder by both the diameters, by 3·1416, and by $3\frac{6}{7}$, the product will give the curve surface of either spheroid.

1. An elliptical dome, 20 feet diameter at the base, and 20 feet high, being half an oblong spheroid, is to be covered with lead at 7 lb. per square foot, having 20 rolls or battens, each inches additional to the surface. Required the superficial content of the lead, its weight, and also the whole expense at s. 8d. per square foot.

Ans. Content 1311 square feet 10 in. 5 pts. Weight 81 cwt. 3 qrs. 27 lb. 13 oz. $11\frac{2}{5}$ drams, and expense £109, 6s. $\frac{1}{4}$d.

2. There is an elliptical dome (half of an oblate spheroid), the diameter at the base is 60 feet, and the perpendicular height 4 feet, how many yards of plaster does it contain?

Ans. 546 yards 4 feet 8 inches.

PROB. IV. To find the vacuity of a spherical dome.

Multiply the area of the base by two-thirds of the height.

NOTE. This rule is true only when the dome is a semicircle.

1. Required the content of a spherical dome, the diameter its circular base being 30 feet.

Ans. $30^2 \times ·7854 \times \frac{2}{3} \times 15 = 7068·6$ cubic feet.

2. Required the solid content of an octagonal dome, of which the height is 21 feet, and each side of the base 20 feet.

Ans. 27039·1912 cubic feet.

3. Required the solid content of a dome upon a nonagonal base, of which the side is 12 feet, and the height 30 feet.

Ans. 17803·653696 cubic feet.

OF SALOONS.

SALOONS are formed by arches connecting the side-walls of a building with a ceiling or platform in the middle.

PROB. I. To find the surface of a saloon.

RULE. Apply a line close to the arch, across the surface, from the side-wall to the platform, for its breadth, then measure along the middle of it quite round the room for its length, and multiply one of these by the other, to get the surface.

1. The girt across the face of a saloon is 4 feet, and the mean length round the room is 108 feet. Required the surface. Ans. 432 square feet.

2. The girt across the face of a saloon is 7 feet 10 inches, and the mean length round the room 140 feet. What will the plastering of it cost, at $6\frac{3}{4}$d. per square yard, and the painting in oil, at 15d. per square yard?

Ans. £3, 8s. $6\frac{1}{2}$d. plastering; £7, 12s. $3\frac{3}{4}$d. $\frac{1}{9}$ painting.

3. The mean length of a saloon is 127 feet 6 inches, and the breadth across the face of the saloon 6 feet. What will the size-painting of it cost, at $4\frac{1}{4}$d. per square yard?

Ans. £1, 10s. $1\frac{1}{4}$d.

PROB. II. To find the area of the concave part of a quadrantal saloon.

RULE I. Multiply 3·1416 by half the radius of the arch for the girt across the arch. Multiply the girt across the arch by the length round the ceiling at the top of the arch, and to this product add the areas of two circles, whose diameters are twice the radius of the arch for a circular saloon; but when the saloon is square or oblong, add the areas of two squares, whose sides are double the radius, and the sum will be the area required.

RULE II. Find the length round the room at the top of the arch, and also at the bottom of the arch. To half the sum of these lengths, add the product of the radius into 1·091 when the room is square or oblong; but into ·8584 when the room is circular; then multiply the sum by the girt across the arch for the area required.

1. The diameter of a circular room is 60 feet, over which springs a quadrantal arch of 5 feet radius. Required the curve-surface of the arch.

By Rule I. $60 - 10 = 50$ feet, the diameter at the flat ceiling; $50 \times 3{\cdot}1416 = 157{\cdot}08$, the circumference at the ceiling; and $3{\cdot}1416 \times 10 \div 4 = 7{\cdot}854$ feet, the girt across the curve. Now, $157{\cdot}08 \times 7{\cdot}854 = 1233{\cdot}70632$ feet, and $10^2 \times 2 \times {\cdot}7854 = 157{\cdot}08$ twice the area of a circle whose diameter is double the radius; hence $1233{\cdot}70632 + 157{\cdot}08 = 1390{\cdot}78632$ square feet, the curve-surface required.

By Rule II. ½ (60 × 3·1416 + 50 × 3·1416) = ½ (188·496 + 157·08) = 172·788 half the sum of the lengths, and 5 × ·8584 = 4·292; then 172·788 + 4·292 × 7·854 = 177·08 × 7·854 = 1390·78632 square feet, the same as before.

2. There is a room 80 feet long and 60 feet wide, over which springs a quadrantal arch of 6 feet radius. Required the curve-surface of the arch.

By Rule I. 80 — 12 = 68 feet, the length at the flat ceiling; 60 — 12 = 48 feet, the breadth at the flat ceiling; 68 + 48 × 2 = 232 feet, the girt at the ceiling; and 3·1416 × 12 ÷ 4 = 9·4248, the girt across the curve. Now, 232 × 9·4248 = 2186·5536 feet, and 12^2 × 2 = 288 feet, twice the square of double the radius; hence 2186·5536 + 288 = 2474·5536 square feet, the curve-surface required.

By Rule II. ½ (232 + 280) = 256 half the sum of the lengths, and 1·091 × 6 = 6·546; then (256 + 6·546) × 9·4248 = 262·546 × 9·4248 = 2474·4435 square feet, nearly the same as before.

3. There is a saloon 60 feet square, having a quadrantal arch of 5 feet radius. Required the concave surface of the saloon.

Ans. 1770·8 square feet by Rule I., and 1770·72357 square feet by Rule II.

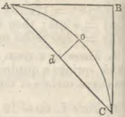
4. There is a circular saloon, 40 feet in diameter, having a quadrantal arch whose radius is 6 feet. Required the curve surface of the saloon. Ans. 897·303792 square feet.

Prob. III. To find the vacuity of a saloon.

Rule I. Multiply the difference between the area of the triangle and the area of the segment by the length of the room, and subtract the product from the cubic content of the room, the remainder is the vacuity of the saloon.

Note. Multiply ⅔ of the chord of the segment by the versed sine, and to this product add the quotient of the cube of the versed sine, divided by twice the chord; the sum will give the area of the segment.

1. Suppose the perpendicular height of a saloon to be 38·4 inches, the horizontal distance from the platform to the side-wall 37·9 inches, the chord of the arch 54 inches, and the distance of its middle point from the arch 9 inches, the chord of half the arch 28·44 inches, and the compass round the middle of the saloon 50 feet. Required the vacuity.



First $37{\cdot}9 \times 19{\cdot}2 \div 144 = 727{\cdot}68 \div 144 = 5{\cdot}05\dot{3}$ feet, area of the triangle ABC; then $(4{\cdot}5 \times \frac{2}{3} \times {\cdot}75) + ({\cdot}75^3 \div 9) = 2{\cdot}25 + ({\cdot}421875 \div 9) = 2{\cdot}25 + {\cdot}046875 = 2{\cdot}296875$ feet, area of the segment A*o*C. Now, $5{\cdot}05\dot{3} - 2{\cdot}296875 \times 50 = 2{\cdot}756458\dot{3} \times 50 = 137{\cdot}82291\dot{6}$ cubic feet, content occupied by the saloon, which, taken from the cubic content of the room, will leave the vacuity of the saloon.

Rule II. When the size of the room, &c. is given.

1. Multiply the height of the arc by its projection by $\frac{1}{4}$ of the perimeter of the ceiling, and by 3·1416, for the first product.

2. From a side or diameter of the room take a like side or diameter of the ceiling, and multiply the square of the remainder by $\frac{2}{3}$ of the height, and by 1 if the room is square or rectangular, but by ·7854 if the room is circular; or, if the room is a regular polygon, multiply by the area of that polygon whose side is unity for the second product.

3. Multiply the area of the flat ceiling by the height of the arch, and add this and the two former products together for the vacuity required.

1. What is the vacuity of a saloon with a circular quadrantal arch, of 2 feet radius, springing over a rectangular room 20 feet long and 16 feet wide, the projection on each side being 2 feet?

Here the flat part of the ceiling is 16 feet by 12, hence the perimeter = 56.

First $3{\cdot}1416 \times 2 \times 2 \times 14 = 175{\cdot}9296$ first product; then $(20 - 16)^2 \times 2 \times \frac{2}{3} = 4^2 \times 2 \times \frac{2}{3} = 21{\cdot}\dot{3}$ feet, second product; and $16 \times 12 \times 2 = 192 \times 2 = 384$. Now, $175{\cdot}9296 + 21{\cdot}\dot{3} + 384 = 581{\cdot}2629\dot{3}$ cubic feet, the vacuity.

2. A circular building of 40 feet diameter, and 25 feet high to the ceiling, is covered with a saloon whose circular arch is 5 feet radius. Required the capacity of the room in cubic feet. Ans. 30779·45948 cubic feet.

Rule III. From the cubic content of the whole void space before the saloon is formed, deduct the cubic content of the space occupied by the arched bracket, the remainder is the cubic content of the vacuity required.

NOTE 1. The cubic content of the space formed by the bracket is found by multiplying the area of a cross section by the length round the room.

NOTE 2. The area of a quadrantal bracket is found by deducting the product of the square of the radius into ·7854 from the square of the radius.

NOTE 3. The length of the space occupied by the bracket is found by taking the girt round the room at the bottom, and also round the ceiling at the top of the arch, and adding to the half of their sum the product of the radius into 1·738173 when the room is circular; but into 2·2132 when the room is square or oblong.

1. A circular room 40 feet in diameter, and 25 feet high, is made into a saloon with a quadrantal arch of 5 feet radius. Required the vacuity of the room after the saloon is finished.

First $40 \times 40 \times \cdot 7854 \times 25 = 31416$ cubic feet, whole vacuity before the saloon is formed; then $\frac{1}{2}(40 + 30) = 35$ mean diameter, and $35 \times 3\cdot 1416 = 109\cdot 956$, and $109\cdot 956 + (1\cdot 738173 \times 5) = 109\cdot 956 + 8\cdot 690865 = 118\cdot 646865 =$ the length of the bracket; also $(5 \times 5) - (5 \times 5 \times \cdot 7854) = 25 - 19\cdot 635 = 5\cdot 365$ area of cross section of the bracket; consequently $31416 - (118\cdot 646865 \times 5\cdot 365) = 31416 - 636\cdot 5404 = 30779\cdot 4596$ cubic feet, the vacuity after the saloon is formed.

2. There is a saloon, 20 feet long, 16 feet wide, and 18 feet high, with a quadrantal arch of 2 feet radius. Required the whole vacuity. Ans. 5701·2628 cubic feet.

STRENGTH OF MATERIALS.

A PIECE of solid matter may be exposed to four distinct kinds of strains. 1st, It may be torn asunder, as in the case of ropes, tie-beams, king-posts, stretchers, &c. 2d, It may be crushed, as in the case of truss-beams, columns, posts, &c. 3d, It may be broken across, as in the case of joists, beams, &c. 4th, It may be twisted or wrenched, as in the case of axles of wheels, the nail of a press, &c.

The subjoined tables of data, with the practical problems, have been deduced from a number of careful experiments made by Barlow, Tredgold, and others.

TABLE I.—OF THE FLEXIBILITY AND STRENGTH OF TIMBER.

Name of the kind of Wood.	Specific Gravity.	Value of U.	Value of E.	Value of S.	Value of C.
Teak,........................	745	818	9657802	2462	15555
Poon,........................	579	596	6759200	2221	14787
English oak,..................	969	598	3494730	1181	9836
Do. specimen 2,..............	934	435	5806200	1672	10853
Canadian oak,................	872	588	8595864	1766	11428
Dantzic oak,..................	756	724	4765750	1457	7386
Adriatic oak,................	993	610	3885700	1583	8808
Ash, ...	760	395	6580750	2026	17337
Beech,........................	696	615	5417266	1556	9912
Elm,...........................	553	509	2799347	1013	5767
Pitch pine,....................	660	588	4900466	1632	10415
Red pine,......................	657	605	7359700	1341	10000
New England fir,............	553	757	5967400	1102	9947
Riga fir,......................	753	588	5314570	1108	10707
Do. specimen 2,..............	738	—	3962800	1051	—
Mar Forest fir,..............	696	588	2581400	1144	9539
Do. specimen 2,..............	693	403	3478328	1262	10691
Larch,.........................	531	411	2465433	653	—
Do. specimen 2,..............	522	518	3591133	832	—
Do. specimen 3,..............	556	518	4210830	1127	7655
Do. specimen 4,..............	560	518	4210830	1149	7352
Norway spar,.................	577	648	5832000	1474	12180

PROB. I. To find the strength of direct cohesion of a piece of timber of any given dimensions.

RULE. Multiply the area of the transverse section, in inches, by the value of C in the table, and the product will be the strength required in pounds.

Let $b =$ the breadth, and $d =$ the depth of the piece of timber in inches; then $b \times d \times$ tabular value of $C = W$, or the weight in pounds, from which equation any of the quantities may be found when the others are given.

NOTE. If the specific gravity differs from the mean tabular specific gravity, multiply the product by the specific gravity, and divide by the specific gravity in the table for the correct strength.

1. What weight will it require to tear asunder a piece of English oak, specimen 1, 4 inches square, the specific gravity being 969? Ans. 157376 lbs.

2. What weight will it require to tear asunder a piece of beech 3 inches square? Ans. 89208 lbs.

3. What weight will tear asunder a cylinder of red pine 6 inches in diameter? Ans. 282744 lbs.

4. What must be the depth of a piece of ash which is 4 inches broad, and requires a weight of 160000 lbs. to tear it asunder? Ans. 2·3071 inches.

5. What must be the diameter of a cylinder of teak, which requires a weight of 200000 lbs. to tear it asunder? Here $d^2 = \frac{W}{\cdot 7854 \times 15555}$. Ans. 4·046 inches.

PROB. II. To find the deflection of a beam *fixed* at one end, and loaded with any given weight at the other.

RULE. Divide 32 times* the weight multiplied by the cube of the length of the beam in inches, by the continued product of the tabular value of E, into the breadth and cube of the depth of the beam, both being in inches.

NOTE. When the beam is loaded uniformly throughout, the rule still applies, only we multiply the cube of the length by 12 times the weight instead of 32 times.

Let l = length, b = the breadth, and d = the depth of the beam in inches, and W = the weight in pounds; then $32 \text{ W} \times l^3 \div$ tabular value of $\text{E} \times b \times d^3$ = the deflection in inches when the beam is loaded at the ends; and $12 \text{ W} \times l^3 \div$ tabular value of $\text{E} \times b \times d^3$ = deflection when the beam is loaded uniformly throughout, from which equation any of the quantities may be found, the others being given.

1. If a weight of 300 lbs. be hung upon the extremity of an ash batten 4 inches square, and projecting 5 feet from the wall where it is fixed, how much will it be deflected? Ans. 1·23 inch.

2. How much would the same beam be deflected, if a prop proceeding from the wall met it at the distance of 2 feet from the wall? Ans. ·266 of an inch.

3. A batten of teak 10 feet long, 5 inches broad, and 6 inches deep, is fixed at one end, and a weight of 700 lbs. suspended from the other. Required its deflection, and also the deflection when loaded uniformly throughout its length.

Ans. 3·711 inches when the load is suspended from the end, and 1·3916 inches when disposed uniformly throughout.

* According to Mr Bevan, this number should be 16; but Mr Barlow says that it is 32, in the usual methods of fixing beams in ordinary buildings.

4. A batten of Dantzic oak 20 feet long, 5 inches broad, and 6 deep, is fixed at one end, and loaded uniformly throughout with 1000 lbs. Required its deflection, and also the deflection when the load is suspended from the end, and the batten supported by a prop from the wall meeting it at 10 feet from the fixed end.

Ans. 32·23 inches in the first case, and 10·7433 inches in the second case.

5. A beam of Dantzic oak, 20 feet long and 5 inches broad, is fixed at one end and loaded uniformly throughout with a weight of 1000 lbs. which causes a deflection of 32·23 inches. Required the depth of the beam. Ans. 5·999 inches.

6. A beam of elm, 30 feet long and 7 inches deep, is fixed at one end and loaded at the other with a weight of 1200 lbs. which produces a deflection of 30 inches. Required its breadth. Ans. 7·7746 inches.

Prob. III. To find the deflection of beams *supported* at both ends, and loaded in the middle with any given weight.

Rule. Divide the product of the cube of the length in inches by the given weight in lbs. by the continued product of the tabular value of E, into the breadth and cube of the depth in inches, for the deflection sought.

Note. When the beam is *fixed* at both ends, the deflection is $\frac{2}{3}$ of that given in the rule.

That is, $l^3 \times W \div$ tabular value of $E \times b \times d^3 =$ the deflection in inches when *supported* at both ends, and $\frac{2}{3}(l^3 \times W \div$ tabular value of $E \times b \times d^3) =$ deflection when the beam is *fixed* at both ends.

1. A beam of pitch pine 8 inches broad, 3 thick, and 30 feet long, is supported at both ends, and loaded in the centre with a weight of 100 lbs. Required its deflection.

Ans. 4·408 inches.

2. A beam of Mar Forest fir, specimen 1, 14 inches broad, 9 deep, and 20 feet long, is supported at both ends. How much will it be deflected with 3000 lbs. suspended at its centre? Ans. 1·574 inch.

3. A beam of Canadian oak 6 inches broad, 8 deep, and 30 feet long, is fixed at both ends in a wall, and loaded at the centre with 4000 lbs. Required its deflection.

Ans. 4·71 inches.

4. A beam of Adriatic oak, 4 inches broad and 5 deep, supported at both ends, and loaded in the middle with a weight of 2000 lbs., is deflected 10 inches. What is its length?

Ans. 17 feet 9·37 inches.

5. A beam of ash, 20 feet long, 6 inches broad, and 7 deep, is deflected 8 inches, when fixed at both ends and loaded in the middle. What is the weight of the load?

Ans. 11756·2356 lbs.

PROB. IV. To find the deflection of beams *supported* at both ends, and loaded uniformly throughout their lengths with a given weight.

RULE. Multiply the deflection found by last problem by 5, and divide the product by 8, and the quotient will be the answer.

That is, $\frac{5}{8}(l^3 \times W \div$ tabular value of $E \times b \times d^3) =$ deflection in inches when the beam is supported at both ends, and $\frac{5}{32}(l^3 \times W \div$ tabular value of $E \times b \times d^3) =$ deflection when fixed at both ends.

1. A beam of Norway spar, 4 inches broad and 5 deep, is supported at both ends, the length being 20 feet. What will be the deflection when it is loaded uniformly throughout its length with a weight of 600 lbs.? Ans. 1·777 inch.

2. A beam of English oak, specimen 1, 9 inches square and 20 feet long, supports a load of 3000 lbs. disposed uniformly throughout its length. Required the deflection.

Ans. 1·13 inch.

3. A beam of larch, specimen 3, 10 inches broad and 1 foot deep, supports the building over a gateway 10 feet wide. What deflection may be expected, supposing the whole weight [illegible]0000 lbs.? Ans. ·742 of an inch.

4. What must be the length of a beam of larch 4 inches broad, 5 inches deep, and supported at both ends, to sustain a load of 6000 lbs. uniformly disposed throughout its length, so that the deflection may not be more than 2 inches?

Ans. 3 feet 10·46 inches.

5. What weight uniformly disposed throughout the length of a beam of teak, fixed at both ends, 20 feet long, 5 inches broad, and 6 inches deep, will produce a deflection of 1·5 inch? Ans. 2716·257 lbs.

PROB. V. To find the ultimate deflection of beams or rods *supported* at both ends, before their fracture.

RULE. Divide the square of the length in inches by the product of the tabular value of U, multiplied by the depth of the beam in inches, and the quotient will be the ultimate deflection.

That is, $l^2 \div$ tabular value of $U \times d =$ the ultimate deflection in inches.

1. A rod of poon, 2 inches square and 10 feet long, is broken by a weight applied to its centre. Required the deflection at the instant of fracture. Ans. 12·08 inches.

2. Required the ultimate deflection of a beam of Adriatic oak 6 inches square and 30 feet long. Ans. 35·41 inches.

3. Required the ultimate deflection of a beam of ash 1 foot broad, 8 inches deep, and 40 feet long. Ans. 72·91 inches.

4. The ultimate deflection of a rod of teak, 20 feet long, is 25 inches. Required its depth. Ans. 2·8117 inches.

5. The ultimate deflection of a beam of larch, 6 inches deep, is 50 inches. Required its length.

Ans. 29 feet 3·138 inches.

Prob. VI. To find the ultimate transverse strength of any rectangular beam of timber *fixed* at one end and loaded at the other.

Rule. Multiply the tabular value of S by the breadth and square of the depth, both in inches, and divide the product by the length in inches, the quotient will be the weight in pounds.

That is, tabular value $S \times b \times d^2 \div l = W$.

1. What weight will it require to break a piece of Riga fir, 1st specimen, fixed at one end and loaded at the other, the breadth being 3 inches, the depth 4 inches, and 5 feet long?

Ans. $886\frac{2}{5}$ lbs.

2. What weight will it require to break a piece of ash fixed at one end and loaded at the other, the breadth being 6 inches, the depth 4 inches, and 7 feet long? Ans. $2315\frac{3}{7}$ lbs.

3. What weight uniformly distributed throughout the length of a beam of English oak, 2d specimen, will break it, the breadth being 6 inches, the depth 9 inches, and its projection from the wall in which it is fixed, 12 feet? Ans. 11286 lbs.

4. A beam of elm, 30 feet long and 4 inches broad, is fixed at one end, and loaded at the other with a weight of 1000 lbs. Required its depth, if this weight is just sufficient to break it. Ans. 9·4257 inches.

5. A weight of 1200 lbs. is suspended at the end of a bar of teak, which is 6 inches broad and 7 deep, and fixed at the other end. Required its length when this weight is just sufficient to break it. Ans. 50·266 feet.

Prob. VII. To find the ultimate transverse strength of any rectangular beam when *supported* at both ends and loaded in the centre.

RULE. Multiply the tabular value of S by 4 times the breadth and square of the depth in inches, and divide the product by the length in inches for the weight.

That is, tabular value $S \times 4b \times d^2 \div l = W$.

NOTE 1. When the beam is *fixed* at each end, and loaded in the middle, the result obtained by the rule must be increased by its half.
NOTE 2. When the beam is loaded uniformly throughout its length, the result obtained by the rule must be doubled.
NOTE 3. When the beam is *fixed* at both ends, and loaded uniformly throughout, the result obtained by the rule must be multiplied by 3.

1. What weight will it require to break a beam of English oak, 2d specimen, supported at both ends and loaded in the middle, the length being 12 feet, the breadth 6 inches, and the depth 8 inches? Ans. $17834\frac{2}{3}$ lbs.

2. What weight will it require to break a piece of larch, 3d specimen, supported at both ends and loaded in the middle, the length being 8 feet 4 inches, the breadth 8 inches, and the depth 10 inches? Ans. 36064 lbs.

3. What weight will it require to break a beam of New England fir, fixed at both ends, and loaded uniformly throughout its length, which is 10 feet, and 6 inches square?
Ans. $23803\frac{1}{5}$ lbs.

4. What weight will it require to break a beam of Riga fir, 1st specimen, fixed at both ends, and loaded at the centre, the length being 15 feet, the breadth 9 inches, and the depth 1 foot? Ans. $47865\frac{3}{5}$ lbs.

5. What must be the length of a beam of beech, which is 6 inches broad and 8 deep, to break with a weight of 4000 lbs. when supported at both ends? Ans. 41 feet 5·92 inches.

6. What must be the depth of a beam of red pine, which is 20 feet long and 5 inches broad, to break with a load of 5000 lbs. uniformly disposed throughout its length when the beam is fixed at both ends? Ans. 8·974 inches.

NOTE. In Barlow's Essay on the Strength of Timber, a second rule is given to each of the two last problems, the angle of deflection being taken into consideration, which gives a greater result. The rules given here are, however, best for practice, as they are simpler, and *two-thirds of their results for a permanent load is reckoned sufficient.*

PROB. VIII. To find the weight under which a given column will begin to bend when placed vertically on a horizontal plane.

RULE. Multiply the tabular value of E by the cube of the least thickness, and by the greatest thickness, both in inches, and that product again by ·2056. Then divide the last product by the square of the length in inches for the weight in pounds.

That is, tabular value $E \times b^3 \times d \times \cdot 2056 \div l^2 = W$, where $b =$ the least, and $d =$ the greatest thickness.

1. What weight will it require to bend a column of ash 4 inches square and 6 feet 8 inches long, when placed vertically on a plane, and the weight applied at its upper extremity?
Ans. 54120·088 lbs.

2. What weight will it require to bend a column of English oak, 2d specimen, 20 feet long, 6 inches thick, and 9 broad?
Ans. 40289·222 lbs.

3. What weight will it require to bend a column of Riga fir, 1st specimen, 15 feet long, and 10 inches in diameter?
Ans. 337245·553 lbs.

4. What weight will it require to bend a column of New England fir, 20 feet long, and 1 foot in diameter?
Ans. 441683·08 lbs.

5. What must be the greatest breadth of a column of English oak, 2d specimen, which is 20 feet high and 6 inches thick, which will begin to bend under a weight of 40000 lbs.?
Ans. 8·935 inches.

6. A cylindrical column of beech, 30 feet high, begins to bend under a weight of 50000 lbs. Required its diameter.
Ans. 8·7336 inches.

TABLE II.—SHOWING THE WEIGHT THAT WILL PULL ASUNDER A PRISM ONE INCH SQUARE OF THE FOLLOWING MATERIALS, ACCORDING TO THE EXPERIMENTS OF M. MUSCHENBROEK :—

Cast gold,................22000	Zinc,.......................2600
Cast silver,...............41000	Bismuth,...................2900
Anglesea copper,.........34000	Good brass,...... 51000
Swedish copper,.........37000	Ivory,..... 16270
Cast-iron,.................50500	Horn, ...8750
Bar-iron, ordinary,......68000	Whalebone,................7500
Ditto, best Swedish,....84000	Compositions.
Bar-steel, soft,.........120000	Gold 5, copper 1,........50000
Ditto, razor-temper,...150000	Silver 5, copper 1,.......48500
Cast-tin, Eng. block,.....5200	Swedish copper 6, tin 1,64000
Ditto, grain,......6500	Block-tin 3, lead 1,.......10200
Cast-lead,....................860	Tin 4, lead 1, zinc 1,...13000
Regulus of Antimony,...1000	Lead 8, zinc 1,............4500

ACCORDING TO THE EXPERIMENTS OF MR RENNIE.

	Weight in lbs. that would tear asunder a prism 1 inch square.	Length in feet that would break with its own weight.
Cast-steel,	134256	39455
Swedish iron,	72064	19740
English iron,	55872	16938
Cast-iron,	19096	6110
Cast-copper,	19072	5092
Yellow brass,	17958	5180
Cast-tin,	4736	1496
Cast-lead,	1824	384
Good hemp rope,	6400	18790
Ditto, 1 inch diameter,	5026	18790

TABLE III.—OF THE COHESIVE FORCE OF A SQUARE INCH OF IRON OF DIFFERENT KINDS.

Iron wire,	113077	English iron,	61600
Ditto,	93964	Ditto,	65772
Swedish iron,	78850	Welsh iron,	64960
Ditto,	72064	Ditto,	55776
Ditto,	54960	French iron,	61000
Ditto,	53244	Russian iron,	59472
German iron,	69133	Cast-iron,	18295
English iron,	66000	Ditto,	19488
Ditto,	55000	Welsh ditto,	16255

TABLE IV.—OF THE LATERAL STRENGTH OF THE FOLLOWING MATERIALS, THE BAR BEING 1 FOOT LONG AND 1 INCH SQUARE.

	Weight that will break them.	Weight which they can bear with safety.
Cast-iron,	3270 lbs.	1090 lbs.
Oak,	627	209
Memel fir,	390	130
American white pine,	206	69

TABLE V.—OF THE WEIGHT IN POUNDS NECESSARY TO CRUSH CUBES OF $1\frac{1}{2}$ INCH IN THE SIDE OF THE FOLLOWING SUBSTANCES, ACCORDING TO MR RENNIE AND OTHERS :—

Aberdeen granite, blue,	24536
White-veined Italian marble,	21738
Very hard freestone,	21254
Purbeck limestone,	20610
Limerick limestone, black,	19924

Peterhead granite,	18636
Compact limestone,	17354
Yorkshire paving-stone,	15856
Craigleith stone with the strata,	15560
Ditto, across the strata,	12346
Dundee sandstone,	14919
Cornish granite,	14302
White statuary marble,	13632
Fine brick,	3864
Yellow baked brick,	2254
Red brick,	1817
Pale red brick,	1265
Chalk,	1127

CUBES OF ONE INCH IN THE SIDE WERE CRUSHED BY THE FOLLOWING WEIGHTS :—

Elm,	1284 lbs.	English oak,	3860 lbs.
White deal,	1928	Craigleith stone,	8688

CUBES OF ONE-FOURTH OF AN INCH IN THE SIDE BY

Iron cast vertically,	11140 lbs.
Ditto, horizontally,	10110
Copper cast,	7318
Cast-tin,	966
Cast-lead,	483

PROB. IX. To find the breadth of a uniform cast-iron beam to sustain a given weight in the middle.

RULE. Multiply the length in feet by the weight to be supported in pounds, and divide the product by 850 times the square of the depth in inches; the quotient will give the breadth in inches.*

That is, $\frac{l \times W}{850 \times d^2} = b$ when $l =$ the length in feet, $W =$ the weight in lbs. to be supported, $d =$ the depth, and $b =$ the breadth in inches. From this equation any of the quantities may be found when the others are given.

1. What is the breadth of a beam 30 feet long and 12 inches deep, which will support a weight of 10 tons placed in the middle?

* The rules for estimating the strength of cast-iron are chiefly from Tredgold's Essay on the Strength of Cast-iron,—a work which should be in the hands of every engineer.

Here $l = 30$, $W = 22400$, and $d = 12$; hence $\frac{30 \times 22400}{850 \times 12^2}$ $= b = 672000 \div 122400 = 5{\cdot}49$ inches, the breadth.

2. What is the depth of a beam, 30 feet long and $5\frac{1}{2}$ inches broad, which will support a weight of 10 tons placed in the middle? Ans. 11·989 inches.

3. What is the length of a beam, 12 inches deep and $5\frac{1}{2}$ inches thick, which will support a weight of 10 tons suspended from its centre? Ans. 30·05357 feet.

4. What weight will a beam 30 feet long, 12 inches deep, and $5\frac{1}{2}$ inches thick, bear suspended from its centre?

Ans. 22440 lbs.

NOTE. When no particular depth or breadth is determined by the nature of the situation for which the beam is intended, it will sometimes be found convenient to assign some proportion; as, for example, let the depth be n times the breadth, then the equation will be $n \times l \times W = 850 \times d^3$, and the breadth will be $\frac{d}{n}$.

5. A beam 30 feet between the supports is required to bear a weight of 10 tons, and the depth is to be 3 times the breadth. Required the depth and breadth.

Ans. Depth, 13·335 inches; breadth, 4·446 inches.

6. A bar 20 feet long is to support a weight of 15 tons applied at its centre, and its depth is to be 5 times its thickness. Required its depth and breadth.

Ans. Depth, 15·8115 inches; breadth, 3·1623 inches.

PROB. X. To find the breadth and depth of a beam of cast-iron, supported at both ends, when the load is not in the middle between the supports.

RULE. Multiply the distance in feet from the point at which the weight is applied to the one support by its distance from the other, and 4 times this product divided by the whole length between the supports will give the effective leverage of the load; which being used instead of the length, the breadth and depth may be found as in last problem.

That is, if AC = the length of the longer, and CD that of the shorter arm of the beam, the others as before; then

$$4(4AC \times CD \times W) \div 850l = d^3.$$

1. What are the depth and breadth of a bar, 20 feet long, which will support a load of 15 tons 5 feet from the one end, the breadth being one-fourth of the depth?

Here $AC = 15$, $CD = 5$, $W = 33600$, and $l = 20$; hence $\frac{4 \times 4 \times 15 \times 5 \times 33600}{850 \times 20} = d^3 = \frac{4 \times 4 \times 15 \times 168}{17} = 40320 \div 17 =$

$2371\cdot764706 = d^3$, and $\sqrt[3]{2371\cdot764706} = 13\cdot335$ inches, the depth, and $13\cdot335 \div 4 = 3\cdot336$ inches, the breadth.

2. What are the depth and breadth of a bar, 30 feet long, which will support a weight of 20 tons, 10 feet from the one end, the breadth being $\frac{1}{4}$ of the depth?

Ans. Depth, 17·7812 inches; breadth, 4·4453 inches.

3. What load will a bar 25 feet long, 12 inches deep, and 4 inches broad, support when applied at 8 feet from the one end? Ans. 22500 lbs.

NOTE. When the load is uniformly distributed over the length of the beam, the equation is $l \times W = 2 \times 850 \times b \times d^2$, or when the breadth is the nth part of the depth $n \times l \times W = 2 \times 850 \times d^3$.

1. A beam, 20 feet long and 3 inches broad, is to support a load of 33600 lbs. uniformly distributed over its length. Required its depth.

Here $l = 20$, $W = 33600$, and $b = 3$; hence $\frac{20 \times 33600}{2 \times 850 \times 3} = d^2 = \frac{2 \times 1120}{17} = \frac{2240}{17} = 131\cdot7647$, and $\sqrt{131\cdot7647} = 11\cdot4788$ inches the depth.

2. The front of a house is to be broken out to make shops, and the front-wall, which is 40 feet long, is to be supported by 2 cast-iron beams, with a prop in the middle. Now suppose there are 4000 cubic feet of wall, Required the breadth and depth of the beams, the breadth being one-fifth of the depth.

Ans. Depth, 20·194 inches; breadth, 4·0388 inches (supposing a cubic foot of masonry to weigh 140 lbs. avoirdupois).

3. What weight will a beam 30 feet long, 10 inches thick, and 5 inches broad, support when the load is uniformly distributed over its length? Ans. $28333\frac{1}{3}$ lbs.

4. What length must a beam be which is 4 inches broad and 6 deep to support a weight of 10 tons uniformly distributed over its length? Ans. 10·928 feet.

PROB. XI. When a beam is fixed at one end, and the load applied at the other, also when a beam is supported upon a centre of motion.

Take the length from the point at which the beam is fixed to the point where the load is suspended, or, when the beam is supported any where between the ends, take the length from the prop, observing to use the weight which is to act on that end in the calculation. Then calculate the strength by the rules in Problem IX. using $850 \div 4 = 212$ instead of 850.

That is, $l \times W = 212 \times d^2 \times b$, or when the breadth is the nth part of the depth $n \times l \times W = 212 \times d^3$.

1. A beam of cast-iron, 4 inches in breadth, projects 5 feet from the wall in which it is fixed. Required its depth to sustain a weight of 8 tons suspended from its projecting end.

Here $l = 5$, $W = 8960$, and $b = 4$; hence $\frac{5 \times 8960}{212 \times 4} = d^2$ $= \frac{44800}{848} = 52{\cdot}83$, and $\sqrt{52{\cdot}83} = 7{\cdot}268$ inches, the depth.

2. A beam of cast-iron, whose breadth is one-half its depth, projects from the wall in which it is fixed 6 feet. What must its depth and breadth be to sustain a weight of 10 tons suspended from its end?

Ans. Depth, 8·59 inches; breadth, 4·295 inches.

3. The length of the arms of the beam of a balance is 2 feet, and the breadth is one-eighth of the depth. What are these dimensions when the extreme weight that can be weighed by it is 5 cwt. or 560 lbs.?

Ans. Depth, 3·483 inches; breadth, 0·435 of an inch.

4. The length of the arms of the beam of a balance is $3\frac{1}{2}$ feet, the breadth $\frac{3}{4}$ of an inch, and the depth 3 inches. What is the heaviest weight which it will weigh? Ans. 408·857 lbs.

NOTE 1. For wrought-iron we should use 238 instead of 212, which would give the answer in last example 459 lbs.

NOTE 2. When the weight is uniformly distributed over the length of the beam, the number 425 must be used instead of 212.

5. The second story of a building is to project over the first in front 3 feet, or exactly the thickness of the wall. What must be the depth of the fixed iron beams, which are 4 inches broad, and placed at the distance of 6 feet from each other, supposing the weight of the superincumbent wall on each, 6 feet in breadth, to be 33600 lbs.? Ans. 7·7 inches.

6. Required the depth for the cantilevers of a balcony which project 5 feet from the wall in which they are fixed; they are 3 inches broad, and placed at the distance of 6 feet from each other, the weight of the material which forms the balcony is 1500 lbs., and the greatest load upon it 3000 lbs. for each 6 feet in length.

Here $1500 + 3000 = 4500$ lbs. the load upon each cantilever uniformly distributed over its length.

Ans. 4·2008 inches.

NOTE 1. The depth thus found should be the depth at the fixed end, and if the breadth be the same throughout its length, the cantilever will be equally strong in every part if the under-side should taper a little toward the projecting end; care, however, should be taken not to reduce the depth at the projecting point too much.

NOTE 2. The strength of the teeth of wheels depends upon this case, the length being the length of the teeth, and the depth the thickness. Great allowance, however, must be made for irregular action, and for wearing by friction. The length of the teeth ought not to exceed their thickness, although the strength is not affected by the greater or less length. The breadth of the teeth should be in proportion to the stress upon them, and this stress should not exceed 400 lbs. for each inch in breadth, as the surface of contact is always small, and the teeth work irregularly when much worn.

Rule. Divide the stress at the pitch-line of the wheel in lbs. by 1500; the quotient is the square of the thickness of the teeth in inches.

That is, $W \div 1500 = d^2$.

1. Let the greatest stress at the pitch-circle of a wheel be 5000 lbs. Required the thickness of the teeth.

Here $\frac{5000}{1500} = 3\cdot\dot{3}$, and $\sqrt{3\cdot\dot{3}} = 1\cdot8257$ inch.

Some writers, taking into consideration the length and breadth of the teeth, give the following formula for the thickness:

$$\frac{W \times l}{212 \times b} = d^2.$$

Suppose that in last example the length was $\frac{1}{4}$ of a foot, and the breadth 3 inches.

Then $\frac{5000 \times \cdot25}{212 \times 3} = 1\cdot9654$, and $\sqrt{1\cdot9654} = 1\cdot402$ inch.

2. Let the greatest stress at the pitch-circle of the wheel be 8000 lbs., the length of the teeth $\frac{1}{8}$ of a foot, and the thickness 1·5 inch. Required the breadth.

Ans. 2·795 inches; but, to allow for wearing by friction, this quotient should be doubled, or 5·590 inches is the breadth of the teeth. According to what was previously stated, the breadth of the teeth should be 1 inch for each 400 lbs. of stress upon them; hence for 8000 lbs. the breadth should be 20 inches, and by the first equation the thickness should be $\sqrt{\frac{8000}{1500}} = \sqrt{5\cdot\dot{3}} = 2\cdot31$ inches.

The pitch of the teeth of a wheel is the distance from middle to middle of the teeth, and should be at least 2·1 times the thickness of the teeth; hence, in this example, the pitch should be 4·85 inches.

Prob. XII. To find the thickness of the teeth of a wheel when the power of the first mover in pounds and the velocity in feet per second are given.

RULE. Multiply 0·073 times the power of the first mover in pounds by its velocity in feet per second, and divide the product by the number of revolutions the wheel is proposed to make per minute, and by the radius the wheel should have, if its pitch were *two inches;* the cube root of the quotient will be the thickness of the teeth in inches.

1. Suppose the effective force acting at the circumference of a water-wheel to be 800 lbs., and its velocity 12 feet per second. What is the thickness for the teeth of a wheel which is to make 15 revolutions, and have 36 teeth?

First $0·073 \times 800 \times 12 = 700·8$. Now the circumference of a wheel with 36 teeth and a pitch of 2 inches is $36 \times 2 =$ 72 inches, consequently its radius is $\frac{72}{3·1416 \times 2} = 11·459$ inches; then $\sqrt[3]{\frac{700·8}{15 \times 11·459}} = \sqrt[3]{4·05356} = 1·6$ inches, very nearly, the thickness required.

2. Suppose the effective force of the piston of a steam-engine to be 15000 lbs. and its velocity 4 feet in a second. What is the thickness for the teeth of a wheel which is to make 20 revolutions per minute, and have 140 teeth?

Ans. 1·7 inches, very nearly.

3. A machine is moved by 6 horses, with an effective power of 400 lbs. each, and a velocity of 3 feet per second. What should be the thickness for the teeth of a wheel which has 60 teeth, and makes 16 revolutions per minute?

Ans. 1·2 inch, very nearly.

PROB. XIII. To find the diameter of a solid cylinder of cast-iron to sustain a given weight, when supported at both ends, and the weight applied at the middle of the length.

RULE. Multiply the weight in pounds by the length in feet, and divide the product by 500; the cube root of the quotient is the diameter in inches.*

That is, $W \times l \div 500 = \text{diam.}^3$

1. What is the diameter of a solid cylinder of cast-iron, 30 feet long, and supported at both ends, which will sustain a pressure of 20000 lbs. in the middle of its length?

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* The figure of equal strength for a solid, of which the cross section is every where circular, is that generated by two cubic parabolas set base to base, the bases being equal, and joining at the section where the strain is the greatest.—(EMERSON'S *Mechanics*, 4to Edition, Cor. 4, Prop. 73.)

Here $\sqrt[3]{\frac{20000 \times 30}{500}} = \sqrt[3]{1200} = 10{\cdot}626$ inches, the diameter required.

2. A solid cylinder of cast-iron, 1 foot in diameter, and supported at both ends, sustains a weight of 33600 lbs. in the middle of its length. What is its length? Ans. $25\frac{5}{7}$ feet.

3. What weight will a cylinder, 9 inches in diameter and 20 feet long, sustain in the middle of its length when supported at both ends? Ans. 18225 lbs.

Prob. XIV. To find the diameter of a solid cylinder of cast-iron supported at both ends to bear a given weight when the strain is not in the middle.

Rule. Multiply the product of the segments of the cylinder by 4 times the weight in pounds, and divide the product by 500 times the length in feet, the cube root of the quotient is the diameter of the cylinder in inches.*

That is, if in the beam AB the weight is applied at the point C, $(AC \times CB \times 4W) \div 500 \times l = \text{diam.}^3$

1. What must be the diameter of a cylindrical bar of cast-iron to resist a pressure of 10000 lbs. applied at 5 feet from the one end, the whole length of the bar being 15 feet?

Here $\sqrt[3]{\frac{5 \times 10 \times 4 \times 10000}{500 \times 15}}\ \sqrt[3]{266{\cdot}\dot{6}} = 6{\cdot}436$ inches, the diameter required.

2. What weight applied at 10 feet from one end of a beam of cast-iron, 30 feet long, will be supported when the diameter is 12 inches? Ans. 32400 lbs.

3. What must be the diameter of a solid cylinder of cast-iron, which is 20 feet long, to sustain a weight of 33600 lbs. when the weight is applied at 5 feet from one of the ends? Ans. 10·0266 inches.

Prob. XV. To find the diameter of a solid cylinder of cast-iron, when supported at both ends to sustain a load uniformly distributed over its length.

Rule. Multiply the length in feet by the weight in pounds, and $\frac{1}{10}$ of the cube root of the product is the diameter in inches.†

That is, $\frac{1}{10}\sqrt[3]{(W \times l)} = d$, or $\frac{W \times l}{1000} = d^3$.

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* The solid of equal strength is the same as in last problem.

† The figure of equal strength for a uniform load, the section being every where circular, is that generated by the revolution of a curve of which the equation is, $a\,(lx - x^2)^{\frac{1}{3}} = y$. (Emerson's *Mechanics*, Cor. 3, Prop. 73.)

1. A load of 20000 lbs. is to be uniformly distributed over the length of a solid cylinder of cast-iron, 15 feet long. Required its diameter.

Here $\frac{1}{10}\sqrt[3]{(20000 \times 15)} = 66{\cdot}943 \div 10 = 6{\cdot}6943$ inches, the diameter.

2. A load of 22400 lbs. is to be uniformly distributed over the length of a solid cylinder of cast-iron whose diameter is 10 inches. Required its length. Ans. 44·6428 feet.

3. A solid cylinder of cast-iron, 25 feet long and 9 inches in diameter, is to be uniformly loaded to its utmost strength. What weight will it bear without breaking? Ans. 29160 lbs.

Prob. XVI. To find the length of a solid cylinder of cast-iron, when fixed at one end and loaded at the other; also when the cylinder is supported on a centre of motion.

Rule. Multiply the leverage with which the weight acts, in feet, by the weight in lbs., and $\frac{1}{5}$ of the cube root of the product is the diameter in inches.

That is, $\frac{1}{5}\sqrt[3]{(W \times l)} = d$, or $d^3 = \frac{W \times l}{125}$.

1. A solid cylinder of cast-iron, fixed at the one end and loaded at the other with a weight of 10000 lbs. is 20 feet in length. Required its diameter.

Here $\frac{1}{5}\sqrt[3]{(10000 \times 20)} = 58{\cdot}481 \div 5 = 11{\cdot}696$ inches, the diameter required.

2. A solid cylinder of cast-iron, 10 inches in diameter, is fixed at one end, and projects 30 feet. What weight will it bear suspended from its end? Ans. $4166{\cdot}\dot{6}$ lbs.

3. A solid cylinder of cast-iron, 5 inches in diameter, is supported in the middle. What is the length of the arms to support 4000 lbs. at the end of each?

Ans. 3·90625 ft., or whole length of the cylinder 7·8125 ft.

Note. This problem may be applied to determine the proportions of gudgeons and axles. The greatest strain on these takes place when by accident that strain is thrown upon the extreme point of their bearing. One-fifth part of the diameter, and in some cases even more, should be allowed for their wearing by friction.

Allowing $\frac{1}{5}$ for wear, the equation becomes $\frac{1}{5}\sqrt[3]{(W \times l)} = d \times \frac{4}{5}$, or when the length is in inches $\sqrt[3]{(\frac{1}{12}l \times W)} = 5 \times \frac{4}{5}d = 4d$, i. e. $\frac{1}{12}l \times W = 64d^3$; whence $\sqrt[3]{\frac{1}{768}l \times W)} = d$, or $\frac{1}{9}\sqrt[3]{l \times W} = d$ nearly, which gives this practical rule.

Multiply the stress in lbs. by the length in inches, and $\frac{1}{9}$ the cube root of the product is the diameter in inches.

4. If the stress on a gudgeon be 33600 lbs. and its length 10 inches. Required its diameter.

Here $\frac{1}{9}\sqrt[3]{(10 \times 33600)} = 69{\cdot}5205 \div 9 = 7{\cdot}7245$ inches, the diameter required.

5. The length of a gudgeon is 8 inches, and its diameter 5 inches. What stress should it bear in lbs.?

Ans. 11390·625 lbs.

The following equations are derived from the supposition, that the stress upon the gudgeon should be limited to a portion of the circumference equal to three-fourths of the diameter, and the pressure not to exceed 1500 lbs. on the square inch.

$\frac{1}{30}\left(\frac{W^2}{d}\right) = d$, or $W = 854d^2$; and $l = {\cdot}854d$. Whence, if the weight to be sustained be 30000 lbs., the diameter should be $= \sqrt{\frac{30000}{854}} = 5{\cdot}927$ inches, and the length $= 5{\cdot}927 \times {\cdot}854 = 5{\cdot}062$ inches.

Again, if the diameter of the gudgeon be 12 inches, its length will be $\cdot854 \times 12 = 10{\cdot}248$ inches, and the stress it may sustain $854 \times 12^2 = 122976$ lbs.

Prob. XVII. To find the exterior diameter of a hollow cylinder of cast-iron (when the proportion between the exterior and interior diameters is given) to resist a given force when supported at the ends, and the weight acts at the middle of the length.

Let the proportion between the exterior and interior diameters be 1 : N; hence N will always be a decimal, and it should never be greater than 0·8.

Rule. Multiply the length in feet by the weight to be supported in lbs. and divide the product by 500, multiplied by the difference between 1 and the fourth power of N, and the cube root of the quotient will give the diameter in inches.

That is, $l \times W = 500 \times (1 - N^4) \times d^3$.

Note. The interior diameter is found by multiplying the number N by the exterior diameter, and the thickness of the metal is half the difference of the diameters.

1. Required the exterior diameter of a hollow cylinder of cast-iron 10 feet long, and supported at both ends to sustain a weight in the middle of its length of 33600 lbs., the proportion of the exterior to the interior diameter being 1 : ·8.

Here the difference between 1 and the fourth power of N is $1 - {\cdot}4096 = {\cdot}5904$; hence $\sqrt[3]{\frac{10 \times 33600}{500 \times {\cdot}59}} = \sqrt[3]{1139} = 10{\cdot}443$ inches, the exterior diameter required; whence $10{\cdot}443 \times {\cdot}8 =$

8·3544 inches, the interior diameter, and 10·443 — 8·3544 ÷ 2 = 2·0886 ÷ 2 = 1·0443 inch, the thickness of the metal.

2. Suppose the weight of a water-wheel, when the buckets are full of water, to be 48000 lbs. and the whole length of the shaft 10 feet, from which deducting 6 feet, the width of the wheel, we have 4 feet for the length of bearing. Required the diameter of a hollow shaft for it, the exterior diameter being to the interior as 1 : ·6.

Ans. 7·613 inches, exterior diameter; 4·5678 inches, interior diameter; and 1·5226 inch, the thickness of the metal.

If, as in this example, the thickness of the metal be always $\frac{1}{5}$ of the exterior diameter, in which case the proportions of the diameter are fixed; that is, N = ·6; then the equation is $\left(\frac{W \times l}{435}\right)^{\frac{1}{3}} = d$; hence $\left(\frac{48000 \times 4}{435}\right)^{\frac{1}{3}} = \sqrt[3]{441·38} = 7·613$ the exterior diameter in inches, the same as before, but by a much simpler computation.

Prob. XVIII. To find the diameter of a hollow cylinder of cast-iron when supported at both ends, but the load nearer the one end than the other.

Rule. Multiply the product of the segments into which the strained point divides the cylinder in feet by 4 times the weight in lbs., and divide this product by 500 times the length in feet multiplied by the difference between 1 and the fourth power of N, and the cube root of the quotient is the diameter in inches.

That is, $4W \times AC \times CB = 500l \times (1 - N^4) \times d^3$; and when the thickness of the metal is one-fifth of the exterior diameter, the equation becomes $4W \times AC \times CB = 435 \times l \times d^3$; AC, CB being the segments into which the strained point divides the cylinder.

1. Suppose the weight of a water-wheel, when the buckets are filled with water, to be 48000, the distance of the straining point from the one end to the point of bearing 3·7 feet, and the distance of the other bearing 1·3 foot, and also N = ·8. Required the exterior and interior diameters of the shaft and the thickness of the metal.

Here $1 - N^4 = ·59$, and $\frac{4 \times 48000 \times 3·7 \times 1·3}{500 \times 5 \times ·59} = 626·115$,

and $\sqrt[3]{626·115} = 8·555$ inches, exterior diameter; whence $8·555 \times ·8 = 6·844$ inches, interior diameter, and $\frac{1}{2}(8·555 - 6·844) = ·8555$ inch, thickness of the metal.

2. Suppose the weight of a wheel and other pressure upon a hollow shaft to be 28000 lbs., the distance from the strain-

ing-point to the point of bearing being at one end 4·5 feet, and at the other 2·5 feet; N being ·6. Required the exterior and interior diameters, and the thickness of the metal.

Ans. Exterior diameter, 7·452 inches; interior diameter, 4.4712 inches; thickness of the metal, 1·4904 inch.

Prob. XIX. To find the diameter of a solid cylinder of cast-iron to resist torsion, with a given flexure.

Rule. Multiply the power in pounds by the length of the shaft in feet, and by the leverage also in feet; then divide the product by 55 times the number of degrees of the angle of flexure, and take the fourth root of the quotient for the diameter in inches.

That is, $W \times l \times \text{leverage} = 55 \times \text{angle of flexure} \times d^4$.

1. What is the diameter of a shaft 40 feet long, which will transmit a power of 5000 lbs. acting at the circumference of a wheel of 3 feet radius, so that the angle of flexure may not be more than $1\frac{1}{2}$ degree?

Here $\left(\frac{5000 \times 40 \times 3}{55 \times 1\frac{1}{2}}\right)^{\frac{1}{4}} = \left(\frac{1000 \times 40 \times 2}{11}\right)^{\frac{1}{4}} = \sqrt[4]{7272\cdot\dot{7}\dot{2}}$ $= 9\cdot235$ inches, the diameter required.

2. What is the diameter for a series of shafts 25 feet long, which are to transmit a power of 6000 pounds, acting with a leverage of $2\frac{1}{2}$ feet, so that the twist or angle of flexure may not exceed $\frac{1}{2}$ a degree? Ans. 10·806 inches.

Prob. XX. To find the diameter of a hollow cylinder of cast-iron to resist torsion, with a given flexure, when the thickness of the metal is one-fifth of the diameter.

Rule. Multiply the power in pounds by the length in feet, and by the leverage also in feet; then divide the product by 48 times the angle of flexure, and take the fourth root of the quotient for the diameter in inches.

That is $W \times l \times \text{leverage} = 48 \times \text{angle of flexure} \times d^4$.

1. What is the diameter of a hollow shaft of cast-iron 20 feet long, sufficient to withstand a force of 3000 lbs. acting at the circumference of a wheel of 6 feet diameter, the angle of flexure being $\frac{3}{4}$ of a degree, and the thickness of the metal one-fifth of the diameter of the shaft.

Here $\left(\frac{3000 \times 20 \times 3}{48 \times \cdot 75}\right)^{\frac{1}{4}} = \left(\frac{3000 \times 5}{3}\right)^{\frac{1}{4}} = \sqrt[4]{5000} = 8\cdot409$ inches, the exterior diameter; $8\cdot409 \div 5 = 1\cdot682$ inch, the thickness of the metal; and $8\cdot409 - (1\cdot682 \times 2) = 5\cdot045$ inches, the interior diameter.

2. What is the diameter of a hollow shaft of cast-iron 30 feet long, to withstand a force of 4000 lbs. acting with a leverage of 2 feet, so that the angle of flexure may not exceed $1\frac{1}{2}$ degree; the thickness of the metal being one-fifth of the exterior diameter?

Ans. 7·598 inches, exterior diameter; 4·559 inches, interior diameter, and 1·5196 inch, the thickness of the metal.

Prob. XXI. To find the side of a square shaft of cast iron to resist torsion with a given flexure.

Rule. Multiply the power in pounds by the length of the shaft in feet, and by the leverage also in feet; then divide the product by 92·5 times the angle of flexure in degrees; and the square root of the quotient is the area of a cross section of the shaft in inches, or the fourth root of the quotient is the side in inches.

That is, $W \times l \times \text{leverage} = 92{\cdot}5 \times \text{angle of flexure} \times s^4$ where $s =$ the side of the shaft.

1. What is the side of a square shaft of cast-iron 15 feet long, to withstand a power of 1000 lbs. acting with a leverage of $1\frac{1}{2}$ foot, so that the angle of flexure may not exceed $1\frac{1}{2}$ degree?

Here $\left(\frac{1000 \times 15 \times 1\frac{1}{2}}{92{\cdot}5 \times 1\frac{1}{2}}\right)^{\frac{1}{2}} = \sqrt{162{\cdot}\dot{1}6\dot{2}} = 12{\cdot}7303$ square inches, area of a cross section, and $\sqrt{12{\cdot}7303} = 3{\cdot}568$ inches, the side of the square shaft.

2. What is the side of a square shaft of cast-iron 20 feet long, which is to be driven by a power of 10000 lbs. acting on a pinion fixed on it of 2 feet radius at the pitch line, so that the flexure may not exceed 2 degrees? Ans. 6·819 inches.

Note 1. The strength of direct cohesion of the materials in Tables II. and III. may be found by Problem I., using the numbers in those Tables opposite to the material, instead of the value of C in Table I.

1. What weight will pull asunder a rod of cast-iron 2 inches square?

Here $2 \times 2 \times 50500 = 202000$ lbs., or according to Mr Rennie's experiments $2 \times 2 \times 19096 = 76384$ lbs.

2. What weight may be suspended from a brass wire $\frac{1}{12}$ inch in diameter?

Ans. 278·1626 lbs., or according to Mr Rennie 97·946 lbs.

3. What weight will pull asunder a hemp rope $1\frac{1}{4}$ inch in diameter? Ans. 11308·5 lbs.

Note 2. The lateral strength of iron may be found by the rules for that of timber, using the number opposite to iron in Table IV., instead of the value of S in Table I.

1. What weight will it require to break a bar of cast-iron 20 feet long, 10 inches deep, and 6 broad, when fixed at one end and the load applied at the other?

Here $3270 \times 6 \times 10^2 \div 240 = 1962000 \div 240 = 8175$ lbs. By Problem XI. the weight which it would bear is 6360 lbs.

2. What weight will it require to break a bar of cast-iron 30 feet long, 8 deep, and 4 broad, when supported at both ends?

Here $3270 \times 4 \times 4 \times 8^2 \div 360 = 9301{\cdot}3$ lbs. By Problem IX. the weight which it would bear is 7253·3 lbs.

NOTE 3. The strength of a column to resist being crushed is directly as the area of its cross section. Hence, to find the weight which will crush any column, multiply the area of its cross section in inches by the numbers in Table V. and divide the product by $2\frac{1}{4}$, the quotient is the number of pounds.

1. What weight will it require to crush a square column of Dundee sandstone, 12 inches in the side?

Here $14919 \times 12^2 \div 2\frac{1}{4} = 954816$ lbs.

2. What weight will it require to crush a cylindrical column of Aberdeen blue granite, 10 inches in diameter?

Ans. 856469·97 lbs.

3. What weight will it require to crush a cylindrical column of Craigleith stone, 12 inches in diameter?

Ans. 777643·847 lbs. with the strata, or 620579·1 lbs. across the strata.

Mr R. Buchanan, in his Essay on the Strength of Shafts, gives the following rule for the diameters of solid shafts of cast-iron, viz.:—

The cube root of the weight in pounds is *nearly* equal to the diameter in inches.

This agrees very nearly with the rule in Prob. XV., or when the weight is uniformly distributed over the length of the shaft, but differs widely from the rule in Prob. XIII., or when the weight acts in the middle of the length; in which case one-third of the result should be added to render the rule safe in practice. Thus,

Suppose the shaft of a water-wheel to be 10 feet long, and the weight of the wheel, when the buckets are full of water, to be 20 tons, or 44800 lbs., what is the diameter of the journal?

By Mr Buchanan's rule, we have $\sqrt[3]{20 \times 20} = 7{\cdot}36$ inches.

By Problem XIII., $\left(\frac{44800 \times 10}{500}\right)^{\frac{1}{3}} = \sqrt[3]{896} = 9{\cdot}64$ inches,

and by Problem XV., $\frac{1}{10}\,(44800 \times 10)^{\frac{1}{3}} = 76{\cdot}5 \div 10 = 7{\cdot}65$ inches, which last agrees very nearly with Mr Buchanan's rule.

Mr Buchanan, in the same Essay, in treating of the torsion of shafts, states, from various experiments, that the fly-wheel of a 50 horse power engine, making 50 revolutions per minute, should be $7\frac{1}{2}$ inches in diameter. Now, as the strength of revolving shafts is directly as the cubes of their diameters and revolutions, and inversely as the resistance they have to overcome, then the cube of $7\frac{1}{2} = 421{\cdot}875$, is the multiplier for all other shafts in the same proportion, and from this as a standard he deduces the following multipliers:—

1. For any shaft connected with a first power in an engine, the multiplier is................................ *400
2. For shafts in the inside of mills, or second movers, the multiplier is.. 200
3. For the small shafts of the machinery, or third movers, the multiplier is................................ 100

Whence the number of horses' power a shaft is equal to, is directly as the cube of the diameter and number of revolutions, and inversely as these multipliers.

1. The velocity of a shaft is 100 revolutions per minute, and the fly-wheel of 60 horse power, what is the diameter of the journal?

Here $\left(\frac{60 \times 400}{100}\right)^{\frac{1}{3}} = \sqrt[3]{240} = 6{\cdot}214$ inches, the diameter.

2. The velocity of a shaft is 60 revolutions per minute, and its diameter 5 inches, what is its power?

Here $5^3 \times 60 \div 400 = 18\frac{3}{4}$ horse power.

3. The velocity of a second mover is 45 revolutions per minute with a 30 horse power, what is its diameter?

Here $\left(\frac{30 \times 200}{45}\right)^{\frac{1}{3}} = \sqrt[3]{133{\cdot}\dot{3}} = 5{\cdot}18$ inches, the diameter.

The diameter of second movers is found by multiplying the diameters of the first movers by ·8, and the diameter of third movers is found by multiplying those of first movers by ·641. Thus,

The diameter of a first mover is 7·5 inches, what should be the diameters of the second and third movers?

Here $7{\cdot}5 \times {\cdot}8 = 6$ inches, diameter of second movers, and $7{\cdot}5 \times {\cdot}641 = 4{\cdot}8075$ inches, diameter of third movers.

NOTE. The rules are for shafts of cast-iron, but it will answer for malleable iron, if we multiply the result by ·963; or for oak, if we multiply by 2·238; if the shaft be of fir, we must multiply the result by 2·06.

* Some authors use 240 as a multiplier for first movers, which gives a somewhat smaller diameter to the shaft.

SPHERICAL TRIGONOMETRY.

DEFINITIONS AND PRINCIPLES.

SPHERICAL TRIGONOMETRY is that branch of Mathematics which shows how to compute the sides and angles of spherical triangles.

A SPHERE is a solid bounded by a curve-surface, every point of which is equally distant from a point within it, called the *centre*.

A sphere may be conceived to be generated by a semicircle revolving about its diameter.

The *axis* or *diameter* of a sphere is a straight line passing through the centre, and both ends terminating at the surface.

Any circle formed from the section of a sphere by a plane passing through its centre, is called a *great circle* of the sphere; and all others *small circles*.

The *pole* of a great circle is a point on the surface of the sphere, equally distant from every point in the circumference of that circle.

A *spherical angle* is the angle made by two arcs of great circles, and is the same with the inclination of the planes of these circles, or with the plane angle made by the tangents to those arcs at the point of intersection.

A *spherical triangle* is a figure formed upon the surface of a sphere by the intersection of the arcs of three great circles.

A spherical triangle is called RIGHT-ANGLED, when it has one right angle; QUADRANTAL, when it has one side equal to 90°; and OBLIQUE-ANGLED, when it has none of its angles right angles. It is also called *equilateral*, when the three sides are equal; *isosceles*, when two sides are equal; and *scalene*, when all the three sides are unequal.

NOTE. A right-angled spherical triangle may have *one*, *two*, or *three* right angles, and in the last case it is likewise quadrantal, and the angles and sides are known.

Arcs or angles are said to be *alike*, or of the *same affection*, when both are less or both greater than a quadrant; and they are said to be *unlike* or of *different affection*, when the one is greater and the other less than a quadrant.

Prop. I. If a sphere be cut by a plane in any direction, the section will be a circle.

Prop. II. If two arcs of circles meet each other, they make two angles, which are together equal to two right angles.

Prop. III. If two arcs of a circle intersect each other, the vertical or opposite angles will be equal.

Cor. All the angles formed about the point in which any number of arcs of circles intersect each other, are together equal to four right angles.

Prop. IV. The arc of a great circle, between the pole and the circumference of another great circle, is a quadrant.

Cor. 1. The straight line drawn from the pole of any great circle to the centre of the sphere, is at right angles to the plane of that circle; and conversely.

Cor. 2. The poles of a great circle are the extremities of the axis of the sphere, which is perpendicular to the plane of that great circle.

Prop. V. A spherical angle at the pole of a great circle is measured by the arc of that great circle intercepted between the circles which contain the angle.

Prop. VI. If two arcs of different great circles be drawn from the same point, and each of them be a quadrant, that point is the pole of the great circle which passes through the extremities of these arcs.

Cor. 1. A great circle drawn through the pole of another great circle cuts it at right angles.

Cor. 2. Great circles, whose planes are perpendicular to the plane of one and the same great circle, meet in the poles of that circle.

Prop. VII. If two spherical triangles have the three sides of the one equal to the three sides of the other, each to each, the angles which are opposite to the equal sides are likewise equal; and conversely.

Prop. VIII. If two sides and the included angle of one spherical triangle be equal to two sides and the included angle in another, these two triangles are equal in every respect.

Prop. IX. The angles at the base of an isosceles spherical triangle are equal to one another.

Cor. 1. If two of the angles of a spherical triangle be equal to one another, the sides opposite to them are also equal.

Cor. 2. If a perpendicular be drawn from the vertex of an isosceles spherical triangle to the base, it will bisect both the vertical angle and the base, except when the two sides are quadrants, in which case the number of perpendiculars is indefinite.

Prop. X. Any two sides of a spherical triangle are together greater than the third side, and the difference of any two sides is less than the third.

Cor. The arc which passes through any two points on the surface of a sphere is the shortest distance between these points.

Prop. XI. The three sides of a spherical triangle are together less than the circumference of a great circle or 360°; and the difference of any two sides is less than half the circumference or 180°.

Prop. XII. The greater angle of a spherical triangle has the greater side opposite to it, and the less angle has the less side opposite to it; and conversely.

Prop. XIII. If two sides of a spherical triangle be together equal to, greater, or less than a semicircle, the sum of their opposite angles will be equal to, greater, or less than two right angles; and conversely.

Cor. 1. If each side of a spherical triangle be equal to, greater, or less than a quadrant, each of the angles will, accordingly, be right, obtuse, or acute; and conversely.

Cor. 2. Half the sum of any two sides of a spherical triangle is of the same affection as half the sum of their opposite angles.

Prop. XIV. If from the angular points of a spherical triangle as poles there be described on the surface of the sphere three arcs of great circles, which by their intersection form another spherical triangle, each side of this new triangle will be the supplement of the measure of the angle which is at its pole; and the measure of each of its angles will be the supplement to that side of the primitive triangle to which it is opposite.

Cor. Hence these two triangles are called supplemental or *polar* triangles.

Prop. XV. The three angles of a spherical triangle are together greater than two and less than six right angles.

Cor. 1. The three angles, together with twice the supplement of the least, are less than six right angles.

Cor. 2. The sum of any two angles is greater than the supplement of the third angle.

PROP. XVI. In any right-angled spherical triangle the sides about the right angle are of the same affection with their opposite angles; and conversely.

Cor. The same is also the case in any quadrantal triangle.

PROP. XVII. In any right-angled spherical triangle the hypotenuse is greater or less than a quadrant, according as the two sides about the right angle are of the same or of different affection; and conversely. If one of the sides be a quadrant, the hypotenuse is also a quadrant.

Cor. The hypotenuse will be greater or less than a quadrant, according as the angles are of the same or of different affection, because the angles are of the same affection as their opposite sides.

PROP. XVIII. In any spherical triangle, if the perpendicular drawn from the vertex to the base fall within the triangle, the angles at the base are of the same affection; and if it fall without the triangle, they are of different affection; and conversely.

STEREOGRAPHIC PROJECTION OF THE SPHERE.

DEFINITIONS AND PRINCIPLES.

To project an object is to represent every point of it upon the same plane as it appears to the eye in a certain position.

The *plane* of projection is that upon which the object is projected, and the point where the eye is situated is called the projecting point.

The *stereographic* projection is a representation of the circles of the sphere upon the plane of one of its great circles, such as they would appear to an observer placed in one of the poles of that circle.

The great circle, upon the plane of which the projection is made, is called the *primitive*.

By the *semitangent* of any arc is meant the tangent of half that arc.

The *line of measures* of any circle of the sphere is that diameter of the primitive produced indefinitely, which is perpendicular to the line of common section of the circle and the primitive.

The representation or projection of any point in the sphere is the point in which the straight line drawn from it to the projecting point intersects the plane of projection.

Prop. I. Every great circle of a sphere, which passes through the projecting point, is projected into a straight line passing through the centre of the primitive; and every arc of it, reckoned from the other pole of the primitive, is projected into its semitangent.

Cor. 1. Every small circle which passes through the projecting point is projected into that straight line which is its common section with the primitive.

Cor. 2. Every straight line in the plane of the primitive, and produced indefinitely, is the projection of some circle on the sphere passing through the projecting point.

Cor. 3. The stereographic projection of any point on the surface of the sphere is distant from the centre of the primitive by the semitangent of the distance of that point from the pole opposite the projecting point.

Prop. II. Every circle of the sphere, which does not pass through the projecting point, is projected into a circle.

Cor. 1. The centres and poles of all circles parallel to the primitive have their projections in its centre.

Cor. 2. The centres and poles of every circle inclined to the primitive have their projections in the line of measures.

Cor. 3. All projected great circles cut the primitive in two points diametrically opposite.

Prop. III. The centre of the projection of a great circle is distant from the centre of the primitive by the tangent of that great circle's inclination to the primitive, and its radius is the secant of the same.

Prop. IV. The centre of projection of a small circle perpendicular to the primitive is distant from the centre of the primitive by the secant of the distance of the circle from its nearest pole, and the radius of projection is the tangent of the same.

Prop. V. The projection of the poles of any great circle inclined to the primitive is in the line of measures distant from the centre of the primitive by the tangent and cotangent of half its inclination.

Prop. VI. Any two circles upon the sphere passing through the poles of two great circles, intercept equal arcs upon these circles.

PROP. VII. If from either pole of a projected great circle two straight lines be drawn to meet the primitive and the projection, they will intercept corresponding arcs of these circles.

SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

EVERY spherical triangle consists of six parts,—three sides and three angles,—any three of which being given, the rest may be found.

In a right-angled spherical triangle the right angle can never be the subject of inquiry; and therefore there are only the three sides and the two oblique angles presented to our consideration, and of these the two sides, containing the right angle and the *complements* of the angles and of the hypotenuse, are called the FIVE CIRCULAR PARTS.

When any one of these is taken as the MIDDLE PART, the two which are immediately adjacent to it on the right and left are called the ADJACENT PARTS; and the other two, each being separated from the middle part by an adjacent part, are called OPPOSITE PARTS.

With this arrangement of the different parts, the solution, in every case, is obtained by the two following equations.

1. Rad. $\times$ sin. middle part $=$ the rectangle of the tangents of the adjacent parts.

2. Rad. $\times$ sin. middle part $=$ the rectangle of the cosines of the opposite parts.

NOTE. In applying these equations to the solution of problems, take that, as the middle part, which is either adjacent to the other two given parts, or is separated from them by the remaining parts of the triangle, and form the equations according as the remaining parts are adjacent or opposite.

These equations may be transformed into proportions having the required part for the last term whence its value will be obtained.

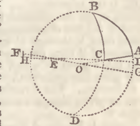
A *quadrantal triangle* may be changed into a right-angled triangle, by calling the supplement of the angle opposite to the quadrantal side, the hypotenuse; the other angles, the sides; the quadrantal side, radius; and the other sides, angles; but in the solution we must substitute *same* for *different affection* in the limitation.

The following Table contains the proportions for the solution of the sixteen cases of any right-angled spherical triangle ABC (see figure, Case 1.)

Given.	Sought.	Solution.	Equations.	Limitation.	Cases.
BC & B	AC	R : sin. BC : : sin. B : sin. AC.	2	of the same affection with B.	1
	AB	R : cos. B : : tan. BC : tan. BA.	1	less than 90°, when BC and B are of the same affection.	2
	C	R : cos. BC : : tan. B : cot. C.	1	otherwise greater than 90°.	3
AC & C	AB	R : sin. AC : : tan. C : tan. AB.	1	of the same affection with C.	4
	BC	cos. C : R : : tan. AC : tan. BC.	1	less than 90°, when AC and C are of the same affection.	5
	B	R : sin. C : : cos. AC : cos. B.	2	of the same affection with AC.	6
AC & B	AB	tan. B : tan. AC : : R : sin. AB.	1	ambiguous; for two triangles may have the given	7
	BC	sin. B : R : : sin. AC : sin. BC.	2	things, but have the things sought in one of them	8
	C	cos. AC : R : : cos. B : sin. C.	2	the supplements of the things sought in the other.	9
AC & CB	AB	cos. AC : R : : cos. BC : cos. BA.	2	less than 90°, if AC and CB be of the same affection.	10
	B	sin. BC : R : : sin. AC : sin. B.	2	of the same affection with AC.	11
	C	tan. BC : tan. AC : : R : cos. C.	1	less than 90°, if AC and CB be of the same affection.	12
AB & AC	BC	R : cos. AC : : cos. AB : cos. BC.	2	less than 90°, if AB and AC be of the same affection.	13
	B	sin. AB : R : : tan. AC : tan. B.	1	of the same affection with AC.	14
	C	sin. AC : R : : tan. AB : tan. C.	1	of the same affection with AB.	14
B & C	AB	sin. B : R : : cos. C : cos. AB.	2	of the same affection with C.	15
	AC	sin. C : R : : cos. B : cos. AC.	2	of the same affection with B.	15
	BC	tan. B : cot. C : : R : cos. BC.	1	less than 90°, if B and C be of the same affection.	16

CASE I. GIVEN THE HYPOTENUSE AND AN ANGLE.

1. In the right-angled spherical triangle ABC are given the hypotenuse BC 63° 30′, and the angle ABC 53° 42′; to find the sides AB, AC, and the angle ACB.



Construction. Draw the radius OF of the primitive BAD. Make OE the semitangent, and OF the tangent of 53° 42′, then E is the pole of the hypotenuse, and F its centre, from which, with the secant of 53° 42′, describe the circle BCD. From B to I lay 63° 30′ on the primitive, draw a straight line from its extremity I to E, cutting BCD in C, and draw the radius OCA; then ACB is the triangle. The side AB is measured on the line of chords. OC measured on the line of semitangents, and subtracted from 90°, or AC reckoned on the line of semitangents from 90° backward, gives the arc AC. Extend the straight line IE to H, and HD, measured on the line of chords, gives the angle ACB.

Calculation. The five parts of this triangle are BC, the angles at B and C, and the complements of AB and AC, which are AG and OC. Of these, BC and B are given; and of the things required, BA and C are adjacent to given things, and are therefore found by Equa. 1.; and AC being separated from given things, is found by Equa. 2.

By Equa. 1. R : cos. BC : : tan. B : cot. C; and R : cos. B : : tan. BC : cot. AG = tan. AB. And by Equa. 2. R : sin. B : : sin. BC : cos. CO = sin. CA. And all the three are acute. For CA is of the same affection with B. And AB and C are acute, because BC and B are of the same affection.

BC 63° 30′	cos. 9·649527	tan. 10·302264	sin. 9·951791
B 53° 42′	tan. 10·133965	cos. 9·772331	sin. 9·906296
C 58° 43′ 28″	cot. 9·783492	AB tan. 10·074595	CA sin. 9·858087
		AB acute = 49° 53′ 48″	CA acute = 46° 9′ 29″

2. Given the hypotenuse BC 126° 24′, and the angle B 57° 22′; to find the rest.

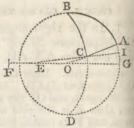
Ans. The angle C 132° 49′ 18″, the sides AB 36° 10′ 59″, and AC 137° 19′ 32″.

3. Given the hypotenuse BC 72° 28′, and the angle B 138° 23′; to find the rest.

Ans. The angle C 104° 58′ 58″, the sides AB 112° 54′ 32″, and AC 140° 42′ 24″.

CASE II. GIVEN A SIDE AND THE ADJACENT ANGLE.

1. In the spherical triangle ABC, right-angled at A, are given the side AB 51° 28′, and the angle ABC 66° 48′; to find the hypotenuse BC, the side AC, and the angle at C.



Construction. Draw the diameter GF of the primitive ABD. Make OE the semitangent of 66° 48′, and OF its tangent. From F, with the secant of 66° 48′ for a radius, describe the circle BCD; make BA 51° 28′, and draw ACO, then ABC is the triangle.

AC, or its complement CO, is measured on the line of semitangents. Draw a line from E through C to I, and the distance of B from the point I, where it cuts BAG gives BC; and the distance of D from its other extremity gives the angle at C.

Calculation. The hypotenuse BC, and the side CA, being adjacent to given things, are found by Equa. 1., and the angle C by Equa. 2.

Thus; 1. R : cos. AG = sin. AB : : tan. B : cot. OC = tan. CA, like B; and cos. B : R : : cot. AG = tan. AB : tan. BC, acute, for BA is like B. Also, 2. R : sin. B : : sin. AG = cos. AB : cos. C, like AB.

AB 51° 28′	sin. 9·893343	R + tan. 20·098876	cos. 9·794467
B 66° 48′	tan. — R 0·367947	cos. 9·595432	sin. 9·963379
AC 61° 16′ 52″	tan. 10·261290	BC tan. 10·503444	C cos. 9·757846
		BC = 72° 34′ 54″	C = 55° 4′ 7″

2. Given the side AB 126° 26′, and the angle B 142° 48″; to find the rest.

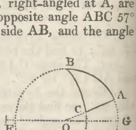
Ans. Hyp. BC 59° 32′ 45″, side AC 148° 35′ 17″, and the angle C 111° 2′ 34″.

3. Given the side AB 57° 44′, and the angle B 112° 26′; to find the rest.

Ans. Hyp. BC 103° 32′ 46″, the side AC 116° 1′ 26″, and the angle C 60° 25′ 54″.

CASE III. GIVEN A SIDE AND THE OPPOSITE ANGLE.

1. In the spherical triangle ABC, right-angled at A, are given the side AC 38° 27′, and the opposite angle ABC 57° 48′; to find the hypotenuse BC, the side AB, and the angle at C.



Construction. On OA the radius of the primitive make OC 51° 33′ the complement of AC. With the tangent of 57° 48′ describe an arc from O, and with the secant of 57° 48′ from C cut that arc in F, from which centre describe the circle BCD, then either ABC or ADC is the triangle. AB is measured on the line of chords, and BC and C as in the last case.

Calculation. AB being adjacent to given things, is found by Equa. 1., and BC and C by Equa. 2. They are all ambiguous, or have two values.

1. R : cos. AG = sin. AB : : tan. B : cot. CO = tan. AC, and tan. B : tan. AC : : R : sin. AB or AD.

2. R : sin. B : : sin. BC : cos. CO = sin. CA, and (inver.) sin. B : R : : sin. CA : sin. CB or CD. R : sin. OC = cos. AC : : sin. C : cos. B, and (inver.) cos. CA : R : : cos. B : sin. ACB or ACD.

B = 57° 48′	tan. 10·200843	sin. 9·927470	R+cos. 19·726626
AC = 38° 27′	R + tan. 19·899827	R+sin. 19 793673	cos. 9·893846
	Sine of AB = 9·698984	Sin. BC 9·866203	Sin. C 9·832780
AB = 30° 0′ 4″		BC = 47° 17′ 43″	C = 42° 52′ 37″
or 149° 59′ 56″		or 132° 42′ 17″	or 137° 7′ 23″

2. Given the side AC 136° 28′, and the angle B 127° 48′; to find the rest.

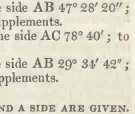
Ans. The hyp. BC 60° 39′ 24″; the side AB 47° 28′ 20″; and the angle C 57° 43′ 1″, or their supplements.

3. Given the angle B 84° 21′, and the side AC 78° 40′; to find the rest.

Ans. The hyp. BC 80° 9′ 35″; the side AB 29° 34′ 42″; and the angle C 30° 3′ 54″ or their supplements.

CASE IV. WHEN THE HYPOTENUSE AND A SIDE ARE GIVEN.

1. In the spherical triangle ABC, right-angled at A, are given the hypotenuse BC 64° 42′, and the side AC 47° 48′; to find the side AB, and the angles at B and C.



Construction. Lay AC 47° 48′ on the primitive, and draw the radii OC, OA. On the former lay the secant of 64° 42′ from O to H, from which, with the tangent of 64° 42′, cut OA in B, and describe the circle CBD, then ABC is the triangle.

Let F be the centre of CBD, then OF measured on the line of tangents gives the angle ACB. Lay the semitangent of it from D to E. Lay a ruler from B through E; the arc of the primitive between it and D is the measure of the angle at B, and OB measured on the line of semitangents gives the complement of AB.

Calculation. The angle at C being adjacent to the given things, is found by Equa. 1; the other two, being separated from them, are found by Equa. 2.

1. Tan. CB : cot. AG = tan. AC : : R : cos. C acute, since AC, AB are alike.

2. Sin. CB : R : : cos. AG = sin. AC : sin. B, like AC. Sin. AG = cos. AC : R : : cos. BC : sin. OB = cos. BA acute, because AC, CB are alike.

BC 64° 42′	R+cos. 19·630792	sin. 9·956208	tang. 10·325416
AC 47° 48′	cos. 9·827189	R + sin. 19·869704	R + tang. 20·042515
AB 50° 29′ 24″	cos. 9·803603	sin. B 9·913496	cos. C 9 717099
		B = 55° 1′ 29″	C = 58° 34′ 46″

2. Given the hypotenuse BC 121° 12′, and the side AC 56° 15′; to find the rest.

Ans. The angles C 155° 0′ 33″, and B 76° 25′ 31″; and the side AB 158° 48′ 56″.

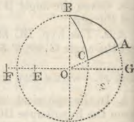
3. Given the hypotenuse BC 72° 28′, and the side AC 123° 16′; to find the rest.

Ans. The angles C 118° 47′ 20″, and B 118° 44′ 2″, and the side AB 123° 18′ 46″.

CASE V. GIVEN THE SIDES ABOUT THE RIGHT ANGLE.

1. Given the sides AB 47° 38′, and AC 67° 30′, about the right angle BAC of the spherical triangle ABC; to find the hypotenuse BC, and the angles at B and C.

Construction. Make AB 47° 38′ on the primitive, and draw the radius OA, on which make OC = 22° 30′ the complement of AC taken from the line of semitangents, and having drawn the diameter BD, describe the circle BCD; then ABC is the triangle.



Let F be the centre of BCD, then OF measured on the line of tangents gives the angle at B. Make OE its semitangent, then E is the pole of BCD, and BC and C are measured as in the 2d Case.

Calculation. The angles at B and C being adjacent to given things, are found by Equa. 1., and the hypotenuse BC by Equa. 2.

1. Cos. AG = sin. AB : R : : cot. OC = tan. AC : tan. B, like AC. Cos. OC = sin. AC : R : : cot. AG = tan. AB : tan. C, like AB. 2. R : sin. OC = cos. AC : : sin. AG = cos. AB : cos. BC acute, for AB, AC are like.

AC 67° 30′	R + tan. 20·382776	sin. 9·965615	cos. 9·582840
AB 47° 38′	sin. 9·868555	R + tan. 20·039977	cos. 9·828578
B 72° 59′ 2″	tan. 10·514221	C tan. 10·074362	BC cos. 9·411418
		C = 49° 52′ 53″	BC = 75° 3′ 21″

2. Given the sides about the right angle AB 108° 44′, and AC 67° 42′; to find the rest.

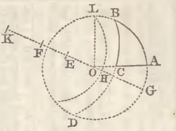
Ans. The angles C 107° 25′ 12″, and B 68° 46′ 25″, and the hyp. BC 97°.

3. Given the sides about the right angle AC 127° 48′, and AB 71° 25′; to find the rest.

Ans. The angles B 126° 19′ 29″, and C 75° 7′ 21″, and the side BC 101° 15′ 49″.

CASE VI. WHEN THE TWO OBLIQUE ANGLES ARE GIVEN.

1. In the spherical triangle ABC, right-angled at A, are given the angles at B 39° 48′, and at C 67° 12′; to find the hypotenuse BC, and the sides AB and AC.



Construction. Draw any diameter of the primitive EOG. Make OE the semitangent, and OF the tangent of 39° 48′, and from F with its secant describe the circle BCD. Add and subtract the angles, and make OK the semi-tangent of their sum, and OH that of their difference; then upon the diameter HK describe a circle, cutting the primitive in L. Join LO, and draw OA perpendicular to it; then ABC is the triangle.

The hypotenuse and the sides are measured as before.

Calculation. The hypotenuse being adjacent to the given angles is found by Equa. 1., and the sides by Equa. 2.

1. Tan. B : cot. C : : R : cos. BC acute, for B and C are alike.

2. Sin. B : R : : cos. C : sin. AG = cos. AB, like C; and sin. C : R : : cos. B : sin. OC = cos. AC, like B.

C 67° 12′	R + cot.	19·623623	R + cos.	19·588289	sin.	9·964666
B 39° 48′	tan.	9·920733	sin.	9·806254	R + cos.	19·885522
BC 59° 41′ 58″	cos.	9·702890	AB cos.	9·782035	AC cos.	9·920856
			AB =	52° 44′ 35″	AC =	33° 33′

2. Given the angles B 112° 38′, and C 63° 40′; to find the sides.

Ans. The hyp. BC 101° 54′ 34″, the sides AC 115° 25′ 44″, and AB 61° 16′ 30″.

3. Given the angles C 102° 28′, and B 118° 30′; to find the sides.

Ans. The hyp. BC 83° 6′ 20″, the sides AB 104° 13′ 11″, and AC 119° 15′ 14″.

SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

When the three sides or the three angles are not the given parts, the solution may always be obtained by drawing a perpendicular from the extremity of a given side and opposite a given angle, and then computing by Napier's rules of the circular parts.

The following Table contains the proportions for the solution of the 12 cases of oblique-angled spherical triangles, where ABC represents any spherical triangle in which the perpendicular AD either falls within the triangle, or meets the base BC produced beyond C.

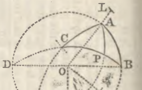
Note. The cases referred to are those of the preceding Table.

Cases.	Given.	Sought.	Solution.
1	AB, AC, and B, opposite to AC.	C, the angle opposite to AB.	Sin. AC : sin. AB : : sin. B : sin. C. If the sum of BA, AC be less than 180°; and AB less than AC; the angle at C is acute; or, if the sum of BA, AC be greater than 180°, and AB greater than AC, ACB is obtuse. In other cases, ACB is ambiguous.
2	AB, AC, and B, opposite to AC.	BC, the third side.	R : cos. B : : tan. AB : tan. BD (case 2), and cos. AB : cos. AC : : cos. BD : cos. DC. When ABC is acute, DC, CA are of the same affection, otherwise they are of different affection. If CD be not less than DB, their sum is CB; if CD be less than DB, but their sum not less than 180°, their difference is CB. In other cases, CB is ambiguous.
3	AB, AC, and B, opposite to AC.	A, the angle contained by the sides.	R : cos. AB : : tan. B : cot. BAD (case 3), and tan. AC : tan. AB : : cos. BAD : cos. DAC. If B be acute, DAC and AC are of the same affection, otherwise they are of different affection. If DAC be not less than BAD, their sum is BAC; if DAC be less than BAD, but their sum not less than 180°, their difference is BAC. In other cases, BAC is ambiguous.
4	B, C, and AB, two angles and the side opposite to one of them C.	AC, the side opposite to B.	Sin. C : sin. B : : sin. AB : sin. AC. If the sum of B and C be less than 180°, and B less than C, AC is acute; or if the sum of B and C be greater than 180°, and B greater than C, AC is obtuse. In other cases, AC is ambiguous.
5	B, C, and AB, two angles and the side opposite to one of them C.	A, the third angle.	R : cos. AB : : tan. B : cot. BAD (case 3), and cos. B : cos. C : : sin. BAD : sin. DAC, which is less than BAD, if B and C be of different affection, or less than the supplement of BAD, if B and C be of the same affection. In other cases it is ambiguous. When B and C are of the same affection, BAC is the sum of BAD, DAC, otherwise it is their difference.
6	B, C, and AB, two angles and the side opposite to one of them C.	BC, the side between the angles.	R : cos. B : : tan. AB : tan. BD (case 2), and tan. C : tan. B : sin. BD : : sin. DC; and DC is less than DB, if B and C be of different affection; or less than the supplement of DB, if B and C be of the same affection. In other cases, DC is ambiguous. If B and C be of the same affection, BC is [illegible]

7	AB, BC, and B, two sides and the included angle.	C, one of the other angles.	R : cos. B : : tan. AB : tan. BD (case 2), and the difference of BC and BD is DC. And sin. DC : sin. DB : tan. B : tan. C, where B and C are of the same affection, if BC be greater than BD; otherwise they are of different affection.
8	AB, BC, and B, two sides and the included angle.	AC, the third side.	Find BD and DC as in the last case, then cos. BD : cos. DC : : cos. BA : cos. AC. If BD and DC be of the same affection, BA and AC are of the same affection; otherwise they are of different affection. Or add the sines of the two given sides, and twice the sine of half the contained angle, and from half the sum of these three logarithms subtract the sine of half the difference of the sides; the remainder is the tangent of an arc, whose sine taken from the half sum will leave the sine of half the required side.
9	A, B, and AB, two angles, and the included side.	C, the third angle.	R : cos. AB : : tan. B : cot. BAD (case 3), and the difference of BAC, BAD, is DAC, then sin. BAD : sin. DAC : : cos. B : cos. C. If BAC be greater than BAD, B and C are of the same affection; otherwise they are of different affection.
10	A, B, and AB, two angles and the included side.	AC, one of the other sides.	Find BAD and DAC, as in the last case; then cos. DAC : cos. BAD : : tan. AB : tan. AC. If DAC and B be of the same affection, AC is less than 90°; otherwise it is greater than 90°.
11	AB, AC, and BC, the three sides.	B, one of the angles.	Let the perp. AD fall within, or be the nearest to B or C that falls without; then tan. ½ BC : tan. ½ sum of BA, AC : : tan. ½ diff. of BA, AC : tan. ½ E, and ½ E added to ½ BC, gives the segment nearest the greater side, if the sum of AB, AC be less than 180°; otherwise it gives the segment nearest the less side. And tan. AB : tan. BD : : R : cos. B (case 12). Otherwise, Let D be ½ the diff. of AB, BC; then the rect. sin. AB, sin. BC : rect. sin. sum and diff. of D and ½ AC : : R^2 : : sin. ^{2}, ½ B. Otherwise, Let P be ½ the perimeter; then rect. sin. AB, sin. BC : rect. sin. P. sin. diff. of P and AC : : R^2 : cos. ^{2}, ½ B.
12	A, B, and C, the three angles.	AC, one of the sides.	With the supplement of either of the angles A or C, and the measures of the other two angles, suppose a triangle made; and in it find the angle opposite to the side which is the measure of the angle at B, and the measure of the angle thus found is AC.

CASE I. GIVEN TWO SIDES, AND THE ANGLE OPPOSITE TO ONE OF THEM.

1. In the oblique-angled spherical triangle ABC are given the two sides AB 43° 30′, and AC 67° 34′, and the angle at B 72° 12′; to find the angles at A and C, and the side BC.



Construction. Draw the diameter of the primitive BOD, and OG perpendicular to it. Make OF the semitangent of 72° 12′, and OG its tangent, and from G describe the circle BCD. Make AB 43° 30′, and draw OA. Lay the secant of 67° 34′ on OA produced to L, and with its tangent describe from L an arc, cutting BCD in C, and describe the circle ACE; then ABC is the triangle.

Let K be the centre of ACE, join KO, then KO is the tangent of the angle BAC, or of its supplement. Lay the semitangent of it from O to H for the pole of ACE. A ruler laid from F to C will cut off an arc on the primitive between it and B equal to BC. Lines from C through F and H will cut off on the primitive the measure of the angle at C. Describe the circle AFE, which will be perpendicular to BC.

Calculation. The angle at C is found thus; Sin. CA : sin. AB : : sin. B : sin. C, which is acute, because AB is less than AC, and AB + AC less than 180°.

To find the other parts; first, find BP thus, R : cos. B : : tan. AB : tan. BP; then cos. AB : cos. AC : : cos. BP : cos. PC, which is acute, because AC is acute, the angle at B being acute. Then CB = BP + PC, because B and C are of the same affection.

Again, Sin. AC : sin. CB : : sin. B : sin. A.

AB 43° 30′	sin. 9·837812	B 72° 12′	cos. 9·485289
B 72° 12′	sin. 9·978696	AB 43° 30′	tan. 9·977250
	19·816508	BP 16° 10′ 38″	tan. 9·462539
CA 67° 34′	sin. 9·965824		
C 45° 9′ 31″	sin. 9·850684		
AC 67° 34′	cos. 9·581618	BC 75° 49′ 44″	sin. 9·986579
BP 16° 10′ 38″	cos. 9·982454	B 72° 12′	sin. 9·978696
	19·564072		19·965275
AB 43° 30′	cos. 9 860562	AC 67° 34′	sin. 9·965824
PC 59° 39′ 6″	cos. 9·703510	A 87° 7′ 6″	sin. 9·999451
BP 16° 10′ 38″		180°	
BC 75° 49′ 44″		A 92° 52′ 54″ obtuse value.	

2. Given the sides AC 80° 5′, and AB 70° 10½′, and the angle B 33° 15′; to find the rest.

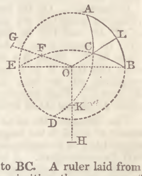
Ans. The angles C 31° 34′ 32″, and A 161° 25′ 19″, and the side BC 145° 4′ 59″.

3. Given the two sides AC 114° 30′, AB 56° 40′, and the opposite angle B 125° 20′; to find the rest.

Ans. BC 83° 11′ 50″, the angles A 62° 53′ 59″, and C 48° 30′ 34″.

CASE II. GIVEN TWO ANGLES, AND THE SIDE OPPOSITE TO ONE OF THEM.

1. In the oblique-angled spherical triangle ABC are given the angles at A 57° 36′, and at B 70° 34′, and the side AC 85° 48′; to find the sides BC and BA, and the angle at C.



Construction. On the radius of the primitive lay OF the semitangent of 57° 36′, and OG its tangent, and with its secant describe from G the circle ACD. Lay a ruler from E to 85° 48′ on the primitive from A, and it will cut ACD in C. With the tangent of 70° 34′ from O describe an arc, and with its secant from C cut that arc in H, from which as a centre describe the circle BCE; and ABC is the triangle.

On OH lay the semitangent of 70° 34′, to K. A ruler from K through C will cut off an arc of the primitive from B equal to BC. A ruler laid from C through K and F will mark off on the primitive the measure of the angle ACB. The radius OCL is the perpendicular on AB.

Calculation. The side CB is found thus; Sin. B : sin. A : : sin. AC : sin. CB acute, because A is less than B.

In the right-angled triangle ACL are given AC and the angle at A, to find AL and ACL. R : cos. A : : tan. AC : tan. AL acute. Also, R : cos. AC : : tan. A : cot. ACL acute. Then tan. B : tan. A : : sin. AL : sin. BL; and cos. A : cos. B : : sin. ACL : sin. BCL. Then AB = AL + LB, and ACB = ACL + LCB.

A 57° 36′	sin.	9·926511	cos. A	9·729024	tan. — R	0·197487
AC 85° 48′	sin.	9·998832	tan. — R	1·134095	cos.	8·864738
		19·925343	AL tan.	10·863119	ACL cot.	9·062225
B 70° 34′	sin.	9·974525	AL =	82° 11′ 46″	ACL =	83° 25′ 1″
CB 63° 14′ 38″	sin.	9·950818				

B	70° 34′	cos.	9·522066	A	57° 36′	tan.	10·197487
ACL	83° 25′ 1″	sin.	9·997127	AL	82° 11′ 46″	sin.	9·995959
			19·519193				20·193446
A	57° 36′	cos.	9·729024	B	70° 34′	tan.	10·452460
BCL =	38° 5′ 7″	sin.	9·790169	BL =	33° 25′ 16″	sin.	9·740986
ACL =	83° 25′ 1″			AL =	82° 11′ 46″		
ACB =	121° 30′ 8″			AB =	115° 37′ 2″		

2. Given the angles A 115° 12′, and B 63° 30′, and the side BC 122° 16′; to find the rest.

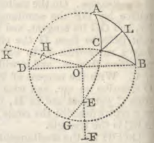
Ans. The sides AB 111° 44′ 43″, AC 56° 45′ 15″, and the angle C 96° 18′ 58″.

3. Given the two angles B 91° 26′ 44″, C 102° 5′ 54″, and the side AC 118° 2′ 14″; to find the rest.

Ans. The sides BC 23° 57′ 13″, AB 120° 18′ 33″, and the angle A 27° 22′ 34″.

CASE III. GIVEN TWO SIDES AND THE INCLUDED ANGLE.

1. In the oblique-angled spherical triangle ABC are given the sides AB 58° 24′, BC 67° 48′, and the included angle ABC 63° 43′; to find the angles at A and C, and the side AC.



Construction. On the radius of the primitive make OE the semitangent of 63° 43′, and OF its tangent, and with its secant from F describe the circle BCD. Make BA 58° 24′. A ruler laid from E to a point in the primitive, 67° 48′ from B, will cut BCD in C. Then describe the great circle ACG, and ACB is the triangle.

The distance OK of O from the centre of ACG is the tangent of the angle BAC, or its supplement. Make OH its semitangent. A line from H through C will cut off on the primitive from A the measure of AC, and lines from C through E and H will cut off on the primitive the measure of ACB.

The radius OCL is perpendicular to AB.

Calculation. In the triangle BCL, right-angled at L are given the side CB 67° 48′, and the angle at B 63° 43′; to find BL and BCL. First R : cos. B : : tan. CB : tan. BL, and the difference of BL and BA is AL. In like manner, R : cos. BC : : tan. B : cot. C. Then sin. AL : sin. BL : : tan. B : tan. A, which is acute if BL be less than BA; otherwise it is obtuse. Also cos. BL : cos. LA : : cos. BC : cos. CA, which is acute, or like BC, if BL, LA be of the same affection, otherwise obtuse. Also tan. BL : tan. LA : : tan. BCL : tan. ACL, and ACB = BCL — ACL.

BC 67° 48′	tan. — R = 0·389241	AL 11° 3′ 49″	cos. 9·991853
B 63° 43′	cos. = 9·646218	BC 67° 48′	cos. 9·577309
BL 47° 20′ 11″	tan. 10·035459		19·569162
BA 58° 24′		BL 47° 20′ 11″	cos. 9·831033
AL 11° 3′ 49″		AC 56° 49′ 35″	cos. 9·738129
BC 67° 48′	cos. 9·577309	AL 11° 3′ 49″	tan. 9·291219
B 63° 43′	tan. — R 0·306388	BCL 52° 34′ 54″	tan. 10·116303
BCL 52° 34′ 54″	cot .9 885697		19·407522
ACL 13° 15′ 13″		B 47° 20′ 11″	tan. 10·035458
ACB 65° 50′ 7″		ACL 13° 15′ 13″	tan. 9·372064

BL 47° 20′ 11″	sin. =	9·866491
B 63° 43′	tan.	10·306388
		20·172879
AL 11° 3′ 49″	sin.	9·283072
BAC 82° 39′ 29″	tan.	10·889807

2. Given the sides AB 41° 9′ 46″, BC 50° 5′ 47″, and the angle at B 114° 7′ 30″; to find the rest.

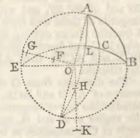
Ans. AC 73° 56′ 40″, the angles at C 38° 41′ 21″, and at A 46° 45′ 49″.

3. Given the two sides AB 61° 14′, BC 58° 27′, and the included angle B 57° 53′ 55″; to find the rest.

Ans. The angles C 77° 22′ 19″, A 71° 33′ 29″, and the side AC 49° 33′.

CASE IV. GIVEN TWO ANGLES, AND THE INCLUDED SIDE.

1. In the oblique-angled spherical triangle ABC are given the side AB 75° 40′, and the angles at A 39° 38′, and B 58° 22′; to find the sides AC and BC, and the angle at C.



Construction. On the primitive make AB 75° 40′, and draw the diameters AD and BE, and perpendicular to them draw OG and OK. Lay the semitangent of 39° 38′ from O to F, and its tangent from O to G. Also lay the semitangent of 58° 22′ from O to H, and its tangent from O to K; and from the centres G and K describe the circles ACD and BCE; then ABC is the triangle.

The unknown parts are measured as before.

Describe the circle AHD, which is perpendicular to BC.

Calculation. In the triangle ABL, right-angled at L, are given the side AB, and the angle at B; to find the angle BAL. And the difference between BAL and BAC is CAL; thus, R : cos. BA : : tan. B : cot. BAL 68° 7′ 53″, whence CAL is 28° 29′ 53″. Then sin. BAL : sin. CAL : : cos. B : cos. C, which is acute if BAC be greater than BAL; otherwise it is obtuse. Also cos. CAL : cos. BAL : : tan. AB : tan. AC acute, if B and CAL be like. Lastly, sin. B : sin. A : : sin. AC : sin. BC.

BA 75° 40′	cos. 9·393685	CAL	28° 28′ 20″	sin.	9·678275
B 58° 22′	tan. — R. 0·210415	B	58° 22′	cos.	9·719730
BAL 68° 6′ 20″	cot. 9·604100				19·398005
BAC 39° 38′		BAL	68° 6′ 20″	sin.	9·967488
CAL 28 28′ 20″			74° 22′ 1″	cos.	9·430517
			180°		
		BCA	105° 37′ 59″		
BAL 68° 6′ 20″	cos. 9·571590	A	39° 38′	sin.	9·804734
AB 75° 40′	tan. 10·592581	AC	58° 56′ 16″	sin.	9·932782
	20·164171				19·737516
CAL 28° 28′ 20″	cos. 9·944013	B	58° 22′	sin.	9·930145
AC 58° 56′ 16″	tan. 10·220158	BC	39° 55′ 23″	sin.	9·807371

2. Given the side AB 124° 12′, and the angles at A 126° 0′, and at B 56° 15′; to find the rest.

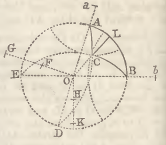
Ans. The sides AC 43° 30′ 31″, BC 138° 9′ 42″, and the angle C 92° 42′ 46″.

3. Given the two angles A 58° 5′ 4″, B 62° 34′ 6″, and the side AB 122°; to find the rest.

Ans. The angle C 130°, the sides AC 79° 17′ 15″, and BC 70° 0′ 2″.

CASE V. WHEN THE THREE SIDES ARE GIVEN.

1. In the oblique-angled spherical triangle ABC are given the sides AB 82° 26′, BC 68° 53′, and AC 57° 30′; to find the angles.



Construction. On the primitive make AB 82° 26′, and draw the diameters AOD, BOE, and make O*a* and O*b* the secants of 57° 30′ and of 68° 53′, and with the tangents of these arcs from *a* and *b* describe circles cutting one another in C, and describe the circles BCE and ACD; then ABC is the triangle.

The distances from O of the centres of ACD and BCE, measured on the line of tangents, give the angles at A and B, and the angle at C is measured as before.

The radius OCL is the perpendicular upon AB.

Calculation. Tan. ½ AB : tan. ½ (BC + AC) : : tan. ½ (BC − AC) : tan. ½ (BL ± AL), and ½ BA ± ½ (BL ± AL) = BL. Then, tan. BC : tan. BL : : R : cos. B, and tan. CA : tan. AL : : R : cos. A, and sin. AC : sin. BA : : sin. B : sin. C.

½(BC+AC) 63°11′30″	tan.	10·296434	BL 53° 54′ 22″	tan.+R.	20·137243
½(BC—AC) 5° 41′ 30″	tan.	8·998548	BC 68° 53′	tan.	10·413185
		19·294982	ABC 58° 0′ 45″	cos.	9·724058
½ BA 41° 13′	tan.	9·942478			
½(BL—AL) 12° 41′ 22″	tan.	9·352504	AL 28° 31′ 38″	tan.+R.	19·735256
BL = 53° 54′ 22″			AC 57° 30′	tan.	10·195813
AL = 28° 31′ 38″			BAC 69° 44′ 21″	cos.	9·539443

AB 82° 26′	sin.	9·996202
B 58° 0′ 45″	sin.	9·928479
		19·924681
AC 57° 30′	sin.	9·926029
85° 29′ 18″	sin.	9·998652
94° 30′ 42″ = the angle ACB.		

Method II. From ½ the sum of the three sides take the side opposite to the angle sought, and add the arithmetical complements of the sines of the two containing sides, and the sines of the ½ sum and remainder; and ½ the sum of these four is the cosine of ½ the angle sought.

Method III. Take the sum and difference of ½ the base and ½ the difference of the sides, and then add the sines of this sum and difference, and the arithmetical complements of the sines of the containing sides; and ½ the sum of these four is the sine of ½ the angle sought.

NOTE. Instead of taking the sum and difference of ½ the base, and ½ the difference of the sides, the two containing sides may be subtracted from the ½ sum of the three sides.

METHOD IV. Add the arithmetical complements of the sines of the half sum, and of its excess above the base, and the sines of its excesses above the other two sides; and ½ the sum of these four is the tangent of ½ the angle sought.

NOTE. In using the common tables of logarithms, the third method is more accurate than the second when the required angle is small, and the second is more accurate when it is large. The fourth method may be used in all cases, except when the angle sought is very nearly equal to two right angles.

BY THE SECOND METHOD.

AB = 82° 26′		
BC = 68° 53′	ar. co. sin.	0·030189
AC = 57° 30′	ar. co. sin.	0·073971
Sum 208° 49′		
½ Sum 104° 24′ 30″	sin.	9·986121
82° 26′		
Diff. 21° 58′ 30″	sin.	9·573106
		½)19·663387
47° 15′ 21·6″	cos.	9·831693
2		
94° 30′ 43·2″ Angle at C.		

BY THE THIRD METHOD.

½ sum	104° 24′ 30″		
BC	68° 53′	ar. co. sin.	0·030189
AC	57° 30′	ar. co. sin.	0·073971
1st rem.	35° 31′ 30″	sin.	9·764220
2d rem.	46° 54′ 30″	sin.	9·863478
			½)19·731858
	47° 15′ 21·6″	sin.	9 865929
	2		
	94° 30′ 43·2″ Angle at C.		

BY THE FOURTH METHOD.

Half sum	104° 24′ 30″	ar. co. sin.	0·013879
Excess above AB	21° 58′ 30″	ar. co. sin.	0·426894
Excess above BC	35° 31′ 30″	sin.	9·764220
Excess above AC	46° 54′ 30″	sin.	9·863478
			½)20·068471
	47° 15′ 21·6″	tan.	10·034235
	2		
	94° 30′ 43·2″ Angle ACB.		

2. Given the sides AC 50° 54′ 32″, CB 37° 47′ 18″, and AB 74° 51′ 50″; to find the angles.

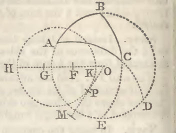
Ans. The angles at B 44° 10′ 41″, at A 33° 22′ 45″, and at C 119° 55′ 5″.

3. Given AB 58° 0′ 5″, AC 88° 12′ 28·8″, and BC 94° 52′ 40·8″; to find the angles.

Ans. The angles at C 57° 40′ 21·6″, at B 84° 49′ 2″, and at A 96° 53′ 9″.

CASE VI. WHEN THE THREE ANGLES ARE GIVEN.

1. In the oblique-angled spherical triangle ABC are given the angles A 78° 25′, B 110° 30′, and C 64° 48′; to find the sides.



Construction. Draw the radius of the primitive, and make OF the semitangent of 69° 30′ = 180° — 110° 30′, and make OG its tangent, and with its secant describe from G the circle BCE. Lay the semitangent of 134° 18′ = 69° 30′ + 64° 48′ from O to H, and the semitangent of 4° 42′ = 69° 30′ — 64° 48′ from O the same way to K, and upon the diameter HK describe the circle HPK, and with the semitangent of 78° 25′ from O cut that circle in P. Join OP, and on it lay OM the tangent of 78° 25′, and with the secant of 78° 25′ from M describe the circle ACD; then ABC is the triangle.

Describe a great circle through the points F and P. The triangle OFP is semi-supplemental to ABC. For OP is the measure of BAC, because O and P are the poles of AB and AC; FP is the measure of ACB, because F and P are the poles of BC and CA, and OF is the supplement of ABC. Also, AB is the measure of the angle POF, because A and B are the poles of PO and OF; and BC is the measure of OFP, because B and C are the poles of OF and FP, and AC is the supplement of OPF.

Calculation. To find BC or the angle OFP. Take OF the supplement of ABC 69° 30′, and the difference between it and C or PF is 4° 42′, and the half of it taken from ½ BAC or OP, and added to it, are 36° 51½′ and 41° 33½′. Add the arithmetical complements of the sines of CBD and of C and the sines of 41° 33½′ and of 36° 51½′; and ½ the sum of these four is the sine of ½ OFP or of ½ BC. In the same manner we find AB and AC.

180 — B	69° 30′	ar. co. sin. 0·028412	A	78° 25′	ar. co. sin. 0·008936
C	64° 48′	ar. co. sin. 0·043434	180 — B	69° 30′	ar. co. sin. 0·028412
½ sum	106° 21′ 30″	sin. 9·982054	½ sum	106° 21½′	sin. 9·982054
A	78° 25′		C	64° 48′	
	27° 56′ 30″	sin. 9·670777	½	41° 33½′	sin. 9·821764
		2)19·724677			2)19·841166
	43° 15′ 7″	cos. 9·862338		33° 36′ 15″	cos. 9·920583
BC	86° 30′ 14″		BC	67° 12′ 30″	

180 — A	101° 35′	arith. comp. sin. 0·008936
C	64° 48′	arith. comp sin. 0·043434
½ sum	138° 26′ 30″	sin. 9·821764
B	110° 30′	
	27° 56′ 30″	sin. 9·670777
		2)19·544911
	53° 41′ 17″	cos. 9·772455
AC	107° 22′ 34″	

2. Given the angles A 44° 10′ 40″, B 33° 22′ 45″, and C 119° 55′ 6″; to find the sides. Ans. The sides BC 50° 54′ 30″, AB 74° 51′ 50″, and AC 37° 47′ 18″.

3. Given the three angles A 87° 46′ 13″, B 46° 34′ 5″, and C 53° 39′ 20″; to find the sides. Ans. The sides AB 31° 23′ 54″, BC 40° 15′ 50″, and AC 28° 0′ 54″.

APPLICATION OF SPHERICAL TRIGONOMETRY
TO THE SOLUTION OF ASTRONOMICAL PROBLEMS.

SPHERICAL TRIGONOMETRY is of great use in Astronomy, Geography, and Navigation; and therefore a few examples of its application to these sciences are given here, after explaining the circles of the sphere.

To lay down the circles of the sphere on the plane of the meridian of Edinburgh, in Lat. 55° 57′ 20″ N.

Let the primitive be the meridian. Draw the diameter HR for the horizon, and the perpendicular diameter ZN; then Z is the zenith, and N the nadir. Make RP, ZE, each 55° 57′ 20″, and draw the diameters Pp and EQ; then P and p are the poles, and EQ the equator, and Pp the hour-circle of six. About the points P and p as poles describe the circles d e f and g h k for the polar circles at the distance of 23° 28′; and in the same manner describe the tropics about the poles P and p at the distance of 66° 32′.

Suppose the time for which the circles are drawn to be the 3d August 1831, at 9h 36m in the morning. The declination for that time is 17° 40′ N. About the pole P, at the distance of 72° 20′, describe the circle a C b, which is the parallel of the sun's declination for that day. Let it meet HR in A, Pp in C, and ZN in F, and describe the great circles PAp, PFp, meeting EQ in B, G, and ZCN meeting HR in D. Describe the great circle PSp, making the angle ZPS 36° = 2h 24m the time from noon, and describe the circle ZSN meeting HR in T, and let PSp meet EQ in M.

The point b is the sun's place at midnight, and a his place at noon; A the point where he rises, C is his place at six, F his place when due east, and S his place at the given time. The circle ZON is the prime, or east and west vertical circle; O the east or west point of the horizon, R its north, and H its south points; Rb is the sun's depression at midnight, a H is his meridian altitude, ST his altitude at the given time, OF his altitude when east, and CD his altitude at six. The arch QB, or the angle QPB, is the time of the sun's rising from midnight, and BO or BPO the time from six, which is called the sun's ascensional difference; BE, or BPE, the time of his rising from noon; OG, or OPG, the time from six, when he is due east; and GE, or GPE, the time from noon. Also OM, or OPM, is the given time from six, and EM, or EPM, the given time from noon. AR, or AZR, is the sun's amplitude from the north; OA, or OZA, his amplitude from the east; and AH, or AZH, from the south. RD, or RZD, is his azimuth from the north at six; and DH, or DZH, from the south. And HT, or HZT, is his azimuth from the south at the given time; and TR, or TZR, that from the north.

The student should apply the data in the following problems to a celestial globe, in order to obtain a correct idea of the figures. He will also find it of great advantage to reverse all the problems assuming the things required, as given, to find the given things; thus performing the exercises in all possible ways.

Prob. I. Given the obliquity of the ecliptic, and the sun's longitude, to find his declination and right ascension.

In the spherical triangle SMO, right-angled at M, are given the angle SOM the obliquity of the ecliptic, and the side SO the sun's longitude; to find SM the declination, and OM the right ascension. These are found by Case 1, Right-angled Triangles.

On the 1st of March 1836, the obliquity of the ecliptic being 23° 27′ 44″, and the sun's longitude 340° 59′ 6″, Required his declination and right ascension.

Here the angle SOM = 23° 27′ 44″, and the side SO = 340° 59′ 6″, but as the sun is in the 4th quadrant, we must take this from 360°, and use the remainder in the computation, taking the result again from 360° for the right ascension; hence

	23° 27′ 44″	sin. 9·600041		cos. + R. 19·962522
	19° 0′ 54″	sin. 9·512972		tan. 9·537341
Decl.	7° 27′ 12·7″ S.	sin. 9·113013	17° 32′ 35·6″	tan. 9·499863

Right ascension = 342° 27′ 24·4″ = 22h. 49m. 49·6s.

Given the Obliquity of Ecliptic.	Given the Sun's Longitude.	Required the Sun's R. A.	Required the Sun's Declination.
1. 23° 27′ 43·84″	78° 35′ 38·0″	5h. 10m. 23·1sec.	22° 58′ 19·1″ N.
2. 23 27 45·15	164 50 37·3	11 4 11·0	5 58 31·2 N.
3. 23 27 45·07	224 8 52·5	14 46 43·9	16 6 0·3 S.
4. 23 27 44·68	264 39 6·8	17 36 41·5	23 21 15·7 S.

Prob. II. Given the latitude of the place, and the sun's declination, to find his amplitude from the east, and the time of his rising from midnight.

In the spherical triangle APR, right-angled at R, are given PR the latitude, and the hypotenuse PA, the polar distance; to find AR the amplitude from the north, and the angle APR, which, converted into time at the rate of 15° to an hour, gives the time from midnight when the sun rises. Wherefore by Case 4 of Right-angled Spherical Triangles, cos. latitude : R : : cos. polar dist. or sin. decl. : cos. RA, or sin. OA, the amplitude.

And tan. polar dist. : tan. Lat. : : R : cos. P, the time of rising.

The polar distance is equal to 90° minus the declination when the latitude and the declination are of the same name, or 90° + the declination when they are of different names.

The same things may be found in the triangle OAB right-angled at B, where AOB or RQ is the co-latitude, and AB the declination; to find AO the amplitude, and OB the ascensional difference, which, subtracted from six hours, gives the time of sun-rising. This is wrought by Case 3 of Right-angled Spherical Triangles.

On the 15th of May 1836, the sun's declination being 18° 56′ 43″ N., Required his amplitude, and the time of his rising at Edinburgh in latitude 55° 57′ 20″ N.

Here 90° — 18° 56′ 43″ = 71° 3′ 17″, sun's north polar distance; hence

71° 3′ 17″	cos. + R. 19·511435		ar. co. tan.	9·535623
55° 57′ 20″ ar. co. cos.	0·251939		tan. + R.	20·170286
54° 33′ 16″	cos. 9·763374	59° 27′ 57″	cos.	9·705909

Amp. 35° 26′ 44″

and 59° 27′ 57″ = 3h. 57m. 51·8sec., time of sun-rising.

Given the Sun's Declination.	Latitude of Place.	Required the Sun's Amp.	Hour of Sunrise.
1. 15° 23′ 5″ N.	57° 8′ 58″ N.	29° 16′ 48″	4h. 19m. 6·5 sec.
2. 12 15 6 N.	33 56 3 S.	14 49 9	6 33 36·3
3. 16 18 27 S.	33 48 50 S.	19 45 9	5 14 47·9
4. 21 15 20 S.	48 50 13 N.	33 25 15	7 45 36·3

Prob. III. Given the latitude of the place and the sun's declination, to find the sun's azimuth and altitude at six o'clock.

In the triangle PCZ, right-angled at P, are given PZ the co-latitude, and PC the polar distance; to find ZC the zenith distance, or complement of CD the altitude, and the angle CZP the azimuth from the north. Wherefore, by Case 5 of Right-angled Spherical Triangles, R : cos. ZP = sin. Lat. : : cos. CP = sin. declination : cos. CZ, or sin. CD the altitude.

And sin. ZP = cos. Lat. : R : : tan. PC, or cot. decl. : tan. Z, the azimuth.

The same things may be found in the triangle OCD, right-angled at D, where COD or PR is the latitude, and OC the declination. This is wrought by Case 1 of Right-angled Spherical Triangles.

On the 1st of June 1836, the sun's declination being 22° 6′ 26″ N., Required his azimuth and altitude at 6 o'clock at Edinburgh.

Co-lat. 34° 2′ 40″	cos. 9·918347		ar. co. sin. 0·251939
N. P. D. 67° 53′ 34″	cos. 9·575582		tan. + R. 20·391255
Altitude 18° 10′ 12″	sin. 9·493929	Az. 77° 11′ 18″	tan. 10·643194

Given the Latitude of Place.	Sun's Declination.	Required at 6 o'clock the Sun's Azimuth.	Sun's Altitude.
1. 51° 45′ 40″ N.	16° 24′ 30″ N.	79° 40′ 14″	12° 49′ 8″
2. 15 55 26 S.	12 18 25 S.	78 9 6	3 21 9
3. 51 28 39 N.	18 25 16 N.	78 16 51	14 18 48
4. 33 48 50 S.	10 11 23 S.	81 30 22	5 39 0

Prob. IV. Given the latitude of the place and the sun's declination, to find the sun's altitude and the time when he is due east.

In the triangle ZPF, right-angled at Z, are given ZP the co-latitude, and PF the polar distance; to find ZF the zenith distance, and the angle ZPF the time from noon. Wherefore, by Case 4 of Right-angled Spherical Triangles, cos. ZP, or sin. Lat. : R : : cos. PF, or sin. decl. : cos. FZ, or sin. FO, the altitude. And tan. FP, or cot. decl. : tan. ZP, or cot. Lat. : : R : cos. P.

The same things may be found in the triangle FOG, right-angled at G, in which are given FOG, or ZE, the latitude, and FG the declination; to find FO the altitude, and OG the complement of GE, the time from noon. This is wrought by Case 3 of Right-angled Spherical Triangles.

On the 1st of May 1836, the sun's declination being 15° 10′ 31″ N., Required his altitude and the time from noon when he is due east at Edinburgh.

Co-lat.	34° 2′ 40″	ar. co. cos. 0·081653		tan + R.	19·829714
N.P.D.	74° 49′ 29″	cos. + R. 19·417925		ar. co. tan.	9·433339
Alt.	18° 24′ 59″	sin. 9·499578	79° 26′ 26″	cos.	9·263053

and 79° 26′ 26″ = 5h. 17m. 45·7sec., time when due east.

Given the Latitude of Place.	Given the Sun's Declination.	Required the Sun's Altitude.	Required the Time when due East.
1. 50° 48′ 3″ N.	17° 10′ 27″ N.	22° 23′ 50″	5h. 1m. 36·2sec.
2. 59 56 31 N.	21 15 40 N.	24 46 9	5 7 56·9
3. 33 4 9 N.	21 25 35 N.	42 1 38	3 31 44·9
4. 33 48 50 S.	15 42 18 S.	29 6 17	4 20 42·7

Prob. V. Given the latitude, declination, and hour; to find the sun's altitude and azimuth at that time.

In the triangle OSM, right-angled at M, are given MS the declination, and MO the time from six, to find the angle MOS. By Case 5 of Right-angled Spherical Triangles, sin. OM : R : : tan. MS : tan. O, and SOM + co-lat. EOH = SOT. Also R : cos. MO : : cos. MS : cos. SO. Then in the triangle OST, right-angled at T, are given SO, and the angle SOT; to find OT, the complement of TH, the azimuth, and TS the altitude, by Case 1 of Right-angled Spherical Triangles.

The same things may be found by resolving the oblique-angled triangle PZS, in which are given PZ the co-latitude, PS the polar distance, and the angle ZPS the hour from noon; to find ZS the zenith distance, and the angle at Z the azimuth, by Case 3, of Oblique-angled Spherical Triangles.

On the 1st July 1836, the sun's declination at 7 o'clock morning being 23° 5′ 50″ N. at Edinburgh, Required his altitude and azimuth at that time.

Time from 6=1h. or 15°		sin.	0·587004		cos. 9·984944
Decl.	23° 5′ 50″	tan. + R.	19·629898		cos. 9·963713
	58° 44′ 51″	tan.	10·216902	SO 27° 18′ 53″	cos. 9·948657
	34° 2′ 40″				
SOT	92° 47′ 1″	cos.	8·686315		sin. 9·999487
SO	27° 18′ 53″	tan.	9·713040		sin. 9·661697
OT	1° 26′ 12″	tan.	8·399355	TS 27° 16′ 47″	sin. 9·661184
	88° 33′ 48″ Azimuth.			Altitude	

Given the Latitude of Place.	Given the Sun's Declination at 10 o'clock A. M.	Required at that time the Sun's Altitude.	Required at that time the Sun's Azimuth.
1. 6° 9′ 0″ S.	15° 57′ 12″ N.	52° 58′ 50″	52° 59′ 2″
2. 22 34 16 N.	23 25 11 N.	62 25 14	82 18 15
3. 15 55 26 S.	10 13 26 S.	60 15 28	82 41 13
4. 51 30 20 N.	5 11 15 S.	27 46 48	34 15 4

Prob. VI. Given the latitude and longitude of the moon, or of a star, and the obliquity of the ecliptic; to find the right ascension and declination.

Suppose HR the equator, and EQ the ecliptic, then the latitude of the moon or any star S is MS, the longitude OM, the right ascension OT, the declination TS, and the obliquity of the ecliptic TOM. Therefore in the triangle OMS, right-angled at M, are given the two sides OM and MS about the right angle, to find the side OS and the angle MOS; these are found by Case 5 of Right-angled Spherical Triangles. Now it is evident that when the moon or star S is without the ecliptic, the angle MOS added to the obliquity of the ecliptic will give the angle TOS, or when S is within the ecliptic, the difference of these angles will be TOS. Hence in the triangle OTS, right-angled at T, are given the angle TOS, and the hypotenuse OS; consequently the sides OT and TS may be found by Case 1 of Right-angled Spherical Triangles.

On the 1st of March 1836, the obliquity of the ecliptic being 23° 27′ 44·2″, the longitude of the moon 137° 51′ 24·1″, and her latitude 5° 0′ 25·2″ N., Required her right ascension and declination.

OM 137° 51′ 24·1″	cos. 9·870093		ar. co. sin.	0·173286
MS 5° 0′ 25·2″	cos. 9·998340		tan. + R	18·942562
OS 42° 23′ 3″	cos. 9·868433	MOS 7° 26′ 21″	tan.	9·115848

And since the moon is without the ecliptic TOM + MOS = TOS = 30° 54′ 5·2″; hence

TOS 30° 54′ 5·2″	cos. 9·933514		sin.	9·710594
OS 42° 23′ 3″	tan. 9·960289		sin.	9·828723
38° 3′ 49″	tan. 9·893803	TS 20° 15′ 16·5″	sin.	9·539317

OT = 180° — 38° 3′ 49″ = 141° 56′ 11″ = 9h. 27m. 44·7sec., right ascension, and TS = 20° 15′ 16·5″, declination N.

Given the Moon's Latitude.	Given the Longitude.	Given the Obliq. of Eclip.	Required Moon's R. A.	Required Moon's Declination.
° ′ ″	° ′ ″	° ′ ″	h. m. sec.	° ′ ″
1. 4 31 10·3 S.	351 27 10·2	23 27 44·13	23 35 47·2	7 32 35·5 S.
2. 0 59 15·0 N.	59 40 32·5	23 27 43·94	3 49 0·1	21 3 54·5 N.
3. 5 1 9·9 N.	124 9 46·0	23 27 45·15	8 31 10·9	24 6 25·5 N.
4. 0 31 57·6 S.	226 30 28·4	23 27 44·63	14 55 30·5	17 17 56·6 S.

Prob. VII. Given the latitude of the place, and the sun's declination; to find the time when twilight begins and ends.

This problem is solved by Case 5 of Oblique-angled Spherical Triangles, since there are given the polar distance, the co-latitude, and the zenith distance = 90° + 18°, which form the three sides of an oblique-angled spherical triangle, from whence to find the angle at the pole opposite the zenith distance, which is the time from noon that twilight begins and ends.

On the 1st of May 1836, the sun's declination being 15° 10′ 31″ N., At what time will twilight begin and end at Edinburgh, in latitude 55° 57′ 20″ N. ?

SZ = 90° + 18° = 108°			
SP = 90° — 15° 10′ 31″ =	74° 49′ 29″	ar. co. sin.	0·015414
PZ co-latitude	34° 2′ 40″	ar. co. sin.	0·251939
½ (SP + PZ + SZ)	108° 26′ 4·5″	sin.	9·977122
½ sum — SZ	0° 26′ 4·5″	sin.	7·891046
			2)18·135521
	83° 17′ 15″	cos.	9·067760
	SPZ 166° 34′ 30″ = 11h. 6m. 18s.		

afternoon ends, 53m. 42sec. after midnight begins.

Required the time of the beginning and end of twilight at Edinburgh when the sun's declination is

Sun's Declination.	Twilight begins.	Twilight ends.
1. 15° 10′ 30″ S.	5h. 20m. 44·1sec.	6h. 39m. 15·9sec.
2. 12 16 30 N.	1 50 2·5	10 9 57·5
3. 22 15 40 S.	6 2 9·5	5 57 50·5
4. 4 16 10 N.	3 13 34·5	8 46 25·5

Prob. VIII. Given the right ascensions and declinations, or the longitudes and latitudes of two celestial objects; to find their distance.

This problem is solved by Case 3 of Oblique-angled Spherical Triangles, since there are given two sides and the contained angle to find the opposite side. The sides are the complements of the declinations, or latitudes, and the contained angle the difference between the right ascensions, or longitudes. By this problem the distances of two places on the globe may be found, of which the latitudes and longitudes are given; for the polar distances are the sides of the spherical triangle, and the difference of longitude is the measure of the contained angle.

On the 1st of April 1836, at noon, on the meridian of Greenwich, the right ascension of Jupiter being 6h. 32m. 3·22sec., and his declination 23° 28′ 41·3″ N.; the right ascension of the moon 12h. 28m. 49·62sec. and her declination 0° 49′ 3·1″ N., Required the distance between them.

N. P. distance of Jupiter	66° 31′ 18·7″	sin. =	9·962470
N. P. distance of the moon	89° 10′ 56·9″	sin. =	9·999996
½ Diff. of their right asc.	44° 35′ 48″	sin. × 2 =	19·692812
			2)39·655278
			19·827639 (*a*)
½ Diff. of their N. P. dist.	11° 19′ 49·1″	sin.	9·293285
	73° 42′ 46″	tan.	10·534354
	73° 42′ 46″	sin.	9·982211 (*b*)
	44° 28′ 11″	sin.	9·845428 (*a* — *b*)
	2		
	88° 56′ 22″ distance between them.		

At noon, on the meridian of Greenwich, are given				Required
Moon's R. A.	Moon's Declination.	Star's R. A.	Star's Declination.	Their Dists.
h. m. sec.	° ′ ″	h. m. sec.	° ′ ″	° ′ ″
1. 14 1 50·23	11 36 38·7 S.	9 23 9·79	16 5 50·8 N.	74 15 53
2. 14 1 50·23	11 36 38·7 S.	9 49 8·95	15 58 52·7 N.	68 18 37
3. 14 1 50·23	11 36 38·7 S.	16 48 14·80	22 26 14·8 S.	41 4 35
4. 14 1 50·23	11 36 38·7 S.	9 59 38·05	12 45 59·1 N.	64 49 31

PROB. IX. Given the latitude of the place and the time when the moon or a planet is on the meridian, and its declination; to find its semidiurnal arc, and thence the time of its rising and setting.

Let A be the object at rising or setting; then in the quadrantal triangle APZ are given AZ a quadrant, ZP the co-latitude, and AP the object's polar distance, to find ZPO the angle whose measure is the semidiurnal arc.

The same thing may be found by Case 3 of Right-angled Spherical Triangles, for in the triangle ABO, right-angled at B, are given BOA the co-latitude, and BA the declination, to find BO the ascensional difference, or the time between the object's rising or setting and its passing the six o'clock hour-circle. The sum or difference of OB and six hours, according as the latitude and declination are of the same or of different names, is the semidiurnal arc.

On the 15th of May 1836, the moon is on the meridian of Edinburgh at 11h. 52m. A. M., and her declination at noon is 19° 1′ 25″ N., Required her semidiurnal arc, and the time of her rising and setting.

Co-latitude	34° 2′ 40″	cot. 10·170286
Declination	19° 1′ 25″	tan. 9·537553
Ascen. diff.	30° 41′ 6″	sin. 9·707839

= 2h. 2m. 44·4 sec., and as the latitude and declination are of the same name 6h. + 2h. 2m. 44·4 sec. = 8h. 2m. 44·4 sec. now by subtracting the semidiurnal arc from the time of the moon's passing the meridian, we obtain the hour of her rising, and by adding these two we get the hour of setting, thus 11h. 52m. ∓ 8h. 2m. 44·4sec. = 3h. 49m. 15·6sec. morning, time of rising, and 7h. 54m. 44·4sec. time of setting. These, however, are only approximate results; for, in order to obtain the rising and setting accurately, the declination must be that at the instant when the object is in the horizon.

Given the		Required the
Latitude of the Place.	Decl. of the Object.	Semidiurnal Arc.
1. 59° 54′ 5″ N.	12° 28′ 35″ S.	4h. 30m. 14·3 sec.
2. 53 23 13 N.	25 50 47 N.	8 42 18·3
3. 33 56 3 S.	22 25 30 S.	7 4 29·1
4. 41 53 52 N.	15 55 40 N.	6 59 20·4

NOTE. Astronomical observations require to be corrected for the effects of Refraction, Parallax, &c.; but as these belong entirely to practical astronomy, it would be improper to introduce tables and rules for them here. The student who wishes to obtain complete information on these subjects is referred to the Introduction to Galbraith's Mathematical and Astronomical Tables,—a work replete with the most valuable scientific instruction.

PRACTICAL EXERCISES.

1. On the 1st of April 1831, the obliquity of the ecliptic being 23° 27′ 34·1″, and the sun's declination 4° 21′ 51″ N., Required his longitude and right ascension.

Ans. Longitude 11° 1′ 10·9″; right ascension 0h. 40m. 30·8sec.

2. On the 1st of July 1831, the obliquity of the ecliptic being 23° 27′ 33·7″, and the sun's right ascension 6h. 38m. 27·3sec., Required his longitude and declination.

Ans. Longitude 3^s 8° 49′ 55·86″; declination 23° 9′ 53·4″ N.

3. On the 1st of January 1831, the obliquity of the ecliptic being 23° 27′ 33″, and the sun's longitude 9^s 10° 23′ 58″, Required his right ascension and declination.

Ans. Right ascension 18h. 45m. 15·2sec.; declination 23° 3′ 5″ S.

4. On the 1st of October 1831, the declination of the sun being 2° 59′ 38″ S., and his right ascension 12h. 27m. 41·3sec., Required his longitude, and the obliquity of the ecliptic.

Ans. Longitude 6^s 7° 32′ 19″; obliquity of the ecliptic 23° 27′ 32·5″.

5. On the 31st of December 1831, the sun's longitude being 9^s 9° 7′ 50″, and his declination 23° 8′ 42″ S., Required his right ascension, and the obliquity of the ecliptic.

Ans. Right ascension 18h. 39m. 45sec.; obliquity 23° 27′ 34·8″.

6. On the 1st of April 1831, the sun's longitude being 11° 1′ 10·9″, and his right ascension 40m. 30·8sec., Required his declination, and the obliquity of the ecliptic.

Ans. Declination 4° 21′ 51″ N.; obliquity 23° 27′ 34″.

7. On the 1st of February 1831, the sun's declination being 17° 13′ 14″ S., Required the time of his rising and amplitude on the parallel of Edinburgh (55° 57′ 20″ N.)

Ans. Amplitude 31° 55′ 32″; time of rising 7h. 49m. 13½sec.

8. On the 1st of April 1831, the sun's declination being 4° 21′ 51″ N., In what latitude does he rise at 9 o'clock?

Ans. Latitude 83° 50′ 24″ S.

9. On the 1st of May 1830, the sun rises at Paris, in latitude 48° 50′ 14″ N., at 4h. 48m. 35sec. Required his declination. Ans. Declination 15° 0′ 20″ N.

10. On the 22d of June 1831, the sun's declination being 23° 27′ 33″ N., Required his altitude at Edinburgh at 6 o'clock. Ans. Altitude 19° 15′ 38″.

11. The same things being given as in the last exercise, Required his altitude at 10 o'clock morning. Ans. 50° 46′ 15″.

12. Given the altitude of the sun 45° 32′, declination as in the last. Required the hour of the day at Edinburgh.

Ans. 9h. 13m. 26sec. morning, or 2h. 46m. 34sec. afternoon.

13. Given the sun's declination 15° 30′ 20″ N. Required his azimuth at 9 o'clock morning for Edinburgh.

Ans. 58° 39′ 39″.

14. Given the altitude of the sun at 6 o'clock 18° 30′ 15″. Required his azimuth for Edinburgh. Ans. 76° 55′ 52·6″.

15. On the 1st of August 1831, the sun's declination being 18° 10′ 22″ N., Required the hour when he is due east at Edinburgh.

Ans. 6h. 51m. 15·3sec. morning, or 5h. 8m. 44·7sec. afternoon.

16. On the 10th of September 1831, the sun's declination being 5° 8′ 26″ N., and his altitude when due east 16° 53′ 0″, Required the latitude of the place.

Ans. Latitude 17° 58′ N.

17. On the 20th of January 1831, the moon's longitude at noon, on the meridian of Greenwich, being 19° 11′ 27″, her latitude 3° 52′ 31″ S., and the obliquity of the ecliptic 23° 27′ 33·4″, Required her right ascension and declination.

Ans. Right ascension 19° 10′ 47″; declination 3° 55′ 53″ N.

18. On the 24th of May 1831, the right ascension of the moon, on the meridian of Greenwich, at noon, being 217° 59′ 4″, her declination 9° 55′ 4″ S., and the obliquity of the ecliptic 23° 27′ 34″, Required her latitude and longitude.

Ans. Latitude 4° 46′ 53″ N.; longitude 7^s 8° 49′ 17″.

19. On the 1st of July 1831, the moon's latitude, on the meridian of Greenwich, at midnight, being 2° 55′ 31″ S., her right ascension 358° 20′ 53″, and the obliquity of the ecliptic 23° 27′ 33·7″, Required her declination and longitude.

Ans. Declination 3° 54′ 20·2″ S.; longitude 11^s 26° 55′ 5·3″.

20. On the 1st of January 1831, the declination of Spica Virginis being 10° 16′ 32·9″ S., the right ascension 13h. 16m. 8sec., and the mean obliquity of the ecliptic 23° 27′ 42·1″, Required the longitude and latitude of the star.

Ans. Longitude 6^s 21° 34′ 32″, latitude 2° 2′ 24·5″ S.

21. On the 1st of January 1831, the mean obliquity of the ecliptic being 23° 27′ 42·1″, the longitude of Aldebaran

$2^s\ 7° 25' 39{\cdot}3''$, and the latitude $5° 28' 45{\cdot}8''$ S., Required his declination and right ascension.

Ans. Declination $16° 9' 44{\cdot}3''$ N.; right ascension 4h. 26m. 14sec.

22. On the 1st of January 1831, the mean obliquity of the ecliptic being $23° 27' 42{\cdot}1''$, the declination of Pollux $28° 25' 39''$ N., and the latitude $6° 40' 20\frac{1}{4}''$ N., Required his longitude and right ascension.

Ans. Longitude $3^s\ 20° 53' 1''$; right ascension 7h. 34m. 58sec.

23. On the 1st of April 1830, at noon on the meridian of Greenwich, the longitude of the moon being $3^s\ 25° 44' 54''$, her latitude $4° 14' 7''$ S., and the longitude of the sun $11° 15' 53''$, Required the distance between them.

Ans. $104° 26' 36''$.

24. On the 28th of April 1830, the distance between the sun and moon's centre being $74° 11' 43''$, the moon's longitude $3^s\ 21° 48' 42''$, and her latitude $4° 17' 5''$ S., Required the longitude of the sun. Ans. $1^s\ 7° 39' 43''$.

25. On the 27th of August 1830, at noon on the meridian of Greenwich, the distance of the moon's centre from the sun's being $100° 10' 41''$, the moon's longitude $8^s\ 13° 52' 41''$, and the sun's longitude $5^s\ 3° 39' 22''$, Required the moon's latitude. Ans. $5° 17'$ N.

26. On the 1st of January 1830, at noon, on the meridian of Greenwich, the distance between the moon and Aldebaran being $64° 43' 20''$, the right ascension of the star 4h. 26m. 10·5sec., its declination $16° 9' 37''$ N., and the right ascension of the moon $2° 52' 10''$, Required the declination of the moon. Ans. $13' 5{\cdot}4''$ N.

27. On the 7th of January 1830, at noon on the meridian of Greenwich, the distance between the moon and Regulus being $61° 15' 32''$, the declination of the moon $18° 22' 42''$ N., her right ascension $86° 11' 58''$, and the declination of the star $12° 47' 43''$ N., Required his right ascension.

Ans. 9h. 59m. 20sec.

28. On the 4th of January 1831, at midnight on the meridian of Greenwich, the right ascension of the moon being $184° 10' 39''$, her declination $1° 6' 27''$ N., the right ascension of Antares 16h. 19m., and his north polar distance $116° 2' 44''$, Required the distance between the moon and star.

Ans. $64° 21' 4''$.

29. On the 7th of January 1831, at noon on the meridian of Greenwich, the moon's latitude being $4° 30' 58''$ N., and

her longitude $7^s\ 3°\ 21'\ 9''$, the latitude of Jupiter 23′ S., and his longitude $9^s\ 26°\ 42'$, Required his distance from the moon.

Ans. 83° 23′ 22″.

30. On the 13th of June 1831, at noon on the meridian of Greenwich, the latitude of Jupiter being 0° 49′ S., his longitude $10^s\ 22°\ 21'$, the latitude of Saturn 1° 33′ N., and his longitude $4^s\ 26°\ 48'$, Required their distance.

Ans. 175° 29′ 30″.

31. At what time will twilight begin and end at Edinburgh on the 20th August 1831, the sun's declination being 12° 38′ 9″ N.?

Ans. 1h. 44m. 40·8sec. morning, and 10h. 15m. 19·2sec. afternoon.

32. In what latitude, on the 1st of September 1831, does the twilight begin at 3h. 20m. in the morning, the sun's declination being 8° 28′ 54″ N.? Ans. 48° 40′ 40″ N.

33. At Edinburgh the twilight begins at 4h. in the morning. Required the declination of the sun.

Ans. 2° 1′ 29″ S.

34. The latitude of Edinburgh is 55° 57′ 20″ N., and the longitude 3° 10′ 21″ W., the latitude of the Cape of Good Hope is 34° 29′ S., and its longitude 18° 23′ 15″ E. Required the distance between them.

Ans. 5537·367 geographical miles.

35. The latitude of Greenwich Observatory is 51° 28′ 38″ N., the latitude of Bombay Church is 18° 57′ 44″ N., and the longitude 72° 54′ 43″ E. Required the distance between them. Ans. 3882·207 geographical miles.

36. The latitude of Batavia is 6° 9′ S., and its longitude 106° 51′ 45″ E., the latitude of the Royal Observatory of Paris is 48° 50′ 14″ N., and its longitude 2° 20′ 15″ E. Required the distance between them.

Ans. 6250·0137 geographical miles.

PROMISCUOUS QUESTIONS.

1. How many stones of a rectangular form, each 3 feet by $2\frac{1}{2}$ feet, will pave a road 40 yards long, and 6 yards broad?
Ans. 288 stones.

2. How many panes of glass, each 18 inches by 14 inches, will be required for 22 windows, each 5 feet by 3 feet 6 inches?
Ans. 220 panes.

3. What is the excess of a floor, 50 feet long by 30 broad, above two others, each of half its dimensions?
Ans. 750 square feet.

4. How much must be cut off from a board 26 inches broad, to contain $1\frac{1}{2}$ square yards? Ans. 6 feet $2\frac{10}{13}$ inches.

5. The ceiling of a room 28 feet broad, contains 114 square yards 6 feet. What is the length of the room?
Ans. $36\frac{6}{7}$ feet.

6. Along one side of a court 47 feet 9 inches square, there is a footpath 4 feet broad. What will be the expense of laying the rest of it with stones, at 6d. per square yard?
Ans. £5, 16s. $0\frac{3}{4}$d.

7. A room is 60 feet in circuit and 12 feet high. How much paper, 2 feet wide, will line it, deducting the door, 8 feet by 4 feet, and 3 windows, each 5 feet by $3\frac{1}{2}$ feet, and the chimney 4 feet square? Ans. $103\frac{1}{4}$ yards.

8. The base of a right-angled triangle is 300 feet, and the sum of the other two sides is 1000 feet. What are their lengths? Ans. 545 and 455 feet.

9. A roof which is 24 feet 8 inches by 14 feet 6 inches, is to be covered with lead, at 8 lbs. to the square foot. Required the expense, at 2 guineas per cwt. Ans. £53, 13s.

10. How many square feet of deal will be required to make a rectangular chest of which the length is to be $3\frac{1}{2}$ feet, the breadth 2 feet, and the depth 20 inches?
Ans. $32\frac{1}{3}$ square feet.

11. A beam is $8\frac{1}{2}$ inches deep and $3\frac{1}{2}$ feet broad. Re-

...uired the depth of another twice as large, which is $4\frac{3}{4}$ inches ...road. Ans. 12·5263 inches deep.

12. The four sides of a trapezium are, 13, 13·4, 24, and 18 ...et, and the two first contain a right angle. Required the ...rea. Ans. 253·38 square feet.

13. What will be the expense of paving a semicircular ...lot, of which the diameter is 14·8 feet, at 2s. 4d. per square ...ot? Ans. £10, 0s. $8\frac{1}{2}$d.

14. The wheels of a chaise, each 4 feet high, in turning ...ithin a ring, moved so that the outer wheel made two turns ...hile the inner made one, and their distance from one an...her was 5 feet. What were the circumferences of the tracks ...escribed by them?

Ans. Outer, 62·8318 feet. Inner, 31·4159 feet.

15. A circular pond occupies half an acre. What was the ...ngth of the cord which struck the circle? Ans. $27\frac{3}{4}$ yards.

16. A right-angled triangle has its base 16, and its perpen...dicular 12, and a triangle is cut off from it by a line parallel ... the base, of which the area is 24. What are the lengths ... the sides of that triangle? Ans. 8, 6, and 10.

17. An ellipse is surrounded by a wall 14 inches thick, its ...xes are 840 links and 612 links. Required the quantity of ...round enclosed, and the quantity occupied by the walls.

Ans. 4 acres 6 perches enclosed, and 1760·4933 square feet ...rea of the space occupied by the wall.

18. What is the length of the side of an equilateral triangle, ...hich cost as much for paving the area of it, at 8d. per square ...oot, as for palisading its 3 sides at a guinea per lineal yard?

Ans. 72·746 feet.

19. How long must be the tether of a horse which will al...w him to graze quite round an acre of ground?

Ans. $39\frac{1}{4}$ yards.

20. How many 3 inch cubes may be cut out of a 12 inch ...ube? Ans. 64 cubes.

21. What will be the expense of painting a conical spire, ...f which the height is 118 feet, and the circumference of the ...ase 64 feet, at 8d. per square yard? Ans. £14, 0s. 8·96d.

22. The diameter of a standard bushel is $18\frac{1}{2}$ inches, and ...s depth 8 inches. What must be the diameter of that ...ushel which is $7\frac{1}{2}$ inches deep? Ans. 19·10672 inches.

23. What will be the expense of gilding a globe, of which ...e diameter is 6 feet, at $3\frac{1}{2}$d. per square inch?

Ans. £237, 10s. 1·19d.

24. A farmer borrowed a cubical piece of hay, which measured 6 feet every way, and he repaid two cubical pieces, of which the sides were 3 feet each. What part of the quantity borrowed has he returned? Ans. The fourth part only.

25. A person wants a cylindrical vessel 3 feet deep, which shall hold twice as much as another 28 inches deep, and 46 inches in diameter. What must be the diameter of the required vessel? Ans. 57·373 inches.

26. What will be the diameter of a globe, of which the superficial and solid contents are both expressed by the same number? Ans. 6.

27. A sack $22\frac{1}{2}$ inches broad when empty, will contain 3 bushels of corn when filled. What will another sack contain, which is twice its breadth, and of the same length?

Ans. 12 bushels.

28. A cable 3 feet long, and 9 inches in circuit, weighs 22 lbs. What will be the weight of a fathom of that cable, of which the circumference is a foot? Ans. $78\frac{2}{3}$ lbs.

29. The distance between the centres of two circles, each 50 feet diameter, is 30 feet. What is the area of the space enclosed by their circumferences? Ans. 559·119 square feet.

30. What is the length of the chord which cuts off $\frac{1}{3}$ of the area from a circle, of which the diameter is 289 feet?

Ans. 278·6538 feet.

31. A sugar-loaf in form of a cone is 20 inches high; it is required to divide it equally among three persons by sections parallel to the base. What is the height of each part?

Ans. Upper 13·8672, next 3·6044, lowest 2·5284 inches.

32. A malt-kiln is $16\frac{1}{2}$ feet square. Required the side of a square kiln, which is capable of drying three times as much malt. Ans. 28·5788 feet.

33. A round cistern is 26·3 inches in diameter, and $52\frac{1}{2}$ inches deep. What should be the diameter of another of the same depth to contain twice the quantity of liquor?

Ans. 37·1938 inches.

34. How many rafters, each $2\frac{1}{2}$ inches by $1\frac{1}{2}$ inch, can be sawed out of a square log $17\frac{1}{2}$ inches by 10 inches?

Ans. $46\frac{2}{3}$ rafters.

35. How many bricks, each 9 inches long, $4\frac{1}{2}$ inches broad, and 3 inches thick, must be taken to build a wall 100 feet long, 20 feet high, and one foot thick? Ans. $28444\frac{4}{9}$ bricks.

36. A piece of round timber, containing 20 solid feet, is to

be hewn into square timber. How much will it contain when squared? Ans. 12·732 solid feet.

37. What must be the dimensions of a cubical chest to hold 00 oranges, each $2\frac{1}{2}$ inches in diameter?

Ans. Each side 14·62 inches.

38. When the price of timber is 16d. per lineal foot, 14d. per superficial foot, and 20d. per solid foot, which of them is best for the seller, and what is the value of a plank 14 feet long, $\frac{1}{2}$ foot broad, and 6 inches thick, at each of these rates?

Ans. 224d. value lineal, 210d. solid, and 294d. superficial.

39. A board is 10 feet long, 8 inches in breadth at the greater end, and 6 inches at the less. How much must be cut off from the less end to make a square foot?

Ans. 23·2493 inches.

40. If a cubic foot of brass be drawn into wire of $\frac{1}{40}$ inch diameter, what will be the length of the wire, supposing no loss of metal in working?

Ans. 97784·5684 yards, or nearly 56 miles.

41. How high above the earth must a man be raised to see $\frac{1}{3}$ of its surface? Ans. One diameter high.

42. A frustum of a cone of marble has its slant side 8 feet, and the diameters of its bases 4 feet and 1·5 foot. What is its value at 12s. per solid foot? Ans. £30, 1s. $11\frac{3}{4}$d.

43. A garden is 100 feet long and 80 feet broad, and a border of equal breadth surrounds the sides of it, which is just $\frac{1}{2}$ of the garden. What is the breadth of the border?

Ans. 12·9844 feet.

44. A carpenter put a curb of oak round a well: the inner diameter of the curb was $3\frac{1}{2}$ feet, and its breadth $7\frac{1}{4}$ inches. What was the expense of it at 8d. per square foot?

Ans. 5s. $2\frac{1}{4}$d.

45. A piece of square timber is 10 feet long, each side of the greater base 9 inches, and each side of the less 6 inches, How much must be cut off from the less end to contain a solid foot? Ans. 3·39214 feet.

46. The girt of a vessel round the outside of the hoop is 22 inches, and the hoop is 1 inch thick. What is the true girt of the vessel? Ans. $15\frac{4}{7}$.

47. Required the superficial and the solid contents of an elliptical ring in the form of a cylinder, the inner diameters of the ellipse being 38 and 28 inches, and the thickness of the metal in the ring 2 inches?

Ans. 694·3826 square inches in surface, 347·1913 cubic inches solidity.

48. Four men bought a grindstone of 30 inches in diameter, and agreed that the first should use it till he ground down $\frac{1}{4}$ of it for his share, deducting 6 inches of diameter in the middle for waste, and then that the second should use it till he ground down another $\frac{1}{4}$ part, and so on. What part of the diameter must each grind down for his share?

Ans. The 1st 3·8466 inches, 2d 4·5201 inches, 3d 5·7588 inches, 4th 9·8745 inches.

49. Given the distance 12 between the focus of an ellipse and the nearest principal vertex, and the ratio of the curve as 4 to 5, to find the area of the ellipse. Ans. 6785·856.

50. Required the area of a parabola, of which the axis is 120, and the distance of the focus from the principal vertex 10·3, or the perimeter 43·2. Ans. 11520.

51. A gentleman has a bowling-green 300 feet long and 200 feet broad, which he wishes to raise a foot higher by means of the earth dug out of a ditch which surrounds it. To what depth must the ditch be dug, supposing its breadth to be 8 feet? Ans. $7\frac{23}{86}$ feet.

52. Of what diameter must a piece of ordnance be, which is cast for a ball of 24 lbs. weight, so that the diameter of the bore may be $\frac{1}{10}$ of an inch more than that of the ball?

Ans. 5·6918 inches.

53. Suppose the windage of a mortar to be $\frac{1}{60}$ of the diameter of the mortar, and the diameter of the hollow part of the shell to be $\frac{7}{10}$ of that of the mortar. It is required to determine the diameter and weight of the shell, and the weight of the powder requisite for the mortars in common use, viz. those of 13, of 10, of 8, of 5·8, and of 4·6 inches in diameter.

Ans. The diameters of the shells are $12{\cdot}78\dot{3}$, $9{\cdot}8\dot{3}$, $7{\cdot}8\dot{6}$, $5{\cdot}70\dot{3}$, and $4{\cdot}52\dot{3}$ inches. Their weights are 183·3012, 83·4325, 42·7174, 16·27868, and 8·12098 lbs., and the weights of the powder 13·1523, 5·998639, 3·065, 1·168, and 0·5827 lbs.

54. How many shot are in a triangular pile, of which a side of the base contains 50? Ans. 22100 balls.

55. How many shot are in an oblong pile, of which the sides of the base contain 49 and 19? Ans. 8170 balls.

56. How many shot are in an unfinished triangular pile, each side of the bottom being 50 and of the top 20?

Ans. 20770 balls.

57. How many shot are in an incomplete oblong pile, the length and breadth of the base being 50 and 20, and the length and breadth at the top 38 and 8? Ans. 8190 balls.

58. Required the weight of lead in a pipe 600 yards long, the diameter of the bore being $1\frac{1}{4}$ inch, and the thickness of the metal $\frac{1}{4}$ inch. Ans. 10448·274375 lbs.

59. Required the content of a frustum of a cone, of which the greatest diameter is 60 inches, the diagonal between the farthest extremities of the diameters 66, and the slant side 30 inches. Ans. 293·61 imp. gallons.

60. If a heavy sphere, of which the diameter is 4 inches, is dropt into a conical glass full of water, of which the diameter is 5 inches, and the altitude 6 inches, How much water will run over? Ans. 26·2721536 cubic inches.

61. Suppose it is found that a ship, with its ordnance, rigging, &c. displaces 50,000 cubical feet of water, What is the weight of the vessel? Ans. 1395·0893 tons.

62. If a solid inch of metal weighs 8 ounces avoirdupois, What is its specific gravity? Ans. 13824.

63. If a man weighs 192 lbs., and the specific gravity of his body be 1200, How much cork must be tied to him to make him swim? Ans. $10\frac{2}{19}$ lbs.

64. If a cube of solid fir, 12 inches each way, sinks 6 inches in water, What is its specific gravity? Ans. 500.

65. Four solid inches of copper are to be made into a hollow cube. How thick must the metal be that it may swim in one inch depth of water? Ans. ·018635 inches.

66. If two solid feet of feathers weigh 4 lbs., What will the same quantity weigh when compressed into the bulk of half a solid foot, supposing a solid foot of air to weigh $1\frac{1}{7}$ oz.?
Ans. 4 lbs. 1·8 oz.

67. If a man, standing at the side of a river, hears his voice reflected from the opposite bank in 3 seconds of time, What is the breadth of the river? Ans. 1713 feet.

68. I saw the flash of a gun fired from a ship at sea, and 33 seconds afterwards I heard the report. How far was the ship distant from me? Ans. $7\frac{11}{80}$ miles.

69. Observing a battery of cannon, I counted 17 seconds on my watch between the times of seeing the flash and of hearing the report. How far was I distant from the battery?
Ans. $3\frac{1787}{2640}$ miles.

70. The frustum of a cone is 5·7 inches in height, the diameter at the top 3·7 inches, and that at the bottom 4·23 inches. Required the difference between the contents of the

hoofs into which it is divided by a plane passing through the opposite extremities of its diameters.

Ans. 7·05321218 cubic inches.

71. Required the contents of the hoofs into which a cone of which the height is 6 inches, the top diameter 3, and the bottom diameter 4 inches, is divided by a plane passing from the edge of the top to the centre of the base.

Ans. The less hoof 15·26281484, the greater 42·85678516 cubic inches.

72. Suppose a cubic inch of common glass to weigh 1·4921 oz. avoirdupois, one of sea-water ·59542 oz., and one of brandy ·5368 oz. How much force will be required to buoy up in the sea an imperial gallon of brandy in a bottle, of which the weight of the glass in air is 3·84 lbs.? Ans. 20·6686721 oz.

73. How far will a body descend from a state of rest in 20 seconds? Ans. $6433\frac{1}{3}$ feet.

74. If a body is projected perpendicularly in free space with a velocity of 10000 feet per second, To what height would it ascend, and in what time would it again reach the earth? Ans. $294\frac{2513}{6369}$ miles, and in $621\frac{147}{193}$ seconds.

75. Suppose that at the moment a body is projected up AB with the velocity acquired by falling down it, another body begins to fall down it, In what point will they meet, AB being $1029\frac{1}{3}$ feet? Ans. 772 feet from the bottom.

76. Suppose that a body is projected downwards with a velocity of $64\frac{1}{3}$ feet per second, and in 2 seconds after, another body is projected down with a velocity of $257\frac{1}{3}$ feet, In what time will it overtake the other? Ans. $1\frac{1}{2}$ second.

77. A person from a window 20 feet high observes in a mirror placed 12 feet from the foundation of the house the top of a spire 100 feet high. Required the distance of the observer from the spire. Ans. 72 feet.

78. Melville's Monument in St Andrew Square, Edinburgh, is 136 feet 4 inches high, and the statue on the top 14 feet high. At what distance from the base of the monument does the statue subtend the greatest angle?

Ans. 143·1622 feet.

79. Two trees, 100 feet asunder, are placed, the one at the distance of 100 feet, and the other 50 feet from a wall. What is the shortest distance that a person must pass over in running from one tree to touch the wall, and then to the other tree? Ans. 173·2048 feet.

80. I took two stations A and B at the distance of 150 feet

from each other, and in the same straight line with an inaccessible spire; then from A, the station nearest the spire, in a line perpendicular to the line AB, I measured AC 160 feet, and set up a pole at the extremity C; and from B, the other station in a line also perpendicular to AB, I measured the distance BD 275·5 feet, when I observed that the spire and the pole at C were in the same straight line with the point D. Required the distance of the spire from the station A.

Ans. 207·7922 feet.

81. What is the weight of a sphere of oak 6 feet in diameter, its specific gravity being 925? Ans. 2·91895 tons.

82. To what depth would a cube of beech 2 feet 6 inches in the side sink in water? Ans. 2·13 feet.

83. A horse's tether of 40 yards in length is fixed in the circumference of a circular field whose diameter is 350 yards. How much will it allow him to graze? And, supposing that the end of the tether is removed to the circumference of the secondary circle, and in a line with the centre of the field, What additional space would he be enabled to graze?

Ans. First 2391·9022 square yards; and afterwards 3061·1712 square yards.

84. The axes of a punch-bowl in the form of the segment of an oblong spheroid are to each other as 3 to 4, the depth is $\frac{1}{4}$ of the longer axis, and the diameter of its top is 20 inches. What number of rounds may a company of 30 persons drink out of it, using a conical glass of which the top diameter is $1\frac{1}{2}$ inch, and the depth 2 inches? Ans. 38·0148325 rounds.

85. A certain island is 73 miles in circumference, and if 2 men set out from the same point in the same direction, the one travelling at the rate of 5 and the other at the rate of 3 miles an hour, In what time will they be together again?

Ans. $36\frac{1}{2}$ hours.

86. Suppose a cone 20 feet high, and the diameter of the base 6 feet, is cut through the axis 5 feet from the bottom, at an angle of 60 degrees. Required the solidity of the sections.

Ans. Solidity of the upper 79·987 feet. Solidity of the under 108·5095 feet.

87. There is a garden 400 feet long and 300 feet broad, all round which, and close by the wall, is a border 10 feet broad; close by the border there is a walk, and also two others crossing each other in the middle of the garden. The walks are all of one breadth, and their superficial area takes up exactly one-tenth of the whole garden. Required the breadth of the walks. Ans. 6·2375 feet.

88. Suppose in a garden 400 feet long and 300 feet broad, there is a walk 10 feet wide, all round the garden parallel to, and equidistant from the wall, and so placed as to divide the garden into two equal parts, that is making the area betwixt the wall and the walk the same as the area within the walk. Required the breadth of the space between the wall and the walk. Ans. 45·100040033 feet.

89. How many roods of slating on the roof of a house 72 feet long and 60 feet wide, with a platform 44 feet long and 32 feet broad; the depth of the sides and ends is 17 feet, the eaves all round measure 264 feet long and 9 inches broad, each of the four hips is 25 feet 3 inches long by 18 inches in breadth? Ans. 11 roods 35 yards 6 feet 6 inches.

90. Suppose the breadth of a circular moat at the top to be 60 feet, at the bottom 35 feet, the outer slope 15 feet, the inner slope 20 feet, respectively. Required its capacity in cubic yards; the diameter of the inner-circle or edge of the moat being 700 feet, and the top and bottom of the moat horizontal. Ans. 50616 yards 26 feet 10 inches.

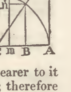
91. Suppose an elliptical garden whose diameters are in the proportion of 5 to 6, contains within the walls an imperial acre, or 43560 superficial feet. The wall is of brick, 15 feet high and 18 inches thick, having four doors, each 8 feet high by 4 feet wide. Required the diameters of the garden, and the superficial content of the wall, after deducting the doors, the wall to be girted in the centre of its thickness.

Ans. The transverse diameter 257·98182 feet; conjugate diameter 214·98485 feet; superficial content of the wall 1234 yards 3 feet 6 inches 8 parts.

APPENDIX.

INSTEAD of introducing into the body of the work demonstrations which would have perplexed the student, it has been considered preferable to give only such as will be easily understood, and to reserve those which require the application of fluxions for the Appendix.

PROPOSITION I. To express the fluxions of circular arcs in terms of the sine, tangent, secant, &c.



Let the radius AC be $= r$, the versed sine AB $= x$, the sine BD $= y$, the tangent AT $= t$, the secant CT $= s$, and the arc AD $= v$. Draw the tangent DS, and the line Sm parallel to BD, and Dn parallel to AC, and let Sm meet the arc in v, then nS $>$ nv. Therefore the ratio of Dn to nv is always greater than that of Dn to nS, but by diminishing Dn it continually approaches to that ratio, and at length comes nearer to it than any given ratio greater than that of Dn to nS; therefore the ratio of Dn to nS is the limit or fluxion of the ratio of Dn to nv, and of course Dn : DS is the fluxion of the ratio of Dn : Dv. But the triangles nDS, CDB are similar, for CDS being a right angle, nDS $=$ BDC; therefore BD : DC : : nD : DS, and $n\mathrm{D} = \dot{x}$, and $\mathrm{DS} = \dot{v}$; therefore $y : r :: \dot{x} : \dot{v} = \frac{r\dot{x}}{y}$. In like manner, BC : CD : : nS : SD, and $n\mathrm{S} = \dot{y}$; therefore $r - x : r :: \dot{y} : \dot{v} = \frac{r\dot{y}}{r - x}$, now $r - x = \sqrt{r^2 - y^2}$, and $y = \sqrt{2rx - x^2}$, therefore $\dot{v} = \frac{r\dot{y}}{\sqrt{r^2 - y^2}} = \frac{r\dot{x}}{\sqrt{2rx - x^2}}$. Again, CB : BD : : CA : AT, or $r - x : y : r : t = \frac{ry}{r - x}$, whence $\dot{t} = \frac{r^2\dot{x}}{y \times (r - x)^2} = \frac{r^2\dot{v}}{(r - x)^2}$

$= \frac{r^2+t^2}{r^2}\dot{v}$; also CB : CD : : CA : CT, or $r-x : r :: r : s = \frac{r^2}{r-x}$, and $\dot{s} = \frac{r^2\dot{x}}{(r-x)^2}$; therefore $\dot{s} : \dot{t} :: y : r :: \dot{x} : \dot{v}$, whence again $\dot{v}$ $= \frac{r^2\dot{s}}{st} = \frac{r^2\dot{s}}{s\sqrt{s^2-r^2}} = \frac{r^2\dot{t}}{s^2} = \frac{r^2\dot{t}}{r^2+t^2}$. If $r = 1$, then $\dot{v} = \frac{\dot{y}}{\sqrt{1-y^2}}$ $= \frac{\dot{x}}{x(1-x)^{\frac{1}{2}}} = \frac{\dot{t}}{1+t^2} = \frac{\dot{s}}{s(s^2-1)^{\frac{1}{2}}}$.

These are the most useful forms of fluxions of circular arcs.

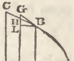
TO FIND THE SINE AND COSINE OF ANY ARC v.

Assume sin. $v = av + bv^2 + cv^3$, &c. and cos. $v = 1 + mv + nv^2 + pv^3$, &c. then $\overline{\text{sin. } \dot{v}|} = a\dot{v} + 2bv\dot{v} + 3cv^2\dot{v}$, &c. and $\overline{\text{cos. } \dot{v}|} = m\dot{v} + 2nv\dot{v} + 3pv^2\dot{v}$, &c. but $\overline{\text{sin. } \dot{v}|} = \dot{v}$ cos. v, and $\overline{\text{cos. } \dot{v}|} = -\dot{v}$ sin. v, whence we have two equations $a + 2bv + 3cv^2$, &c. $= 1 + mv + nv^2 + pv^3$, &c. and $av + bv^2 + cv^3$, &c. $= -m - 2nv - 3pv^2$, &c. and equating the coefficients, we have $a = 1, -m = 0, b = 0, n = -\frac{1}{2}, c = \frac{-1}{1\cdot2\cdot3}, p = 0, d = 0,$ $q = \frac{+1}{1\cdot2\cdot3\cdot4}, e = \frac{+1}{1\cdot2\cdot3\cdot4\cdot5}$, &c.; therefore, substituting these values, we get sin. $v = v - \frac{v^3}{1\cdot2\cdot3} + \frac{v^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{v^7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}$, &c.
Cos. $v = 1 - \frac{v^2}{1\cdot2} + \frac{v^4}{1\cdot2\cdot3\cdot4} - \frac{v^6}{1\cdot2\cdot3\cdot4\cdot5\cdot6} + \frac{v^8}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}$, &c. which are Newton's series for finding the sines and cosines of an arc, given page 158.

TO FIND THE LENGTH OF ANY ARC OF WHICH THE TANGENT IS t.

Assume $v = at + bt^2 + ct^3$, &c. then $\dot{v} = a\dot{t} + 2bt\dot{t} + 3ct^2\dot{t}$ + &c. But $\dot{v} = \frac{\dot{t}}{1+t^2} = \dot{t} - t^2\dot{t} + t^4\dot{t} - t^6\dot{t}$, and, equating the coefficients, we have $a = 1$, $b = 0$, $c = \frac{-1}{3}$, $d = 0$, $e = \frac{+1}{5}$, &c.; therefore $v = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7}$, &c. which is the series given in the note, page 184.

PROP. II. Problem. To determine the length of any curve ABC.



Let AE $= x$, EB $= y$, and the curve AB $= z$. Draw GF parallel to BE and BG to touch the curve at B. and draw BL parallel to AD. Then, while AE has increased to AF, BE has increased to FH, and the tangent is BG, hence GL is always greater than LH. But as BL decreases, GL becomes more nearly equal to HL, and at length they will differ from each other by a quantity less than any given quantity; therefore, representing BL by $\dot{x}$, LG by $\dot{y}$, and BG by $\dot{z}$, we have $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$, or $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$. And the fluent of this equation will be the value of z, which fluent must be determined from the nature of the curve.

EXAMPLES.

1. Let the curve be a parabola, of which the principal vertex is A, and $p =$ the parameter, then $px = y^2$, and $p\dot{x} = 2y\dot{y}$, and $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{y}}{\frac{1}{2}p}\sqrt{\frac{1}{4}p^2 + y^2} = \frac{2y^2\dot{y} + 2d^2\dot{y}}{2d\sqrt{y^2 + d^2}}$ (where $d = \frac{1}{2}p$) $= \frac{2y^3\dot{y} + 2d^2y\dot{y}}{2d\sqrt{y^4 + y^2d^2}} = \frac{2y^3\dot{y} + d^2y\dot{y}}{2d\sqrt{y^4 + y^2d^2}} + \frac{d^2y\dot{y}}{2d\sqrt{y^4 + y^2d^2}}$. Here the fluent of the first term is $\frac{y\sqrt{y^2 + d^2}}{2d}$, and that of the second is $\frac{1}{2}d \times$ hyp. log. of $\frac{y + \sqrt{y^2 + d^2}}{d}$; therefore the length of the curve is $\frac{y\sqrt{y^2 + d^2}}{2d} + \frac{1}{2}d \times \log. \frac{y + \sqrt{y^2 + d^2}}{d}$.

2. Let the curve be a circle, then $\dot{z} = \frac{\dot{x}}{y}$ (see last Prop.), which, being reduced to a series, and the fluent taken, becomes $2y \times (\frac{1}{2} + \frac{x^2}{3y^2} - \frac{x^4}{2 \cdot 3y^4} + \frac{x^6}{5 \cdot 7y^6} - \frac{x^8}{7 \cdot 9y^8}$, &c.) or putting $v^2 = \frac{x^2}{y^2}$, it becomes for the arc of which the chord is $2y = 4y \times (\frac{1}{2} + \frac{1}{3}v^2 - \frac{v^4}{2 \cdot 3} + \frac{v^6}{5 \cdot 7} - \frac{v^8}{7 \cdot 9}$, &c.) and this series is nearly equal to $2y \times \frac{15 + 13v^2}{15 + 3v^2}$, but more nearly equal to $\frac{4y}{3} \times$

$\left(\frac{3}{2}+v^2-v^4\times\frac{\frac{1}{8}v^2+1}{\frac{5}{8}v^2+1}\right)$, which are the two approximations given in Prob. 17, Mensuration of Surfaces.

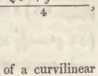
3. Let the curve be an ellipse, then by the 11th Formula, Prop. 7 of Conic Sections, $y=\frac{b}{a}\times\sqrt{2ax-x^2}=\frac{b}{a}\sqrt{a^2-v^2}$ (where $v=$ distance of the ordinate from the centre, $=a-x$); therefore $\dot{y}=\frac{-bv\dot{v}}{a\sqrt{a^2-v^2}}$, and therefore $\dot{z}=\frac{\dot{v}\sqrt{\left(a^2-\frac{a^2-b^2}{a^2}v^2\right)}}{\sqrt{a^2-v^2}}$, or (putting $d=1-\frac{b^2}{a}$), $\dot{z}=\frac{a\dot{v}\sqrt{a^2-dv^2}}{\sqrt{a^2-v^2}}=\frac{a\dot{v}}{\sqrt{a^2-v^2}}\times\left(1-\frac{dv^2}{2a^2}-\frac{d^2v^4}{2\cdot4a^4}-\frac{3d^3v^6}{2\cdot4\cdot6a^6},\ \&c.\right)$ by throwing $\sqrt{a^2-dv^2}$ into a series. But the fluent of $\frac{a\dot{v}}{\sqrt{a^2-v^2}}$, is the corresponding arc of the circle, and therefore the whole fluent (putting $t^2=a^2-v^2$) is $A-\frac{d}{2a^2}\times\frac{a^2A-tv}{2}-\frac{d^2}{2\cdot4a^4}\times\frac{3a^2B-tv^3}{4}-\frac{3d^3}{2\cdot4\cdot6a^6}\times\frac{5a^2C-tv^5}{6}$, &c. where $B=\frac{a^2A-tv}{2}$, $C=\frac{3a^2B-tv^3}{4}$, $D=\frac{5a^2C-tv^5}{6}$.

If the whole quadrant be required, $v=a$, and $t=0$, and then $z=A\times\left(1-\frac{d}{2\cdot2}-\frac{3d^2}{2\cdot2\cdot4\cdot4}-\frac{3\cdot3\cdot5d^3}{2\cdot2\cdot4\cdot4\cdot6\cdot6},\ \&c.\right)$ and this is nearly the fourth part of the series to which the rule in Prob. 2, Conic Sections, might be reduced.

4. Let the curve be a hyperbola, then $y=\frac{b}{a}\sqrt{ax+x^2}$, whence $x=\frac{a}{b}\times\sqrt{b^2+y^2}-a$, and $\dot{x}=\frac{ay\dot{y}}{b\sqrt{b^2+y^2}}$, whence $\dot{z}=\dot{y}\frac{\sqrt{b^2+\frac{b^2+a^2}{b^2}y^2}}{\sqrt{b^2+y^2}}$, or (putting $q=\frac{b^2+a^2}{b^4}$), $\dot{z}=\frac{b\dot{y}}{\sqrt{b^2+y^2}}\times\sqrt{1+qy^2}=\frac{b\dot{y}}{\sqrt{b^2+y^2}}\times\left(1+\frac{qy^2}{2}+\frac{q^2y^4}{2\cdot4}+\frac{3q^3y^6}{2\cdot4\cdot6}+\frac{3\cdot5q^4y^8}{2\cdot4\cdot6\cdot8},\right.$

&c.). Now the fluent of $\frac{b\dot{y}}{\sqrt{b^2+y^2}} = b \times$ hyp. log. of $\frac{y+\sqrt{b^2+y^2}}{b}$ $=$ A, and therefore $z = b \times \left(A + \frac{q}{2}B - \frac{q^2}{2\cdot4}C + \frac{3q^3}{2\cdot4\cdot6}D + \frac{3\cdot5q^4}{2\cdot4\cdot6\cdot8}\right.$ $\left.E, \text{\&c.}\right)$ where $B = \frac{y\sqrt{b^2+y^2}-b^2A}{2}$, $C = \frac{y^3\sqrt{b^2+y^2}-3b^2B}{4}$, $D = \frac{y^5\sqrt{b^2+y^2}-5b^2C}{6}$, &c.

Prop. III. Problem. To find the area of a curvilinear figure.



Let AE $= x$, and ED $= y$. Draw HF parallel to DE, and DG to AC. The parallelogram GFED is always less than the curvilinear HFED, but it continually approaches to an equality with it, as HF approaches to DE, and at length would differ from it by a quantity less than any given quantity. Therefore GE is the limit of the increment of HFED; that is, $y\dot{x}$ is the fluxion of the area AED, and its fluent found from the nature of the curve, and properly corrected, will be the area.

EXAMPLES.

1. Let the curve be a parabola, and p the parameter, then $px = y^2$; therefore $p\dot{x} = 2y\dot{y}$, and $y\dot{x} = \frac{2y^2\dot{y}}{p}$, and the fluent of this or the area $= \frac{2y^3}{3p} = \frac{2xy}{3}$, which is the rule in Prob. 3, Conic Sections.

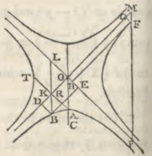
If X be another abscissa, Y its ordinate, and $X - x = d$, then $Y^2 : Y^2 - y^2 :: X : d :: Xp : dp$, and $Xp = Y^2$; therefore $dp = Y^2 - y^2$ and $p = \frac{Y^2 - y^2}{d}$, and the area of the frustum is $\frac{2}{3}\left(\frac{Y^3-y^3}{p}\right) = \frac{2}{3}d\left(\frac{Y^3-y^3}{Y^2-y^2}\right) = \frac{2}{3}d\left(\frac{Y^2+Yy+y^2}{Y+y}\right)$ $= \frac{2}{3}d\left(Y + \frac{y^2}{Y+y}\right)$, which is the rule in Prob. 4, Conic Sections.

2. Let the curve be the segment of a circle, of which the radius is r, then $2rx - x^2 = y^2$; therefore $\dot{x} = \frac{y\dot{y}}{r-x} =$

$\frac{y\dot{y}}{\sqrt{r^2-y^2}}$, and $y\dot{x} = \frac{y^2\dot{y}}{\sqrt{r^2-y^2}}$, which, being reduced to a series, and the fluent taken, becomes $2xy \times \left(\frac{1}{3} + \frac{x^2}{3\ 5y^2} - \frac{x^4}{3\cdot5\cdot7y^4} + \frac{x^6}{5\cdot7\cdot9y^6}, \&c.\right)$ a series which coincides very nearly with $\frac{2yx}{15} \times \left(5 + q^2 - \frac{q^4}{21} \times \frac{4q^2+33}{5q^2+11}\right)$, supposing $q = \frac{x}{y}$, which is the second approximation in Prob. 22, Mensuration of Surfaces.

3. Let the curve be an ellipse, of which the semiaxes are a and b; then (by the 11th Formula, Prop. 7, Conic Sections) $y^2 = \frac{b}{a} \times (2ax - x^2)$, which is the equation for the circle multiplied by $\frac{b}{a}$; therefore the area of the circle, or of any portion of it, multiplied by $\frac{b}{a}$, will give the ellipse, or a similar portion of it, as in Prob. 25, Mensuration of Surfaces.

4. Let the curve be a hyperbola, of which the semiaxes are a and b, then the equation is $\frac{b^2}{a^2} \times (2ax + x^2) = y^2$, or taking $v = a + x$, then $\frac{b}{a}\sqrt{v^2 - a^2} = y$, and $y\dot{v} = \frac{b\dot{v}}{a}\sqrt{v^2 - a^2}$, and the fluent of its double is $\frac{b}{a}\sqrt{v^2 - a^2} - ab \times$ hyp. log. of $\frac{v + \sqrt{v^2 - a^2}}{a} = vy - ab \times$ hyp. log. of $\frac{ay + bv}{ab}$, which is the rule in Prob. 29, Mensuration of Surfaces.

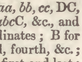


5. To find the area between the hyperbola and the asymptotes.

Let OR = RA = c, RD = x, and DB = y, then OD = $c + x$, and OD × DB = OR × RA; therefore $y - \frac{c^2}{c+x} = c - x + \frac{x^2}{c} - \frac{x^3}{c^2}$, &c. and $y\dot{x} = c\dot{x} - x\dot{x}$

$+\frac{x^2\dot{x}}{c}-\frac{x^3\dot{x}}{c^2}$, &c.; therefore the area RABD $= c^2 \times \left(\frac{x}{c} - \frac{x^2}{2c^2}+\frac{x^3}{3c^3}-\frac{x^4}{4c^4}\right)$, &c. $= c^2 \times$ hyp. log. $\frac{c+x}{c}$.

PROP. IV. If a right line AD be divided into an even number of equal parts, A*a*, *ab*, *bc*, *c*D, &c. and from the points of division perpendicular ordinates, AB, *aa*, *bb*, *cc*, DC, &c. be erected and terminated by any curve B*abc*C, &c., and if A be put for the sum of the first and last ordinates; B for the sum of the even ordinates, that is the second, fourth, &c.; C for all the rest, or odd ordinates, wanting the first and last; and D for the common distance between the ordinates; then $(A+4B+2C)\times\frac{1}{3}D =$ the area of the space ABCD.

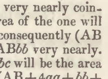


Conceive a parabola to be drawn through the first three points B*ab* of the curve, having its axis parallel to the ordinates, then the parabolic area will be $(AB+4aa+bb)\times\frac{1}{3}Aa$. Now, when the points B, *a*, *b*, are near to each other, the parabolic curve will very nearly coincide with the given curve, and hence the area of the one will be very nearly equal to that of the other; consequently $(AB+4aa+bb)\times\frac{1}{3}Aa$ will be the area of AB*bb* very nearly. In like manner $(bb+4cc+CD)\times\frac{1}{3}Aa$, or *bc* will be the area of *bb*CD; therefore the sum of these areas, or $(AB+4aa+bb+bb+4CC+CD)\times\frac{1}{3}Aa=(A+4B+2C)\times\frac{1}{3}D$, will be the whole area ABCD very nearly; whence, if D equal the whole length of the line AD, and n the number of parts into which it is divided, then $(A+4B+2C)\times\frac{D}{3n}=$ the area, which is Rule II. page 196.

Cor. 1. The same rule will also obtain for the contents of all solids by using the areas of the sections perpendicular to the axis instead of the ordinates.

Cor. 2. It is evident that the greater the number of ordinates or sections which are used, the more accurately will the area or solidity be determined.

PROP. V. Problem. To find the surface of a solid generated by the revolution of a curve about an axis.



Let the curve ADB revolve about the axis AC, then the point D will describe a circle, and the straight line DH will describe the surface of a cylinder, which will be always less than the surface described by DG, but will differ less from it the less that the length of DH is, and will ultimately be the limit of the surface described by DG; therefore, if $p = 3{\cdot}1416$, DE $= y$, and AD $= v$, the fluxion of the surface is $2py\dot{v}$, or if AE $= x$, then $\dot{v} = \sqrt{\dot{x}^2 + \dot{y}^2}$, and the fluxion is $2py\sqrt{\dot{x}^2 + \dot{y}^2}$, and the fluent of this derived from the nature of the curve will be the surface.

In the cylinder y is constant, and the fluent is $2pyv$, where v is the length of the cylinder.

EXAMPLES.

1. To find the surface of a cone. Here ADB is a straight line $= a$, B$c = b$, and $a : b :: v : y = \frac{bv}{a}$. Therefore $2py\dot{v} = \frac{2pbv\dot{v}}{a}$; hence the surface ADE $= \frac{pbv^2}{a}$, and the surface of the whole cone ABC, where $v = a$, becomes $pba = 3{\cdot}1416 \times$ BC $\times$ AB, which is the rule in Prob. 7, Mensuration of Solids.

2. To find the surface of a sphere, where ADB is a circle, of which the radius AC $= a$. (By Prop. 1, App.) $\dot{v} = \frac{a\dot{x}}{y}$, and $2py\dot{v} = 2ap\dot{x}$; therefore the surface of the segment ADE $= 2pax = 3{\cdot}1416 \times 2$AC $\times$ AE, and the whole surface, where AE becomes $= 2$AC, $= 3{\cdot}1416 \times (2\text{AC})^2$, which is the rule in Prob. 13, Mensuration of Solids.

3. To find the surface of a parabolic conoid. Let $a =$ parameter, then $ax = y^2$ (Prop. 7, Conic Sections), and $\dot{x}^2 = \frac{4y^2\dot{y}^2}{a^2}$; whence $\dot{v} = \frac{\dot{y}}{a}\sqrt{a^2 + 4y^2}$, and $2py\dot{v} = \frac{2py\dot{y}}{a} \times \sqrt{a^2 + 4y^2}$; wherefore the corrected fluent is $\frac{p}{6a} \times (a^2 + 4y^2)^{\frac{3}{2}} - \frac{1}{6}pa^2$, the surface generated by AD.

4. To find the surface of a spheroid. Let $2a =$ fixed axis, $2b =$ revolving axis, $y =$ ordinate, and $x =$ distance of

the ordinate from the centre, then $y = \frac{b}{a}\sqrt{a^2 - x^2}$, $\dot{y} = \frac{-b x \dot{x}}{a\sqrt{a^2 - x^2}}$, and $\dot{v} = \frac{\dot{x}\sqrt{a^4 - x^2 \times (a^2 - b^2)}}{a\sqrt{a^2 - x^2}}$; therefore $2py\dot{v} = \frac{2pb\dot{x}}{a^2} \times \sqrt{a^4 - x^2 \times (a^2 - b^2)}, = \frac{2pb\dot{x}}{a^2} \times \sqrt{a^4 \mp d^2x^2}$ (putting $d^2 = a^2 - b^2$), the upper sign belongs to the oblong spheroid, where $a > b$, and the under sign to the oblate spheroid, where $a < b$. Suppose P = the arc, of which the sine is $\frac{dx}{a^2}$, or $= \cdot 017453 \times$ degrees in that arc (radius $= 1$), when $a > b$, or let P = hyp. log. of $\frac{dx + \sqrt{a^4 + d^2x^2}}{a^2}$, when $a < b$, and the surface will be $= \frac{pbx}{a^2} \times \sqrt{a^4 \mp d^2x^2} + \frac{pba^2\text{P}}{d}$. And for the hemispheroid where $x = a$, the arc is to be taken, of which the sine is $\frac{d}{a}$, or the log. of $\frac{b+d}{a}$, and the surface of the hemisphere will be $\frac{2pb}{d} \times (a^2\text{P} + bd)$.

5. To find the surface of a hyperboloid. Let $2a$ = transverse axis, $2b$ = the conjugate, y = ordinate, and x = its distance from the centre, then $y = \frac{b}{a} \times \sqrt{x^2 - a^2}$, and $\dot{y} = \frac{b x \dot{x}}{a\sqrt{x^2 - a^2}}$; therefore $\dot{v} = \frac{\dot{x}\sqrt{d^2x^2 - a^4}}{a\sqrt{x^2 - a^2}}$ (putting $d^2 = a^2 + b^2$), and $2py\dot{v} = \frac{2pb\dot{x}}{a^2} \times \sqrt{d^2x^2 - a^4}$; and therefore the surface will be $\frac{pbx}{a^2} \times \sqrt{d^2x^2 - a^4} - \frac{pba^2}{d} \times$ hyp. log. of $dx + \sqrt{d^2x^2 - a^4}$, and the correction is $- pb^2 +$ log. $a \times (b+d)$; therefore the whole surface will be $\frac{pbx}{a^2} \times \sqrt{d^2x^2 - a^4} - pb^2 + \frac{pba^2}{d} \times \log. \frac{a \times (b+d)}{ax + \sqrt{d^2x^2 - a^4}}$.

Prop. VI. Problem. To find the content of a circular spindle, described by the revolution of the segment ABC about its chord AC.

Let BO $= r$, OE $= d$, AE $= c$, EH $= x$, and HN $= y$, then $c^2 = r^2 - d^2$, and $(d+y)^2 = r^2 - x^2$; whence $y^2 = r^2 - x^2 - d^2 - 2dy = c^2 - x^2 - 2dy$. Now the fluxion of the solidity is $= \dot{x} \times$ the circle described by NH $= py^2\dot{x} = pc^2\dot{x} - px^2\dot{x} - 2pdy\dot{x}$, and $y\dot{x}$ is the fluxion of the area BEHN; therefore, taking the fluent, the content of the zone described by BEHN $= p \times (c^2x - \frac{1}{3}x^3 - 2d \times$ BEHN). This is the rule for the zone, Prob. 19, Mensuration of Solids.

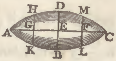
And when x becomes $= c$, the content of half the spindle will be $2p \times (\frac{1}{3}c^3 - d \times \text{ABE})$, which is the rule for the spindle in Prob. 18, Mensuration of Solids.

Cor. 1. If ABC be a segment of an ellipse, and $a =$ semi-axis parallel to AC, y^2 will be found to be $= \frac{r^2}{a^2}(c^2 - x^2) - 2dy$, and $py^2\dot{x} = \frac{pr^2}{a^2} \times (c^2\dot{x} - x^2\dot{x}) - 2pdy\dot{x}$, where, as before, $y\dot{x} =$ fluxion of BEHN; therefore the content of the zone described by BEHN $= \frac{pr^2}{a^2} \times (c^2x - \frac{1}{3}x^3) - 2pd \times$ BEHN, which is the rule for the middle zone of an elliptic spindle, in Prob. 14, Conic Sections. And when $x = c$, the content of half the spindle is $2p \times \left(\frac{\frac{1}{3}r^2c^3}{a^2} - d \times \text{ABE}\right)$.

If $r - d = m$, and S $=$ area ABE, the half spindle $= \frac{2}{3}pc \times \left\{m^2 - d\left(\frac{3S}{c} - m\right)\right\}$, which is the rule for the elliptic spindle, in Prob. 13, Conic Sections.

Cor. 2. If the frustum be taken from half the spindle, there will remain the segment described by the revolution of AHN about AH, and if AH $= h$, it will be in the circle $= p \times \{(\frac{1}{3}h^2 \times (3c - h)\} - 2d \times \text{AHN})$. And in the ellipse $= p \times \left\{\left(\frac{r^2h^2}{3a^2} \times (3c - h)\right\} - 2d \times \text{AHN}\right)$.

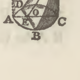
Prop. VII. Problem. To find the content of a parabolic spindle, described by the revolution of the parabola ADC, about its ordinate AC.



Let AE $= a$, ED $= c$, AG $= x$, and GH $= y$, then by the property of the parabola $c : c - y :: a^2 : (a - x)^2$; therefore $c - y = \frac{c \times (a-x)^2}{a^2}$ and $y = c \times \frac{2ax - x^2}{a^2}$; therefore $py^2\dot{x} = \frac{pc^2x^2\dot{x}}{a^4} \times 2(a - x)^2 = \frac{4pc^2a^2x^2\dot{x}}{a^4} - \frac{4pc^2ax^3\dot{x}}{a^4} + \frac{pc^2x^4\dot{x}}{a^4}$, and the fluent or value of the segment described by AHG is $= \frac{4pc^2a^2x^3}{3a^4} - \frac{pc^2ax^4}{a^4} + \frac{pc^2x^5}{5a^4}$. And when $x = a$, the half spindle described by AED $= \frac{4pc^2a^5}{3a^4} - \frac{pc^2a^5}{a^4} + \frac{pc^2a^5}{5a^4} = \frac{8pc^2a}{15} = \frac{8}{15}$ of the circumscribing cylinder, which is the rule for the parabolic spindle in Prob. 15, Conic Sections.

Cor. If the segment described by AGH be taken from half the spindle, there will remain the zone or frustum described by DEGH $= pc^2 \times \left(\frac{8a}{15} - \frac{4x^3}{3a^2} + \frac{x^4}{a^3} - \frac{x^5}{5a^4}\right)$, or by substituting for $a - x$ its equal $a\sqrt{\frac{c-y}{c}}$, or $x = a - a\sqrt{\frac{c-y}{c}}$, it becomes $p \times (a-x) \times \frac{8c^2 + 4cy + 3y^2}{15} = \frac{1}{3}\, p \times (a - x) \times \{2c^2 + y^2 - \frac{2}{5} \times (c - y)^2\}$, which is the rule for the middle zone of the parabolic spindle in Prob. 16, Conic Sections.

Prop. VIII. Problem. To find the content of the hoof of a cylinder ABC–FHG, cut off by the plane DFB.



Suppose the hoof to be generated by the triangle ECF, moving parallel to itself along BD. Let FC $= h$, CE $= v$, EB $= s$, AC $= 2r$, cosine CB $= c = r - v$ or $v - r$. The area of the segment DCB $=$ A. Let $x =$ distance of the moving triangle from AC $=$ sine of the arc between it and C, and let $y =$ cosine of the same arc. Then $y - c =$ the base of the moving triangle, and $v : h :: y - c$: its height $= \frac{h}{v} \times (y - c)$; therefore the area of the moving triangle is $\frac{h}{2v}(y - c)^2$, and the fluxion of the hoof is $\frac{h\dot{x}}{2v}(y - c)^2$, but $(y - c)^2 = y^2 - c^2 - 2c \times (y - c) = s^2 - x^2 - 2c \times (y - c)$; therefore the fluxion becomes $\frac{h\dot{x}}{2v} \times \{s^2 - x^2 - 2c \times (y - c)\}$, and $\frac{ch\dot{x}}{v} \times (y - c)$ is $= \frac{ch}{v} \times$ the fluxion of the area generated by the base of the triangle, between that base and CE. Let this area be called B, and the content will be $\frac{hx}{2v} \times (s^2 - \frac{1}{3}x^2) - \frac{hc\text{B}}{v}$, and when $x = s$, the half-hoof becomes $\frac{h}{2v} \times (\frac{2}{3}s^2 - c\text{A})$.

Cor. If E coincide with the centre O, then $c = 0$, and the hoof becomes $\frac{2}{3}r^2h$.

Prop. IX. Problem. To find the content of the hoof EBF–C of a cone AEB–V, cut off from it by the oblique plane ECF.

Draw CR, VP perpendicular to AB and VK, Be perpendicular to CD, and CG parallel to AB. As BR : BP :: CR : PV, also BR : CS = RP :: CR : VS, and because CR : VS :: BC : CV :: Be : VK; therefore VS : VK :: CR : Be :: CD : DB. Join EV, FV. The solid EBF–V is pyramidal or conical, of which the base is EBF, and its height VP; it is therefore $= \frac{1}{3}$ VP $\times$ EBF. And the solid ECF–V has EFC for its base, and VK for its altitude; it is therefore $= \frac{1}{3}$ VK $\times$ ECF. Wherefore the hoof EBF–C, which is the difference of these solids, is $= \frac{1}{3}$ VP $\times$ EBF $- \frac{1}{3}$ VK $\times$ ECF.

Let AB $=$ D, CG $= d$, DB $= v$, DC $= m$, CR $= h$, and $a = \text{D} - d$, then PV $= \frac{\text{D}h}{a}$, VS $= \frac{dh}{a}$, VK $= \frac{vdh}{am}$, and if A be the tabular area of the segment similar to EBF, (the diameter $= 1$, and the versed sine $= \frac{v}{D}$), then $D^2 \times A = EBF$. And these values being substituted, the hoof becomes $\frac{\frac{1}{3}h}{a} \times (D^3 \times A - \frac{vd}{m} \times ECF)$.

Case 1. If DC is parallel to AV, or $AD = CG$, the base ECF is a parabola, and its area is $= \frac{2}{3}\, EF \times CD = \frac{2}{3}\, CD \times 2\sqrt{AD \times DB}$, and if this is substituted, the hoof becomes $\frac{\frac{1}{3}h}{a} \times (D^3 \times A - \frac{4}{3} vd \sqrt{dv})$, or because $v = a = D - d$, the hoof is $\frac{1}{3}\, h \times \left(\frac{D^3 \times A}{a} - \frac{4d}{3}\sqrt{ad}\right)$.

Case 2. If DC meets AV, or if ECF is a segment of an ellipse, then $v > a$; the whole axis is $= \frac{md}{v-a}$, and its conjugate $= d\sqrt{\frac{v}{v-a}}$. And if B is the tabular segment of which the diameter is 1, and the versed sine $m \div \frac{md}{v-a} = \frac{v-a}{d}$, then the area $ECF = \frac{md^2 v^{\frac{1}{2}}}{(v-a)^{\frac{3}{2}}}B$. And therefore the hoof EBFC $= \frac{\frac{1}{3}h}{a} \times \left\{D^3 \times A - d^3 \times \left(\frac{v}{v-a}\right)^{\frac{3}{2}}\right\} B$. This is the rule in Case 2, page 253.

Case 3. If D coincides with A, then $v = D$, the segment EBF is a circle, and ECF an ellipse, the area of the circle is $D^2 p$, ($p = \cdot 7854$), and of the ellipse $pm\sqrt{Dd}$; and therefore the hoof is $\frac{1}{3}\, hpD \times \frac{D^2 - d\sqrt{Dd}}{D-d}$. And the other hoof ACG $= \frac{1}{3}\, hpD \times \frac{D\sqrt{Dd} - d^2}{D-d}$.

Case 4. If the segment ECF is a hyperbola, the transverse is $\frac{md}{a-v}$, the conjugate $d\sqrt{\frac{v}{a-v}}$, and $FG = 2\sqrt{\overline{D-v} \times v}$, and the area may be found by Prob. 29, Mensuration of Surfaces, and if it is called B, the hoof becomes $\frac{\frac{1}{3}h}{a} \times (D^3 \times A - \frac{vd}{m} B)$.

Case 5. If CD is perpendicular to AB, or coincides with CR, then $v = \frac{1}{2}(D - d)$, $m = h$, the transverse $= \frac{2 ih}{a}$, and the conjugate $= d$, and the hoof becomes $\frac{\frac{1}{3}hD^3 \times A}{a} - \frac{1}{6}dB$.

CONSTRUCTION AND USE OF THE TABLE OF JOISTING.

In calculating this Table, we may begin with joists of any size. Let us, therefore, take joists 12 inches by 3 inches, and 18 inches from centre to centre, where there are 12 spaces and an *extra* joist in 18 feet of breadth. Now 18 feet broad by 6 inches long make exactly a square yard, and the cubic timber in 6 inches of length of these 13 joists is

13 × 12 in. × 3 in. × 6 in. = 1 ft. 7 in. 6 pts. also the cub. timb. in 12 joists is
12 × 12 in. × 3 in. × 6 in. = 1 6 0

Consequently 0 1 6 = cubic timber in one joist.

Hence if we wish to find the cubic contents in a square yard for any other distance between centres, we must multiply the cubic timber in 12 joists by 18, and divide by the given distance between centres, which gives the cubic content of 12 joists, to which add 1 in. 6 pts. for the cubic content of 13 joists. Thus if the distance between centres is 22 inches, we have 1 ft. 6 in. × 18 ÷ 22 = 1 ft. 2 in. 8 pts. 9 sec. cubic timber in 6 inches long of 12 joists, to which we add 1 in. 6 pts. and the sum 1 ft. 4 in. 2 pts. 9 sec. is the cubic content of the 13 joists in one square yard of the floor.

If the joists are 11 inches by $2\frac{1}{2}$ inches, then the cubic content in one square yard of the floor, when the distance between centres is 18 inches, is

For 13 joists, 13 × 11 in. × $2\frac{1}{2}$ in. × 6 in. = 1 ft. 2 in. 10 pts. 9 sec.
For 12 joists, 12 × 11 in. × $2\frac{1}{2}$ in. × 6 in. = 1 1 9 0
Therefore the cubic timber in one joist = 0 1 1 9

And for any other distance between centres, say 20 inches, we have 1 ft. 1 in. 9 pts. × 18 ÷ 20 = 1 ft. 0 in. 4 pts. 6 sec., cubic in 12 joists, to which we add 1 in. 1 pt. 9 sec. and the sum 1 ft. 1 in. 6 pts. 3 sec. is the cubic content of the timber in a square yard of such joisting. From these examples the rest of the Table may be easily computed, or it may be extended to scantlings of other sizes and distances.

The use of the Table in finding the cubic content of the joisting is manifest, for we have only to multiply the number in the Table answering to the size of the joist, and the distance between centres by the number of square yards in the floor (allowing for the holds in the walls) to obtain the cubic content. Thus,

The floor of a room, including the holds of the joists in the walls, measures 48 square yards, the joists are 12 inches by 3 inches, and the distance between their centres is 20 inches. Required the solid content of the joisting. The number in the Table answering to 12 inches by 3 inches, and 20 inches between centres, is 1 ft. 5 in. 8 pts. 5 sec., and this multiplied by 48 gives 70 cubic feet 9 in. 8 pts., for the solid content of the joisting.

But the use of the Table is not limited to finding the content only; the value of the timber may likewise be readily found from it, for the feet in the Table are shillings, the inches pence, the parts 12th parts of a penny, &c. when the price of timber is one shilling per cubic foot. For example, when the price of timber is 2s. 7½d. per cubic foot, what is the value of a square yard of joisting 11 inches by 2½ inches, and 17 inches between centres?

The number in the Table answering to 11 inches by 2½ inches, and 17 inches between centres, is 1 ft. 3 in. 8 pts. 5 sec.; therefore the value of a square yard at 1s. per cubic foot is 1s. 3d. 8 : 5, and

	1s.	3d.	8	: 5	
				2	
	2	7	4	10	value at 2s.
6 = ½ of 1s. =		7	10	2 6	value at 6d.
1½ = ¼ of 6d. =		1	11	6 7½	value at 1½d.
Very nearly 3s. 5¼d.,	3s.	5d.	2	7 1½	value at 2s. 7½d.

Again, the floor of a room contains 42 square yards, including the holds of the joists in the walls. What is the value of the joisting, which is 10 inches by 3 inches, and 14 inches between centres, when timber is at 3s. 9d. per cubic foot?

The number in the Table answering to 10 inches by 3 inches, and 14 inches between centres, is, 1 : 8 : 6 : 5, whence

	1	8	6	5
				42
Whole cubic content of the joisting, and also the value at 1s. per cubic foot,.........	71	10	5	6

	Then 71	10	5	6	
				3	
9d. = $\frac{1}{4}$ of 3s.	215	7	4	6	value at 3s.
	53	10	10	$1\frac{1}{2}$	value at 9d.
Very nearly £13, 9s. $6\frac{1}{4}$d.,	269s. 6d.	2		$7\frac{1}{2}$	value at 3s. 9d.

USE OF THE TABLE OF THE SIDES OF POLYGONS.

This Table will be found of great use to practical men, both in laying off regular polygons, and in finding their contents, which will be manifest from the following examples:—

A polygon of 11 sides is to be inscribed in a circle of 50 feet 6 inches diameter. Required the length of the side of the polygon.

Tabular number ·2817325
50·5

14086625
14086625

14·22749125 = 14 feet $2\frac{7}{10}$ inches nearly.

The side of an octagon is 32 feet. Required the diameter of the circumscribing circle.

Tabular number ·3826834; whence 32 ÷ ·3826834 = 83·62 feet = 83 feet $7\frac{1}{2}$ inches nearly, the diameter of the circumscribing circle.

The diameter of a circle is 100 feet. What is the length of the side of a circumscribing polygon of 36 sides?

Tabular number ·0874887 × 100 = 8·74887 feet = 8 feet 9 inches nearly.

Required the area of a polygon of 11 sides circumscribed about a circle whose diameter is 100 feet.

Tabular number ·1989124 × 100 = 19·89124 feet, length of one side, and 19·89124 × 25 (half the radius) × 16 (the number of sides) = 497·281 × 16 = 7956·496 feet = 884 yards $0\frac{1}{2}$ foot very nearly, the area required.

A

TABLE,

CONTAINING

THE LOGARITHMS OF NUMBERS

FROM 1 TO 10,000.

Numbers from 1 to 100 and their Logarithms, with their Indices.

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
1	0.000000	21	1.322219	41	1.612784	61	1.785330	81	1.908485
2	0.301030	22	1.342423	42	1.623249	62	1.792392	82	1.913814
3	0.477121	23	1.361728	43	1.633468	63	1.799341	83	1.919078
4	0.602060	24	1.380211	44	1.643453	64	1.806180	84	1.924279
5	0.698970	25	1.397940	45	1.653213	65	1.812913	85	1.929419
6	0.778151	26	1.414973	46	1.662758	66	1.819544	86	1.934498
7	0.845098	27	1.431364	47	1.672098	67	1.826075	87	1.939519
8	0.903090	28	1.447158	48	1.681241	68	1.832509	88	1.944483
9	0.954243	29	1.462398	49	1.690196	69	1.838849	89	1.949390
10	1.000000	30	1.477121	50	1.698970	70	1.845098	90	1.954243
11	1.041393	31	1.491362	51	1.707570	71	1.851258	91	1.959041
12	1.079181	32	1.505150	52	1.716003	72	1.857332	92	1.963788
13	1.113943	33	1.518514	53	1.724276	73	1.863323	93	1.968483
14	1.146128	34	1.531479	54	1.732394	74	1.869232	94	1.973128
15	1.176091	35	1.544068	55	1.740363	75	1.875061	95	1.977724
16	1.204120	36	1.556303	56	1.748188	76	1.880814	96	1.982271
17	1.230449	37	1.568202	57	1.755875	77	1.886491	97	1.986772
18	1.255273	38	1.579784	58	1.763428	78	1.892095	98	1.991226
19	1.278754	39	1.591065	59	1.770852	79	1.897627	99	1.995635
20	1.301030	40	1.602060	60	1.778151	80	1.903090	100	2.000000

NOTE. In the following part of the Table the Indices are omitted, as they can be very easily supplied by the directions given in the Section on Logarithms, page 93.

N.	0	1	2	3	4	5	6	7	8	9
100	000000	000434	000868	001301	001734	002166	002598	003029	003461	003891
1	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174
2	8600	9026	9451	9876	010300	010724	011147	011570	011993	012415
3	012837	013259	013680	014100	4521	4940	5360	5779	6197	6616
4	7033	7451	7868	8284	8700	9116	9532	9947	020361	020775
5	021189	021603	022016	022428	022841	023252	023664	024075	4486	4896
6	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978
7	9384	9789	030195	030600	031004	031408	031812	032216	032619	033021
8	033424	033826	4227	4628	5029	5430	5830	6230	6629	7028
9	7426	7825	8223	8620	9017	9414	9811	040207	040602	040998
110	041393	041787	042182	042576	042969	043362	043755	044148	044540	044932
1	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830
2	9218	9606	9993	050380	050766	051153	051538	051924	052309	052694
3	053078	053463	053846	4230	4613	4996	5378	5760	6142	6524
4	6905	7286	7666	8046	8426	8805	9185	9563	9942	060320
5	060698	061075	061452	061829	062206	062582	062958	063333	063709	4083
6	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815
7	8186	8557	8928	9298	9668	070038	070407	070776	071145	071514
8	071882	072250	072617	072985	073352	3718	4085	4451	4816	5182
9	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819
120	079181	079543	079904	080266	080626	080987	081347	081707	082067	082426
1	082785	083144	083503	3861	4219	4576	4934	5291	5647	6004
2	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552
3	9905	090258	090611	090963	091315	091667	092018	092370	092721	093071
4	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562
5	6910	7257	7604	7951	8298	8644	8990	9335	9681	100026
6	100371	100715	101059	101403	101747	102091	102434	102777	103119	3462
7	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871
8	7210	7549	7888	8227	8565	8903	9241	9579	9916	110253
9	110590	110926	111263	111599	111934	112270	112605	112940	113275	3609
130	113943	114277	114611	114944	115278	115611	115943	116276	116608	116940
1	7271	7603	7934	8265	8595	8926	9256	9586	9915	120245
2	120574	120903	121231	121560	121888	122216	122544	122871	123198	3525
3	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781
4	7105	7429	7753	8076	8399	8722	9045	9368	9690	130012
5	130334	130655	130977	131298	131619	131939	132260	132580	132900	3219
6	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403
7	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564
8	9879	140194	140508	140822	141136	141450	141763	142076	142389	142702
9	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818
140	146128	146438	146748	147058	147367	147676	147985	148294	148603	148911
1	9219	9527	9835	150142	150449	150756	151063	151370	151676	151982
2	152288	152594	152900	3205	3510	3815	4120	4424	4728	5032
3	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061
4	8362	8664	8965	9266	9567	9868	160168	160469	160769	161068
5	161368	161667	161967	162266	162564	162863	3161	3460	3758	4055
6	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022
7	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968
8	170262	170555	170848	171141	171434	171726	172019	172311	172603	172895
9	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802
150	176091	176381	176670	176959	177248	177536	177825	178113	178401	178689
1	8977	9264	9552	9839	180126	180413	180699	180986	181272	181558
2	181844	182129	182415	182700	2985	3270	3555	3839	4123	4407
3	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239
4	7521	7803	8084	8366	8647	8928	9209	9490	9771	190051
5	190332	190612	190892	191171	191451	191730	192010	192289	192567	2846
6	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623
7	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382
8	8657	8932	9206	9481	9755	200029	200303	200577	200850	201124
9	201397	201670	201943	202216	202488	2761	3033	3305	3577	3848
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9	D.
60	204120	204391	204663	204934	205204	205475	205746	206016	206286	206556	271
1	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
2	9515	9783	210051	210319	210586	210853	211121	211388	211654	211921	267
3	212188	212454	2720	2986	3252	3518	3783	4049	4314	4579	266
4	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
5	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
6	220108	220370	220631	220892	221153	221414	221675	221936	222196	222456	261
7	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
8	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
9	7887	8144	8400	8657	8913	9170	9426	9682	9938	230193	256
70	230449	230704	230960	231215	231470	231724	231979	232234	232488	232742	255
1	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
2	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
3	8046	8297	8548	8799	9049	9299	9550	9800	240050	240300	250
4	240549	240799	241048	241297	241546	241795	242044	242293	2541	2790	249
5	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
6	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
7	7973	8219	8464	8709	8954	9198	9443	9687	9932	250176	245
8	250420	250664	250908	251151	251395	251638	251881	252125	252368	2610	243
9	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
80	255273	255514	255755	255996	256237	256477	256718	256958	257198	257439	241
1	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
2	260071	260310	260548	260787	261025	261263	261501	261739	261976	262214	238
3	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
4	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
5	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
6	9513	9746	9980	270213	270446	270679	270912	271144	271377	271609	233
7	271842	272074	272306	2538	2770	3001	3233	3464	3696	3927	232
8	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
9	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
90	278754	278982	279211	279439	279667	279895	280123	280351	280578	280806	228
1	281033	281261	281488	281715	281942	282169	2396	2622	2849	3075	227
2	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
3	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
4	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
5	290035	290257	290480	290702	290925	291147	291369	291591	291813	292034	222
6	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
7	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
8	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
9	8853	9071	9289	9507	9725	9943	300161	300378	300595	300813	218
200	301030	301247	301464	301681	301898	302114	302331	302547	302764	302980	217
1	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
2	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
3	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
4	9630	9843	310056	310268	310481	310693	310906	311118	311330	311542	212
5	311754	311966	2177	2389	2600	2812	3023	3234	3445	3656	211
6	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
7	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
8	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
9	320146	320354	320562	320769	320977	321184	321391	321598	321805	322012	207
210	322219	322426	322633	322839	323046	323252	323458	323665	323871	324077	206
1	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
2	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
3	8380	8583	8787	8991	9194	9398	9601	9805	330008	330211	203
4	330414	330617	330819	331022	331225	331427	331630	331832	2034	2236	202
5	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
6	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
7	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
8	8456	8656	8855	9054	9253	9451	9650	9849	340047	340246	199
9	340444	340642	340841	341039	341237	341435	341632	341830	2028	2225	198
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D
220	342423	342620	342817	343014	343212	343409	343606	343802	343999	344196	19
1	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	19
2	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	19
3	8305	8500	8694	8889	9083	9278	9472	9666	9860	350054	19
4	350248	350442	350636	350829	351023	351216	351410	351603	351796	1989	19
5	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	19
6	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	19
7	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	19
8	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	19
9	9835	360025	360215	360404	360593	360783	360972	361161	361350	361539	18
230	361728	361917	362105	362294	362482	362671	362859	363048	363236	363424	18
1	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	18
2	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	18
3	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	18
4	9216	9401	9587	9772	9958	370143	370328	370513	370698	370883	18
5	371068	371253	371437	371622	371806	1991	2175	2360	2544	2728	18
6	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	18
7	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	18
8	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	18
9	8398	8580	8761	8943	9124	9306	9487	9668	9849	380030	18
240	380211	380392	380573	380754	380934	381115	381296	381476	381656	381837	18
1	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	18
2	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	17
3	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	17
4	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	17
5	9166	9343	9520	9698	9875	390051	390228	390405	390582	390759	17
6	390935	391112	391288	391464	391641	1817	1993	2169	2345	2521	17
7	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	17
8	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	17
9	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	17
250	397940	398114	398287	398461	398634	398808	398981	399154	399328	399501	17
1	9674	9847	400020	400192	400365	400538	400711	400883	401056	401228	17
2	401401	401573	1745	1917	2089	2261	2433	2605	2777	2949	17
3	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	17
4	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	17
5	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	17
6	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	16
7	9933	410102	410271	410440	410609	410777	410946	411114	411283	411451	16
8	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	16
9	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	16
260	414973	415140	415307	415474	415641	415808	415974	416141	416308	416474	16
1	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	16
2	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	16
3	9956	420121	420286	420451	420616	420781	420945	421110	421275	421439	15
4	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082	16
5	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	16
6	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	16
7	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	16
8	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	16
9	9752	9914	430075	430236	430398	430559	430720	430881	431042	431203	16
270	431364	431525	431685	431846	432007	432167	432328	432488	432649	432809	16
1	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	16
2	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	15
3	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	15
4	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	15
5	9333	9491	9648	9806	9964	440122	440279	440437	440594	440752	15
6	440909	441066	441224	441381	441538	1695	1852	2009	2166	2323	15
7	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	15
8	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	15
9	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	15
N.	0	1	2	3	4	5	6	7	8	9	D

N.	0	1	2	3	4	5	6	7	8	9	D.
80	447158	447313	447468	447623	447778	447933	448088	448242	448397	448552	155
1	8706	8861	9015	9170	9324	9478	9633	9787	9941	450095	154
2	450249	450403	450557	450711	450865	451018	451172	451326	451479	1633	154
3	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
4	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
5	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
6	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
7	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
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7	630428	630530	630631	630733	630835	630936	1038	1139	1241	1342	102
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6	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
7	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
8	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
9	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	85
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1	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	85
2	9270	9355	9440	9524	9609	9694	9779	9863	9948	710033	85
3	710117	710202	710287	710371	710456	710540	710625	710710	710794	0879	85
4	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	84
5	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	84
6	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
7	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
8	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
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2	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	83
3	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
4	9331	9414	9497	9580	9663	9745	9828	9911	9994	720077	83
5	720159	720242	720325	720407	720490	720573	720655	720738	720821	0903	83
6	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
7	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
8	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
9	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
30	724276	724358	724440	724522	724604	724685	724767	724849	724931	725013	82
1	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
2	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
3	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
4	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
5	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
6	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
7	9974	730055	730136	730217	730298	730378	730459	730540	730621	730702	81
8	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
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3	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	80
4	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	80
5	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
6	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
7	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
8	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
9	9572	9651	9731	9810	9889	9968	740047	740126	740205	740284	79
50	740363	740442	740521	740600	740678	740757	740836	740915	740994	741073	79
1	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
2	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
3	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
4	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
5	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
6	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
7	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
8	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
9	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
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1	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
2	9736	9814	9891	9968	750045	750123	750200	750277	750354	750431	77
3	750508	750586	750663	750740	0817	0894	0971	1048	1125	1202	77
4	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
5	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
6	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
7	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
8	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
9	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
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2	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
3	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
4	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
5	9668	9743	9819	9894	9970	760045	760121	760196	760272	760347	75
6	760422	760498	760573	760649	760724	0799	0875	0950	1025	1101	75
7	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
8	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
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2	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
3	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
5	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
8	9377	9451	9525	9599	9673	9746	9820	9894	9968	770042	74
9	770115	770189	770263	770336	770410	770484	770557	770631	770705	0778	74
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1	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
2	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
4	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
5	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
600	778151	778224	778296	778368	778441	778513	778585	778658	778730	778802	72
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2	9596	9669	9741	9813	9885	9957	780029	780101	780173	780245	72
3	780317	780389	780461	780533	780605	780677	0749	0821	0893	0965	72
4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
5	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
7	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
8	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
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1	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
5	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
6	9581	9651	9722	9792	9863	9933	790004	790074	790144	790215	70
7	790285	790356	790426	790496	790567	790637	0707	0778	0848	0918	70
8	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
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2	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
3	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
5	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
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9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
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2	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
4	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	68
5	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
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7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
9	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
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1	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	68
2	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
3	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
4	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
5	9560	9627	9694	9762	9829	9896	9964	810031	810098	810165	67
6	810233	810300	810367	810434	810501	810569	810636	0703	0770	0837	67
7	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
8	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
9	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
0	812913	812980	813047	813114	813181	813247	813314	813381	813448	813514	67
1	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
5	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
6	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
7	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
8	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
60	819544	819610	819676	819741	819807	819873	819939	820004	820070	820136	66
1	820201	820267	820333	820399	820464	820530	820595	0661	0727	0792	66
2	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	66
3	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
4	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	65
5	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
70	826075	826140	826204	826269	826334	826399	826464	826528	826593	826658	65
1	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	65
2	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
3	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
4	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
5	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	64
6	9947	830011	830075	830139	830204	830268	830332	830396	830460	830525	64
7	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
8	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
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1	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	64
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3	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	64
4	5056	5120	5183	5247	5319	5373	5437	5500	5564	5627	63
5	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
7	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	63
8	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
90	838849	838912	838975	839038	839101	839164	839227	839289	839352	839415	63
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2	840106	840169	840232	840294	840357	840420	840482	840545	840608	0671	63
3	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	63
4	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	63
5	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
7	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
9	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	62
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1	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
2	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
3	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
4	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
5	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
6	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
7	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
8	850033	850095	850156	850217	850279	850340	850401	850462	850524	850585	61
9	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
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1	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	61
2	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
3	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	61
4	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
5	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
6	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
7	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
8	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
9	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
720	857332	857393	857453	857513	857574	857634	857694	857755	857815	857875	60
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2	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	60
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4	9739	9799	9859	9918	9978	860038	860098	860158	860218	860278	60
5	860338	860398	860458	860518	860578	0637	0697	0757	0817	0877	60
6	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
7	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
8	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
9	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
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3	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
4	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
5	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
6	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
7	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
8	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	59
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3	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
4	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
5	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
6	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
7	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
8	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
9	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
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4	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
5	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	57
6	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
7	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
8	9669	9726	9784	9841	9898	9956	880013	880070	880127	880185	57
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4	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
5	9302	9358	9414	· 9470	9526	9582	9638	9694	9750	9806	56
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6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
7	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
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9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
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3	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
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5	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
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8	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
9	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
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5	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
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2	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
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4	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
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9	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
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1	9601	9653	9706	9758	9810	9862	9914	9967	920019	920071	52
2	920123	920176	920228	920280	920332	920384	920436	920489	0541	0593	52
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5	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
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7	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
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3	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
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5	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
6	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
7	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
8	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	61
9	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
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3	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
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5	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
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7	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
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6	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
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4	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
5	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
6	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
7	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
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9	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
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4	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
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6	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
7	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
8	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
9	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
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3	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
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5	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
6	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
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2	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
3	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
4	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
5	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
6	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
7	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
8	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
9	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
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5	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
6	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
7	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
8	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
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2	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
3	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
4	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
5	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
6	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
7	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
8	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
9	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	977724	977769	977815	977861	977906	977952	977998	978043	978089	978135	46
1	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
2	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
3	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
4	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
5	980003	980049	980094	980140	980185	980231	980276	980322	980367	980412	45
6	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
7	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
8	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
9	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	982271	982316	982362	982407	982452	982497	982543	982588	982633	982678	45
1	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
2	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
3	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
4	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
5	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
6	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
7	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
8	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
9	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	986772	986817	986861	986906	986951	986996	987040	987085	987130	987175	45
1	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
2	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
3	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
4	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
5	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
6	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
7	9895	9939	9983	990028	990072	990117	990161	990206	990250	990294	44
8	990339	990383	990428	0472	0516	0561	0605	0650	0694	0738	44
9	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	991226	991270	991315	991359	991403	991448	991492	991536	991580	991625	44
1	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
2	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
3	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
4	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
5	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
6	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
7	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
8	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
9	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	995635	995679	995723	995767	995811	995854	995898	995942	995986	996030	44
1	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
2	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
3	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
4	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
5	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
6	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
7	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
8	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
9	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6	7	8	9	D.

TABLES

OF

LOGARITHMIC SINES AND TANGENTS

FOR

EVERY DEGREE AND MINUTE

OF THE

QUADRANT;

AND OF

NATURAL SINES AND TANGENTS

FOR

EVERY FIVE MINUTES

OF A

DEGREE.

A TABLE OF THE ANGLES WHICH EVERY POINT AND QUARTER POINT OF THE COMPASS MAKES WITH THE MERIDIAN.

North.		Points.	° ′ ″	Points.	South.	
		0 ¼	2 48 45	0 ¼		
		0 ½	5 37 30	0 ½		
		0 ¾	8 26 15	0 ¾		
N. b. E.	N. b. W.	1	11 15 0	1	S. b. E.	S. b. W.
		1 ¼	14 3 45	1 ¼		
		1 ½	16 52 30	1 ½		
		1 ¾	19 41 15	1 ¾		
N.N.E.	N.N.W.	2	22 30 0	2	S.S.E.	S.S.W.
		2 ¼	25 18 45	2 ¼		
		2 ½	28 7 30	2 ½		
		2 ¾	30 56 15	2 ¾		
N.E. b. N.	N.W. b. N.	3	33 45 0	3	S.E. b. S.	S.W. b. S.
		3 ¼	36 33 45	3 ¼		
		3 ½	39 22 30	3 ½		
		3 ¾	42 11 15	3 ¾		
N.E.	N.W.	4	45 0 0	4	S.E.	S.W.
		4 ¼	47 48 45	4 ¼		
		4 ½	50 37 30	4 ½		
		4 ¾	53 26 15	4 ¾		
N.E. b. E.	N.W. b. W.	5	56 15 0	5	S.E. b. E.	S.W. b. W.
		5 ¼	59 3 45	5 ¼		
		5 ½	61 52 30	5 ½		
		5 ¾	64 41 15	5 ¾		
E.N.E.	W.N.W.	6	67 30 0	6	E.S.E.	W.S.W.
		6 ¼	70 18 45	6 ¼		
		6 ½	73 7 30	6 ½		
		6 ¾	75 56 15	6 ¾		
E. b. N.	W. b. N.	7	78 45 0	7	E. b. S.	W. b. S.
		7 ¼	81 33 45	7 ¼		
		7 ½	84 22 30	7 ½		
		7 ¾	87 11 15	7 ¾		
East.	West.	8	90 0 0	8	East.	West.

A TABLE OF LOGARITHMIC SINES, TANGENTS, AND SECANTS, TO EVERY POINT AND QUARTER POINT OF THE COMPASS.

Points.	Sine.	Cosine.	Tang.	Cotang.	Secant.	Cosec.	Points.
0	0.000000	10.000000	0.000000	Infinite.	10.000000	Infinite.	8
0 ¼	8.690796	9.999477	8.691319	11.308681	10.000523	11.309204	7 ¾
0 ½	8.991302	9.997904	8.993398	11.006602	10.002096	11.008698	7 ½
0 ¾	9.166520	9.995274	9.171247	10.828753	10.004726	10.833480	7 ¼
1	9.290236	9.991574	9.298662	10.701338	10.008426	10.709764	7
1 ¼	9.385571	9.986786	9.398785	10.601215	10.013214	10.614429	6 ¾
1 ½	9.462824	9.980885	9.481939	10.518061	10.019115	10.537176	6 ½
1 ¾	9.527488	9.973841	9.553647	10.446353	10.026159	10.472512	6 ¼
2	9.582840	9.965615	9.617224	10.382776	10.034385	10.417160	6
2 ¼	9.630992	9.956163	9.674829	10.325171	10.043837	10.369008	5 ¾
2 ½	9.673387	9.945430	9.727957	10.272043	10.054570	10.326613	5 ½
2 ¾	9.711050	9.933350	9.777700	10.222300	10.066650	10.288950	5 ¼
3	9.744739	9.919846	9.824893	10.175107	10.080154	10.255261	5
3 ¼	9.775027	9.904828	9.870199	10.129801	10.095172	10.224973	4 ¾
3 ½	9.802359	9.888185	9.914173	10.085827	10.111815	10.197641	4 ½
3 ¾	9.827084	9.869790	9.957295	10.042705	10.130210	10.172916	4 ¼
4	9.849485	9.849485	10.000000	10.000000	10.150515	10.150515	4
	Cosine.	Sine.	Cotang.	Tang.	Cosec.	Secant.	

0 Degree.				1 Degree.				
Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
0.000000	10.000000	0.000000	Infinite.	8.241855	9.999934	8.241921	11.758079	60
6.463726	000000	6.463726	13.536274	249033	999932	249102	750898	59
764756	000000	764756	235244	256094	999929	256165	743835	58
940847	000000	940847	059153	263042	999927	263115	736885	57
7.065786	000000	7.065786	12.934214	269881	999925	269956	730044	56
162696	000000	162696	837304	276614	999922	276691	723309	55
241877	9.999999	241878	758122	283243	999920	283323	716677	54
308824	999999	308825	691175	289773	999918	289856	710144	53
366816	999999	366817	633183	296207	999915	296292	703708	52
417968	999999	417970	582030	302546	999913	302634	697366	51
463725	999998	463727	536273	308794	999910	308884	691116	50
7.505118	9.999998	7.505120	12.494880	8.314954	9.999907	8.315046	11.684954	49
542906	999997	542909	457091	321027	999905	321122	678878	48
577668	999997	577672	422328	327016	999902	327114	672886	47
609853	999996	609857	390143	332924	999899	333025	666975	46
639816	999996	639820	360180	338753	999897	338856	661144	45
667845	999995	667849	332151	344504	999894	344610	655390	44
694173	999995	694179	305821	350181	999891	350289	649711	43
718997	999994	719003	280997	355783	999888	355895	644105	42
742477	999993	742484	257516	361315	999885	361430	638570	41
764754	999993	764761	235239	366777	999882	366895	633105	40
7.785943	9.999992	7.785951	12.214049	8.372171	9.999879	8.372292	11.627708	39
806146	999991	806155	193845	377499	999876	377622	622378	38
825451	999990	825460	174540	382762	999873	382889	617111	37
843934	999989	843944	156056	387962	999870	388092	611908	36
861662	999988	861674	138326	393101	999867	393234	606766	35
878695	999988	878708	121292	398179	999864	398315	601685	34
895085	999987	895099	104901	403199	999861	403338	596662	33
910879	999986	910894	089106	408161	999858	408304	591696	32
926119	999985	926134	073866	413068	999854	413213	586787	31
940842	999983	940858	059142	417919	999851	418068	581932	30
7.955082	9.999982	7.955100	12.044900	8.422717	9.999848	8.422869	11.577131	29
968870	999981	968889	031111	427462	999844	427618	572382	28
982233	999980	982253	017747	432156	999841	432315	567685	27
995198	999979	995219	004781	436800	999838	436962	563038	26
8.007787	999977	8.007809	11.992191	441394	999834	441560	558440	25
020021	999976	020045	979955	445941	999831	446110	553890	24
031919	999975	031945	968055	450440	999827	450613	549387	23
043501	999973	043527	956473	454893	999823	455070	544930	22
054781	999972	054809	945191	459301	999820	459481	540519	21
065776	999971	065806	934194	463665	999816	463849	536151	20
8.076500	9.999969	8.076531	11.923469	8.467985	9.999812	8.468172	11.531828	19
086965	999968	086997	913003	472263	999809	472454	527546	18
097183	999966	097217	902783	476498	999805	476693	523307	17
107167	999964	107202	892797	480693	999801	480892	519108	16
116926	999963	116963	883037	484848	999797	485050	514950	15
126471	999961	126510	873490	488963	999793	489170	510830	14
135810	999959	135851	864149	493040	999790	493250	506750	13
144953	999958	144996	855004	497078	999786	497293	502707	12
153907	999956	153952	846048	501080	999782	501298	498702	11
162681	999954	162727	837273	505045	999778	505267	494733	10
8.171280	9.999952	8.171328	11.828672	8.508974	9.999774	8.509200	11.490800	9
179713	999950	179763	820237	512867	999769	513098	486902	8
187985	999948	188036	811964	516726	999765	516961	483039	7
196102	999946	196156	803844	520551	999761	520790	479210	6
204070	999944	204126	795874	524343	999757	524586	475414	5
211895	999942	211953	788047	528102	999753	528349	471651	4
219581	999940	219641	780359	531828	999748	532080	467920	3
227134	999938	227195	772805	535523	999744	535779	464221	2
234557	999936	234621	765379	539186	999740	539447	460553	1
241855	999934	241921	758079	542819	999735	543084	456916	0
Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
89 Degrees.				88 Degrees.				

′	2 Degrees.				3 Degrees.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	8.542819	9.999735	8.543084	11.456916	8.718800	9.999404	8.719396	11.280604	60
1	546422	999731	546691	453309	721204	999398	721806	278194	59
2	549995	999726	550268	449732	723595	999391	724204	275796	58
3	553539	999722	553817	446183	725972	999384	726588	273412	57
4	557054	999717	557336	442664	728337	999378	728959	271041	56
5	560540	999713	560828	439172	730688	999371	731317	268683	55
6	563999	999708	564291	435709	733027	999364	733663	266337	54
7	567431	999704	567727	432273	735354	999357	735996	264004	53
8	570836	999699	571137	428863	737667	999350	738317	261683	52
9	574214	999694	574520	425480	739969	999343	740626	259374	51
10	577566	999689	577877	422123	742259	999336	742922	257078	50
11	8.580892	9.999685	8.581208	11.418792	8.744536	9.999329	8.745207	11.254793	49
12	584193	999680	584514	415486	746802	999322	747479	252521	48
13	587469	999675	587795	412205	749055	999315	749740	250260	47
14	590721	999670	591051	408949	751297	999308	751989	248011	46
15	593948	999665	594283	405717	753528	999301	754227	245773	45
16	597152	999660	597492	402508	755747	999294	756453	243547	44
17	600332	999655	600677	399323	757955	999286	758668	241332	43
18	603489	999650	603839	396161	760151	999279	760872	239128	42
19	606623	999645	606978	393022	762337	999272	763065	236935	41
20	609734	999640	610094	389906	764511	999265	765246	234754	40
21	8.612823	9.999635	8.613189	11.386811	8.766675	9.999257	8.767417	11.232583	39
22	615891	999629	616262	383738	768828	999250	769578	230422	38
23	618937	999624	619313	380687	770970	999242	771727	228273	37
24	621962	999619	622343	377657	773101	999235	773866	226134	36
25	624965	999614	625352	374648	775223	999227	775995	224005	35
26	627948	999608	628340	371660	777333	999220	778114	221886	34
27	630911	999603	631308	368692	779434	999212	780222	219778	33
28	633854	999597	634256	365744	781524	999205	782320	217680	32
29	636776	999592	637184	362816	783605	999197	784408	215592	31
30	639680	999586	640093	359907	785675	999189	786486	213514	30
31	8.642563	9.999581	8.642982	11.357018	8.787736	9.999181	8.788554	11.211446	29
32	645428	999575	645853	354147	789787	999174	790613	209387	28
33	648274	999570	648704	351296	791828	999166	792662	207338	27
34	651102	999564	651537	348463	793859	999158	794701	205299	26
35	653911	999558	654352	345648	795881	999150	796731	203269	25
36	656702	999553	657149	342851	797894	999142	798752	201248	24
37	659475	999547	659928	340072	799897	999134	800763	199237	23
38	662230	999541	662689	337311	801892	999126	802765	197235	22
39	664968	999535	665433	334567	803876	999118	804758	195242	21
40	667689	999529	668160	331840	805852	999110	806742	193258	20
41	8.670393	9.999524	8.670870	11.329130	8.807819	9.999102	8.808717	11.191283	19
42	673080	999518	673563	326437	809777	999094	810683	189317	18
43	675751	999512	676239	323761	811726	999086	812641	187359	17
44	678405	999506	678900	321100	813667	999077	814589	185411	16
45	681043	999500	681544	318456	815599	999069	816529	183471	15
46	683665	999493	684172	315828	817522	999061	818461	181539	14
47	686272	999487	686784	313216	819436	999053	820384	179616	13
48	688863	999481	689381	310619	821343	999044	822298	177702	12
49	691438	999475	691963	308037	823240	999036	824205	175795	11
50	693998	999469	694529	305471	825130	999027	826103	173897	10
51	8.696543	9.999463	8.697081	11.302919	8.827011	9.999019	8.827992	11.172008	9
52	699073	999456	699617	300383	828884	999010	829874	170126	8
53	701589	999450	702139	297861	830749	999002	831748	168252	7
54	704090	999443	704646	295354	832607	998993	833613	166387	6
55	706577	999437	707140	292860	834456	998984	835471	164529	5
56	709049	999431	709618	290382	836297	998976	837321	162679	4
57	711507	999424	712083	287917	838130	998967	839163	160837	3
58	713952	999418	714534	285465	839956	998958	840998	159002	2
59	716383	999411	716972	283028	841774	998950	842825	157175	1
60	718800	999404	719396	280604	843585	998941	844644	155356	0
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	87 Degrees.				86 Degrees.				

4 Degrees.				5 Degrees.				
Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
8.843585	9.998941	8.844644	11.155356	8.940296	9.998344	8.941952	11.058048	60
845387	998932	846455	153545	941738	998333	943404	056596	59
847183	998923	848260	151740	943174	998322	944852	055148	58
848971	998914	850057	149943	944606	998311	946295	053705	57
850751	998905	851846	148154	946034	998300	947734	052266	56
852525	998896	853628	146372	947456	998289	949168	050832	55
854291	998887	855403	144597	948874	998277	950597	049403	54
856049	998878	857171	142829	950287	998266	952021	047979	53
857801	998869	858932	141068	951696	998255	953441	046559	52
859546	998860	860686	139314	953100	998243	954856	045144	51
861283	998851	862433	137567	954499	998232	956267	043733	50
8.863014	9.998841	8.864173	11.135827	8.955894	9.998220	8.957674	11.042326	49
864738	998832	865906	134094	957284	998209	959075	040925	48
866455	998823	867632	132368	958670	998197	960473	039527	47
868165	998813	869351	130649	960052	998186	961866	038134	46
869868	998804	871064	128936	961429	998174	963255	036745	45
871565	998795	872770	127230	962801	998163	964639	035361	44
873255	998785	874469	125531	964170	998151	966019	033981	43
874938	998776	876162	123838	965534	998139	967394	032606	42
876615	998766	877849	122151	966893	998128	968766	031234	41
878285	998757	879529	120471	968249	998116	970133	029867	40
8.879949	9.998747	8.881202	11.118798	8.969600	9.998104	8.971496	11.028504	39
881607	998738	882869	117131	970947	998092	972855	027145	38
883258	998728	884530	115470	972289	998080	974209	025791	37
884903	998718	886185	113815	973628	998068	975560	024440	36
886542	998708	887833	112167	974962	998056	976906	023094	35
888174	998699	889476	110524	976293	998044	978248	021752	34
889801	998689	891112	108888	977619	998032	979586	020414	33
891421	998679	892742	107258	978941	998020	980921	019079	32
893035	998669	894366	105634	980259	998008	982251	017749	31
894643	998659	895984	104016	981573	997996	983577	016423	30
8.896246	9.998649	8.897596	11.102404	8.982883	9.997984	8.984899	11.015101	29
897842	998639	899203	100797	984189	997972	986217	013783	28
899432	998629	900803	099197	985491	997959	987532	012468	27
901017	998619	902398	097602	986789	997947	988842	011158	26
902596	998609	903987	096013	988083	997935	990149	009851	25
904169	998599	905570	094430	989374	997922	991451	008549	24
905736	998589	907147	092853	990660	997910	992750	007250	23
907297	998578	908719	091281	991943	997897	994045	005955	22
908853	998568	910285	089715	993222	997885	995337	004663	21
910404	998558	911846	088154	994497	997872	996624	003376	20
8.911949	9.998548	8.913401	11.086599	8.995768	9.997860	8.997908	11.002092	19
913488	998537	914951	085049	997036	997847	999188	000812	18
915022	998527	916495	083505	998299	997835	9.000465	10.999535	17
916550	998516	918034	081966	999560	997822	001738	998262	16
918073	998506	919568	080432	9.000816	997809	003007	996993	15
919591	998495	921096	078904	002069	997797	004272	995728	14
921103	998485	922619	077381	003318	997784	005534	994466	13
922610	998474	924136	075864	004563	997771	006792	993208	12
924112	998464	925649	074351	005805	997758	008047	991953	11
925609	998453	927156	072844	007044	997745	009298	990702	10
8.927100	9.998442	8.928658	11.071342	9.008278	9.997732	9.010546	10.989454	9
928587	998431	930155	069845	009510	997719	011790	988210	8
930068	998421	931647	068353	010737	997706	013031	986969	7
931544	998410	933134	066866	011962	997693	014268	985732	6
933015	998399	934616	065384	013182	997680	015502	984498	5
934481	998388	936093	063907	014400	997667	016732	983268	4
935942	998377	937565	062435	015613	997654	017959	982041	3
937398	998366	939032	060968	016824	997641	019183	980817	2
938850	998355	940494	059506	018031	997628	020403	979597	1
940296	998344	941952	058048	019235	997614	021620	978380	0
Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
85 Degrees.				84 Degrees.				

	6 Degrees.				7 Degrees.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
0	9.019235	9.997614	9.021620	10.978380	9.085894	9.996751	9.089144	10.910856	60
1	020435	997601	022834	977166	086922	996735	090187	909813	59
2	021632	997588	024044	975956	087947	996720	091228	908772	58
3	022825	997574	025251	974749	088970	996704	092266	907734	57
4	024016	997561	026455	973545	089990	996688	093302	906698	56
5	025203	997547	027655	972345	091008	996673	094336	905664	55
6	026386	997534	028852	971148	092024	996657	095367	904633	54
7	027567	997520	030046	969954	093037	996641	096395	903605	53
8	028744	997507	031237	968763	094047	996625	097422	902578	52
9	029918	997493	032425	967575	095056	996610	098446	901554	51
10	031089	997480	033609	966391	096062	996594	099468	900532	50
11	9.032257	9.997466	9.034791	10.965209	9.097065	9.996578	9.100487	10.899513	49
12	033421	997452	035969	964031	098066	996562	101504	898496	48
13	034582	997439	037144	962856	099065	996546	102519	897481	47
14	035741	997425	038316	961684	100062	996530	103532	896468	46
15	036896	997411	039485	960515	101056	996514	104542	895458	45
16	038048	997397	040651	959349	102048	996498	105550	894450	44
17	039197	997383	041813	958187	103037	996482	106556	893444	43
18	040342	997369	042973	957027	104025	996465	107559	892441	42
19	041485	997355	044130	955870	105010	996449	108560	891440	41
20	042625	997341	045284	954716	105992	996433	109559	890441	40
21	9.043762	9.997327	9.046434	10.953566	9.106973	9.996417	9.110556	10.889444	39
22	044895	997313	047582	952418	107951	996400	111551	888449	38
23	046026	997299	048727	951273	108927	996384	112543	887457	37
24	047154	997285	049869	950131	109901	996368	113533	886467	36
25	048279	997271	051008	948992	110873	996351	114521	885479	35
26	049400	997257	052144	947856	111842	996335	115507	884493	34
27	050519	997242	053277	946723	112809	996318	116491	883509	33
28	051635	997228	054407	945593	113774	996302	117472	882528	32
29	052749	997214	055535	944465	114737	996285	118452	881548	31
30	053859	997199	056659	943341	115698	996269	119429	880571	30
31	9.054966	9.997185	9.057781	10.942219	9.116656	9.996252	9.120404	10.879596	29
32	056071	997170	058900	941100	117613	996235	121377	878623	28
33	057172	997156	060016	939984	118567	996219	122348	877652	27
34	058271	997141	061130	938870	119519	996202	123317	876683	26
35	059367	997127	062240	937760	120469	996185	124284	875716	25
36	060460	997112	063348	936652	121417	996168	125249	874751	24
37	061551	997098	064453	935547	122362	996151	126211	873789	23
38	062639	997083	065556	934444	123306	996134	127172	872828	22
39	063724	997068	066655	933345	124248	996117	128130	871870	21
40	064806	997053	067752	932248	125187	996100	129087	870913	20
41	9.065885	9.997039	9.068846	10.931154	9.126125	9.996083	9.130041	10.869959	19
42	066962	997024	069938	930062	127060	996066	130994	869006	18
43	068036	997009	071027	928973	127993	996049	131944	868056	17
44	069107	996994	072113	927887	128925	996032	132893	867107	16
45	070176	996979	073197	926803	129854	996015	133839	866161	15
46	071242	996964	074278	925722	130781	995998	134784	865216	14
47	072306	996949	075356	924644	131706	995980	135726	864274	13
48	073366	996934	076432	923568	132630	995963	136667	863333	12
49	074424	996919	077505	922495	133551	995946	137605	862395	11
50	075480	996904	078576	921424	134470	995928	138542	861458	10
51	9.076533	9.996889	9.079644	10.920356	9.135387	9.995911	9.139476	10.860524	9
52	077583	996874	080710	919290	136303	995894	140409	859591	8
53	078631	996858	081773	918227	137216	995876	141340	858660	7
54	079676	996843	082833	917167	138128	995859	142269	857731	6
55	080719	996828	083891	916109	139037	995841	143196	856804	5
56	081759	996812	084947	915053	139944	995823	144121	855879	4
57	082797	996797	086000	914000	140850	995806	145044	854956	3
58	083832	996782	087050	912950	141754	995788	145966	854034	2
59	084864	996766	088098	911902	142655	995771	146885	853115	1
60	085894	996751	089144	910856	143555	995753	147803	852197	0
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	83 Degrees.				82 Degrees.				

8 Degrees.				9 Degrees.				
Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
9.143555	9.995753	9.147803	10.852197	9.194332	9.994620	9.199713	10.800287	60
144453	995735	148718	851282	195129	994600	200529	799471	59
145349	995717	149632	850368	195925	994580	201345	798655	58
146243	995699	150544	849456	196719	994560	202159	797841	57
147136	995681	151454	848546	197511	994540	202971	797029	56
148026	995664	152363	847637	198302	994519	203782	796218	55
148915	995646	153269	846731	199091	994499	204592	795408	54
149802	995628	154174	845826	199879	994479	205400	794600	53
150686	995610	155077	844923	200666	994459	206207	793793	52
151569	995591	155978	844022	201451	994438	207013	792987	51
152451	995573	156877	843123	202234	994418	207817	792183	50
9.153330	9.995555	9.157775	10.842225	9.203017	9.994398	9.208619	10.791381	49
154208	995537	158671	841329	203797	994377	209420	790580	48
155083	995519	159565	840435	204577	994357	210220	789780	47
155957	995501	160457	839543	205354	994336	211018	788982	46
156830	995482	161347	838653	206131	994316	211815	788185	45
157700	995464	162236	837764	206906	994295	212611	787389	44
158569	995446	163123	836877	207679	994274	213405	786595	43
159435	995427	164008	835992	208452	994254	214198	785802	42
160301	995409	164892	835108	209222	994233	214989	785011	41
161164	995390	165774	834226	209992	994212	215780	784220	40
9.162025	9.995372	9.166654	10.833346	9.210760	9.994191	9.216568	10.783432	39
162885	995353	167532	832468	211526	994171	217356	782644	38
163743	995334	168409	831591	212291	994150	218142	781858	37
164600	995316	169284	830716	213055	994129	218926	781074	36
165454	995297	170157	829843	213818	994108	219710	780290	35
166307	995278	171029	828971	214579	994087	220492	779508	34
167159	995260	171899	828101	215338	994066	221272	778728	33
168008	995241	172767	827233	216097	994045	222052	777948	32
168856	995222	173634	826366	216854	994024	222830	777170	31
169702	995203	174499	825501	217609	994003	223607	776393	30
9.170547	9.995184	9.175362	10.824638	9.218363	9.993982	9.224382	10.775618	29
171389	995165	176224	823776	219116	993960	225156	774844	28
172230	995146	177084	822916	219868	993939	225929	774071	27
173070	995127	177942	822058	220618	993918	226700	773300	26
173908	995108	178799	821201	221367	993897	227471	772529	25
174744	995089	179655	820345	222115	993875	228239	771761	24
175578	995070	180508	819492	222861	993854	229007	770993	23
176411	995051	181360	818640	223606	993832	229773	770227	22
177242	995032	182211	817789	224349	993811	230539	769461	21
178072	995013	183059	816941	225092	993789	231302	768698	20
9.178900	9.994993	9.183907	10.816093	9.225833	9.993768	9.232065	10.767935	19
179726	994974	184752	815248	226573	993746	232826	767174	18
180551	994955	185597	814403	227311	993725	233586	766414	17
181374	994935	186439	813561	228048	993703	234345	765655	16
182196	994916	187280	812720	228784	993681	235103	764897	15
183016	994896	188120	811880	229518	993660	235859	764141	14
183834	994877	188958	811042	230252	993638	236614	763386	13
184651	994857	189794	810206	230984	993616	237368	762632	12
185466	994838	190629	809371	231714	993594	238120	761880	11
186280	994818	191462	808538	232444	993572	238872	761128	10
9.187092	9.994798	9.192294	10.807706	9.233172	9.993550	9.239622	10.760378	9
187903	994779	193124	806876	233899	993528	240371	759629	8
188712	994759	193953	806047	234625	993506	241118	758882	7
189519	994739	194780	805220	235349	993484	241865	758135	6
190325	994719	195606	804394	236073	993462	242610	757390	5
191130	994700	196430	803570	236795	993440	243354	756646	4
191933	994680	197253	802747	237515	993418	244097	755903	3
192734	994660	198074	801926	238235	993396	244839	755161	2
193534	994640	198894	801106	238953	993374	245579	754421	1
194332	994620	199713	800287	239670	993351	246319	753681	0
Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
81 Degrees.				80 Degrees.				

	10 Degrees.				11 Degrees.				
'	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	'
0	9.239670	9.993351	9.246319	10.753681	9.280599	9.991947	9.288652	10.711348	60
1	240386	993329	247057	752943	281248	991922	289326	710674	59
2	241101	993307	247794	752206	281897	991897	289999	710001	58
3	241814	993284	248530	751470	282544	991873	290671	709329	57
4	242526	993262	249264	750736	283190	991848	291342	708658	56
5	243237	993240	249998	750002	283836	991823	292013	707987	55
6	243947	993217	250730	749270	284480	991799	292682	707318	54
7	244656	993195	251461	748539	285124	991774	293350	706650	53
8	245363	993172	252191	747809	285766	991749	294017	705983	52
9	246069	993149	252920	747080	286408	991724	294684	705316	51
10	246775	993127	253648	746352	287048	991699	295349	704651	50
11	9.247478	9.993104	9.254374	10.745626	9.287687	9.991674	9.296013	10.703987	49
12	248181	993081	255100	744900	288326	991649	296677	703323	48
13	248883	993059	255824	744176	288964	991624	297339	702661	47
14	249583	993036	256547	743453	289600	991599	298001	701999	46
15	250282	993013	257269	742731	290236	991574	298662	701338	45
16	250980	992990	257990	742010	290870	991549	299322	700678	44
17	251677	992967	258710	741290	291504	991524	299980	700020	43
18	252373	992944	259429	740571	292137	991498	300638	699362	42
19	253067	992921	260146	739854	292768	991473	301295	698705	41
20	253761	992898	260863	739137	293399	991448	301951	698049	40
21	9.254453	9.992875	9.261578	10.738422	9.294029	9.991422	9.302607	10.697393	39
22	255144	992852	262292	737708	294658	991397	303261	696739	38
23	255834	992829	263005	736995	295286	991372	303914	696086	37
24	256523	992806	263717	736283	295913	991346	304567	695433	36
25	257211	992783	264428	735572	296539	991321	305218	694782	35
26	257898	992759	265138	734862	297164	991295	305869	694131	34
27	258583	992736	265847	734153	297788	991270	306519	693481	33
28	259268	992713	266555	733445	298412	991244	307168	692832	32
29	259951	992690	267261	732739	299034	991218	307815	692185	31
30	260633	992666	267967	732033	299655	991193	308463	691537	30
31	9.261314	9.992643	9.268671	10.731329	9.300276	9.991167	9.309109	10.690891	29
32	261994	992619	269375	730625	300895	991141	309754	690246	28
33	262673	992596	270077	729923	301514	991115	310398	689602	27
34	263351	992572	270779	729221	302132	991090	311042	688958	26
35	264027	992549	271479	728521	302748	991064	311685	688315	25
36	264703	992525	272178	727822	303364	991038	312327	687673	24
37	265377	992501	272876	727124	303979	991012	312967	687033	23
38	266051	992478	273573	726427	304593	990986	313608	686392	22
39	266723	992454	274269	725731	305207	990960	314247	685753	21
40	267395	992430	274964	725036	305819	990934	314885	685115	20
41	9.268065	9.992406	9.275658	10.724342	9.306430	9.990908	9.315523	10.684477	19
42	268734	992382	276351	723649	307041	990882	316159	683841	18
43	269402	992359	277043	722957	307650	990855	316795	683205	17
44	270069	992335	277734	722266	308259	990829	317430	682570	16
45	270735	992311	278424	721576	308867	990803	318064	681936	15
46	271400	992287	279113	720887	309474	990777	318697	681303	14
47	272064	992263	279801	720199	310080	990750	319329	680671	13
48	272726	992239	280488	719512	310685	990724	319961	680039	12
49	273388	992214	281174	718826	311289	990697	320592	679408	11
50	274049	992190	281858	718142	311893	990671	321222	678778	10
51	9.274708	9.992166	9.282542	10.717458	9.312495	9.990644	9.321851	10.678149	9
52	275367	992142	283225	716775	313097	990618	322479	677521	8
53	276024	992118	283907	716093	313698	990591	323106	676894	7
54	276681	992093	284588	715412	314297	990565	323733	676267	6
55	277337	992069	285268	714732	314897	990538	324358	675642	5
56	277991	992044	285947	714053	315495	990511	324983	675017	4
57	278645	992020	286624	713376	316092	990485	325607	674393	3
58	279297	991996	287301	712699	316689	990458	326231	673769	2
59	279948	991971	287977	712023	317284	990431	326853	673147	1
60	280599	991947	288652	711348	317879	990404	327475	672525	0
'	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	'
	79 Degrees.				78 Degrees.				

′	12 Degrees.				13 Degrees.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
0	9.317879	9.990404	9.327474	10.672526	9.352088	9.988724	9.363364	10.636636	60
1	318473	990378	328095	671905	352635	988695	363940	636060	59
2	319066	990351	328715	671285	353181	988666	364515	635485	58
3	319658	990324	329334	670666	353726	988636	365090	634910	57
4	320249	990297	329953	670047	354271	988607	365664	634336	56
5	320840	990270	330570	669430	354815	988578	366237	633763	55
6	321430	990243	331187	668813	355358	988548	366810	633190	54
7	322019	990215	331803	668197	355901	988519	367382	632618	53
8	322607	990188	332418	667582	356443	988489	367953	632047	52
9	323194	990161	333033	666967	356984	988460	368524	631476	51
10	323780	990134	333646	666354	357524	988430	369094	630906	50
11	9.324366	9.990107	9.334259	10.665741	9.358064	9.988401	9.369663	10.630337	49
12	324950	990079	334871	665129	358603	988371	370232	629768	48
13	325534	990052	335482	664518	359141	988342	370799	629201	47
14	326117	990025	336093	663907	359678	988312	371367	628633	46
15	326700	989997	336702	663298	360215	988282	371933	628067	45
16	327281	989970	337311	662689	360752	988252	372499	627501	44
17	327862	989942	337919	662081	361287	988223	373064	626936	43
18	328442	989915	338527	661473	361822	988193	373629	626371	42
19	329021	989887	339133	660867	362356	988163	374193	625807	41
20	329599	989860	339739	660261	362889	988133	374756	625244	40
21	9.330176	9.989832	9.340344	10.659656	9.363422	9.988103	9.375319	10.624681	39
22	330753	989804	340948	659052	363954	988073	375881	624119	38
23	331329	989777	341552	658448	364485	988043	376442	623558	37
24	331903	989749	342155	657845	365016	988013	377003	622997	36
25	332478	989721	342757	657243	365546	987983	377563	622437	35
26	333051	989693	343358	656642	366075	987953	378122	621878	34
27	333624	989665	343958	656042	366604	987922	378681	621319	33
28	334195	989637	344558	655442	367131	987892	379239	620761	32
29	334766	989609	345157	654843	367659	987862	379797	620203	31
30	335337	989582	345755	654245	368185	987832	380354	619646	30
31	9.335906	9.989553	9.346353	10.653647	9.368711	9.987801	9.380910	10.619090	29
32	336475	989525	346949	653051	369236	987771	381466	618534	28
33	337043	989497	347545	652455	369761	987740	382020	617980	27
34	337610	989469	348141	651859	370285	987710	382575	617425	26
35	338176	989441	348735	651265	370808	987679	383129	616871	25
36	338742	989413	349329	650671	371330	987649	383682	616318	24
37	339307	989385	349922	650078	371852	987618	384234	615766	23
38	339871	989356	350514	649486	372373	987588	384786	615214	22
39	340434	989328	351106	648894	372894	987557	385337	614663	21
40	340996	989300	351697	648303	373414	987526	385888	614112	20
41	9.341558	9.989271	9.352287	10.647713	9.373933	9.987496	9.386438	10.613562	19
42	342119	989243	352876	647124	374452	987465	386987	613013	18
43	342679	989214	353465	646535	374970	987434	387536	612464	17
44	343239	989186	354053	645947	375487	987403	388084	611916	16
45	343797	989157	354640	645360	376003	987372	388631	611369	15
46	344355	989128	355227	644773	376519	987341	389178	610822	14
47	344912	989100	355813	644187	377035	987310	389724	610276	13
48	345469	989071	356398	643602	377549	987279	390270	609730	12
49	346024	989042	356982	643018	378063	987248	390815	609185	11
50	346579	989014	357566	642434	378577	987217	391360	608640	10
51	9.347134	9.988985	9.358149	10.641851	9.379089	9.987186	9.391903	10.608097	9
52	347687	988956	358731	641269	379601	987155	392447	607553	8
53	348240	988927	359313	640687	380113	987124	392989	607011	7
54	348792	988898	359893	640107	380624	987092	393531	606469	6
55	349343	988869	360474	639526	381134	987061	394073	605927	5
56	349893	988840	361053	638947	381643	987030	394614	605386	4
57	350443	988811	361632	638368	382152	986998	395154	604846	3
58	350992	988782	362210	637790	382661	986967	395694	604306	2
59	351540	988753	362787	637213	383168	986936	396233	603767	1
60	352088	988724	363364	636636	383675	986904	396771	603229	0
	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	77 Degrees.				76 Degrees.				

	14 Degrees.				15 Degrees.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
0	9.383675	9.986904	9.396771	10.603229	9.412996	9.984944	9.428052	10.571948	60
1	384182	986873	397309	602691	413467	984910	428557	571443	59
2	384687	986841	397846	602154	413938	984876	429062	570938	58
3	385192	986809	398383	601617	414408	984842	429566	570434	57
4	385697	986778	398919	601081	414878	984808	430070	569930	56
5	386201	986746	399455	600545	415347	984774	430573	569427	55
6	386704	986714	399990	600010	415815	984740	431075	568925	54
7	387207	986683	400524	599476	416283	984706	431577	568423	53
8	387709	986651	401058	598942	416751	984672	432079	567921	52
9	388210	986619	401591	598409	417217	984638	432580	567420	51
10	388711	986587	402124	597876	417684	984603	433080	566920	50
11	9.389211	9.986555	9.402656	10.597344	9.418150	9.984569	9.433580	10.566420	49
12	389711	986523	403187	596813	418615	984535	434080	565920	48
13	390210	986491	403718	596282	419079	984500	434579	565421	47
14	390708	986459	404249	595751	419544	984466	435078	564922	46
15	391206	986427	404778	595222	420007	984432	435576	564424	45
16	391703	986395	405308	594692	420470	984397	436073	563927	44
17	392199	986363	405836	594164	420933	984363	436570	563430	43
18	392695	986331	406364	593636	421395	984328	437067	562933	42
19	393191	986299	406892	593108	421857	984294	437563	562437	41
20	393685	986266	407419	592581	422318	984259	438059	561941	40
21	9.394179	9.986234	9.407945	10.592055	9.422778	9.984224	9.438554	10.561446	39
22	394673	986202	408471	591529	423238	984190	439048	560952	38
23	395166	986169	408997	591003	423697	984155	439543	560457	37
24	395658	986137	409521	590479	424156	984120	440036	559964	36
25	396150	986104	410045	589955	424615	984085	440529	559471	35
26	396641	986072	410569	589431	425073	984050	441022	558978	34
27	397132	986039	411092	588908	425530	984015	441514	558486	33
28	397621	986007	411615	588385	425987	983981	442006	557994	32
29	398111	985974	412137	587863	426443	983946	442497	557503	31
30	398600	985942	412658	587342	426899	983911	442988	557012	30
31	9.399088	9.985909	9.413179	10.586821	9.427354	9.983875	9.443479	10.556521	29
32	399575	985876	413699	586301	427809	983840	443968	556032	28
33	400062	985843	414219	585781	428263	983805	444458	555542	27
34	400549	985811	414738	585262	428717	983770	444947	555053	26
35	401035	985778	415257	584743	429170	983735	445435	554565	25
36	401520	985745	415775	584225	429623	983700	445923	554077	24
37	402005	985712	416293	583707	430075	983664	446411	553589	23
38	402489	985679	416810	583190	430527	983629	446898	553102	22
39	402972	985646	417326	582674	430978	983594	447384	552616	21
40	403455	985613	417842	582158	431429	983558	447870	552130	20
41	9.403938	9.985580	9.418358	10.581642	9.431879	9.983523	9.448356	10.551644	19
42	404420	985547	418873	581127	432329	983487	448841	551159	18
43	404901	985514	419387	580613	432778	983452	449326	550674	17
44	405382	985480	419901	580099	433226	983416	449810	550190	16
45	405862	985447	420415	579585	433675	983381	450294	549706	15
46	406341	985414	420927	579073	434122	983345	450777	549223	14
47	406820	985381	421440	578560	434569	983309	451260	548740	13
48	407299	985347	421952	578048	435016	983273	451743	548257	12
49	407777	985314	422463	577537	435462	983238	452225	547775	11
50	408254	985280	422974	577026	435908	983202	452706	547294	10
51	9.408731	9.985247	9.423484	10.576516	9.436353	9.983166	9.453187	10.546813	9
52	409207	985213	423993	576007	436798	983130	453668	546332	8
53	409682	985180	424503	575497	437242	983094	454148	545852	7
54	410157	985146	425011	574989	437686	983058	454628	545372	6
55	410632	985113	425519	574481	438129	983022	455107	544893	5
56	411106	985079	426027	573973	438572	982986	455586	544414	4
57	411579	985045	426534	573466	439014	982950	456064	543936	3
58	412052	985011	427041	572959	439456	982914	456542	543458	2
59	412524	984978	427547	572453	439897	982878	457019	542981	1
60	412996	984944	428052	571948	440338	982842	457496	542504	0
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	75 Degrees.				74 Degrees.				

	16 Degrees.				17 Degrees.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
0	9.440338	9.982842	9.457496	10.542504	9.465935	9.980596	9.485339	10.514661	60
1	440778	982805	457973	542027	466348	980558	485791	514209	59
2	441218	982769	458449	541551	466761	980519	486242	513758	58
3	441658	982733	458925	541075	467173	980480	486693	513307	57
4	442096	982696	459400	540600	467585	980442	487143	512857	56
5	442535	982660	459875	540125	467996	980403	487593	512407	55
6	442973	982624	460349	539651	468407	980364	488043	511957	54
7	443410	982587	460823	539177	468817	980325	488492	511508	53
8	443847	982551	461297	538703	469227	980286	488941	511059	52
9	444284	982514	461770	538230	469637	980247	489390	510610	51
10	444720	982477	462242	537758	470046	980208	489838	510162	50
11	9.445155	9.982441	9.462714	10.537286	9.470455	9.980169	9.490286	10.509714	49
12	445590	982404	463186	536814	470863	980130	490733	509267	48
13	446025	982367	463658	536342	471271	980091	491180	508820	47
14	446459	982331	464128	535872	471679	980052	491627	508373	46
15	446893	982294	464599	535401	472086	980012	492073	507927	45
16	447326	982257	465069	534931	472492	979973	492519	507481	44
17	447759	982220	465539	534461	472898	979934	492965	507035	43
18	448191	982183	466008	533992	473304	979895	493410	506590	42
19	448623	982146	466476	533524	473710	979855	493854	506146	41
20	449054	982109	466945	533055	474115	979816	494299	505701	40
21	9.449485	9.982072	9.467413	10.532587	9.474519	9.979776	9.494743	10.505257	39
22	449915	982035	467880	532120	474923	979737	495186	504814	38
23	450345	981998	468347	531653	475327	979697	495630	504370	37
24	450775	981961	468814	531186	475730	979658	496073	503927	36
25	451204	981924	469280	530720	476133	979618	496515	503485	35
26	451632	981886	469746	530254	476536	979579	496957	503043	34
27	452060	981849	470211	529789	476938	979539	497399	502601	33
28	452488	981812	470676	529324	477340	979499	497841	502159	32
29	452915	981774	471141	528859	477741	979459	498282	501718	31
30	453342	981737	471605	528395	478142	979420	498722	501278	30
31	9.453768	9.981700	9.472068	10.527932	9.478542	9.979380	9.499163	10.500837	29
32	454194	981662	472532	527468	478942	979340	499603	500397	28
33	454619	981625	472995	527005	479342	979300	500042	499958	27
34	455044	981587	473457	526543	479741	979260	500481	499519	26
35	455469	981549	473919	526081	480140	979220	500920	499080	25
36	455893	981512	474381	525619	480539	979180	501359	498641	24
37	456316	981474	474842	525158	480937	979140	501797	498203	23
38	456739	981436	475303	524697	481334	979100	502235	497765	22
39	457162	981399	475763	524237	481731	979059	502672	497328	21
40	457584	981361	476223	523777	482128	979019	503109	496891	20
41	9.458006	9.981323	9.476683	10.523317	9.482525	9.978979	9.503546	10.496454	19
42	458427	981285	477142	522858	482921	978939	503982	496018	18
43	458848	981247	477601	522399	483316	978898	504418	495582	17
44	459268	981209	478059	521941	483712	978858	504854	495146	16
45	459688	981171	478517	521483	484107	978817	505289	494711	15
46	460108	981133	478975	521025	484501	978777	505724	494276	14
47	460527	981095	479432	520568	484895	978737	506159	493841	13
48	460946	981057	479889	520111	485289	978696	506593	493407	12
49	461364	981019	480345	519655	485682	978655	507027	492973	11
50	461782	980981	480801	519199	486075	978615	507460	492540	10
51	9.462199	9.980942	9.481257	10.518743	9.486467	9.978574	9.507893	10.492107	9
52	462616	980904	481712	518288	486860	978533	508326	491674	8
53	463032	980866	482167	517833	487251	978493	508759	491241	7
54	463448	980827	482621	517379	487643	978452	509191	490809	6
55	463864	980789	483075	516925	488034	978411	509622	490378	5
56	464279	980750	483529	516471	488424	978370	510054	489946	4
57	464694	980712	483982	516018	488814	978329	510485	489515	3
58	465108	980673	484435	515565	489204	978288	510916	489084	2
59	465522	980635	484887	515113	489593	978247	511346	488654	1
60	465935	980596	485339	514661	489982	978206	511776	488224	0
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	73 Degrees.				72 Degrees.				

	18 Degrees.				19 Degrees.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
0	9.489982	9.978206	9.511776	10.488224	9.512642	9.975670	9.536972	10.463028	60
1	490371	978165	512206	487794	513009	975627	537382	462618	59
2	490759	978124	512635	487365	513375	975583	537792	462208	58
3	491147	978083	513064	486936	513741	975539	538202	461798	57
4	491535	978042	513493	486507	514107	975496	538611	461389	56
5	491922	978001	513921	486079	514472	975452	539020	460980	55
6	492308	977959	514349	485651	514837	975408	539429	460571	54
7	492695	977918	514777	485223	515202	975365	539837	460163	53
8	493081	977877	515204	484796	515566	975321	540245	459755	52
9	493466	977835	515631	484369	515930	975277	540653	459347	51
10	493851	977794	516057	483943	516294	975233	541061	458939	50
11	9.494236	9.977752	9.516484	10.483516	9.516657	9.975189	9.541468	10.458532	49
12	494621	977711	516910	483090	517020	975145	541875	458125	48
13	495005	977669	517335	482665	517382	975101	542281	457719	47
14	495388	977628	517761	482239	517745	975057	542688	457312	46
15	495772	977586	518185	481815	518107	975013	543094	456906	45
16	496154	977544	518610	481390	518468	974969	543499	456501	44
17	496537	977503	519034	480966	518829	974925	543905	456095	43
18	496919	977461	519458	480542	519190	974880	544310	455690	42
19	497301	977419	519882	480118	519551	974836	544715	455285	41
20	497682	977377	520305	479695	519911	974792	545119	454881	40
21	9.498064	9.977335	9.520728	10.479272	9.520271	9.974748	9.545524	10.454476	39
22	498444	977293	521151	478849	520631	974703	545928	454072	38
23	498825	977251	521573	478427	520990	974659	546331	453669	37
24	499204	977209	521995	478005	521349	974614	546735	453265	36
25	499584	977167	522417	477583	521707	974570	547138	452862	35
26	499963	977125	522838	477162	522066	974525	547540	452460	34
27	500342	977083	523259	476741	522424	974481	547943	452057	33
28	500721	977041	523680	476320	522781	974436	548345	451655	32
29	501099	976999	524100	475900	523138	974391	548747	451253	31
30	501476	976957	524520	475480	523495	974347	549149	450851	30
31	9.501854	9.976914	9.524939	10.475061	9.523852	9.974302	9.549550	10.450450	29
32	502231	976872	525359	474641	524208	974257	549951	450049	28
33	502607	976830	525778	474222	524564	974212	550352	449648	27
34	502984	976787	526197	473803	524920	974167	550752	449248	26
35	503360	976745	526615	473385	525275	974122	551152	448848	25
36	503735	976702	527033	472967	525630	974077	551552	448448	24
37	504110	976660	527451	472549	525984	974032	551952	448048	23
38	504485	976617	527868	472132	526339	973987	552351	447649	22
39	504860	976574	528285	471715	526693	973942	552750	447250	21
40	505234	976532	528702	471298	527046	973897	553149	446851	20
41	9.505608	9.976489	9.529119	10.470881	9.527400	9.973852	9.553548	10.446452	19
42	505981	976446	529535	470465	527753	973807	553946	446054	18
43	506354	976404	529950	470050	528105	973761	554344	445656	17
44	506727	976361	530366	469634	528458	973716	554741	445259	16
45	507099	976318	530781	469219	528810	973671	555139	444861	15
46	507471	976275	531196	468804	529161	973625	555536	444464	14
47	507843	976232	531611	468389	529513	973580	555933	444067	13
48	508214	976189	532025	467975	529864	973535	556329	443671	12
49	508585	976146	532439	467561	530215	973489	556725	443275	11
50	508956	976103	532853	467147	530565	973444	557121	442879	10
51	9.509326	9.976060	9.533266	10.466734	9.530915	9.973398	9.557517	10.442483	9
52	509696	976017	533679	466321	531265	973352	557913	442087	8
53	510065	975974	534092	465908	531614	973307	558308	441692	7
54	510434	975930	534504	465496	531963	973261	558702	441298	6
55	510803	975887	534916	465084	532312	973215	559097	440903	5
56	511172	975844	535328	464672	532661	973169	559491	440509	4
57	511540	975800	535739	464261	533009	973124	559885	440115	3
58	511907	975757	536150	463850	533357	973078	560279	439721	2
59	512275	975714	536561	463439	533704	973032	560673	439327	1
60	512642	975670	536972	463028	534052	972986	561066	438934	0
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	71 Degrees.				70 Degrees.				

20 Degrees.				21 Degrees.				
Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
9.534052	9.972986	9.561066	10.438934	9.554329	9.970152	9.584177	10.415823	60
534399	972940	561459	438541	554658	970103	584555	415445	59
534745	972894	561851	438149	554987	970055	584932	415068	58
535092	972848	562244	437756	555315	970006	585309	414691	57
535438	972802	562636	437364	555643	969957	585686	414314	56
535783	972755	563028	436972	555971	969909	586062	413938	55
536129	972709	563419	436581	556299	969860	586439	413561	54
536474	972663	563811	436189	556626	969811	586815	413185	53
536818	972617	564202	435798	556953	969762	587190	412810	52
537163	972570	564592	435408	557280	969714	587566	412434	51
537507	972524	564983	435017	557606	969665	587941	412059	50
9.537851	9.972478	9.565373	10.434627	9.557932	9.969616	9.588316	10.411684	49
538194	972431	565763	434237	558258	969567	588691	411309	48
538538	972385	566153	433847	558583	969518	589066	410934	47
538880	972338	566542	433458	558909	969469	589440	410560	46
539223	972291	566932	433068	559234	969420	589814	410186	45
539565	972245	567320	432680	559558	969370	590188	409812	44
539907	972198	567709	432291	559883	969321	590562	409438	43
540249	972151	568098	431902	560207	969272	590935	409065	42
540590	972105	568486	431514	560531	969223	591308	408692	41
540931	972058	568873	431127	560855	969173	591681	408319	40
9.541272	9.972011	9.569261	10.430739	9.561178	9.969124	9.592054	10.407946	39
541613	971964	569648	430352	561501	969075	592426	407574	38
541953	971917	570035	429965	561824	969025	592798	407202	37
542293	971870	570422	429578	562146	968976	593171	406829	36
542632	971823	570809	429191	562468	968926	593542	406458	35
542971	971776	571195	428805	562790	968877	593914	406086	34
543310	971729	571581	428419	563112	968827	594285	405715	33
543649	971682	571967	428033	563433	968777	594656	405344	32
543987	971635	572352	427648	563755	968728	595027	404973	31
544325	971588	572738	427262	564075	968678	595398	404602	30
9.544663	9.971540	9.573123	10.426877	9.564396	9.968628	9.595768	10.404232	29
545000	971493	573507	426493	564716	968578	596138	403862	28
545338	971446	573892	426108	565036	968528	596508	403492	27
545674	971398	574276	425724	565356	968479	596878	403122	26
546011	971351	574660	425340	565676	968429	597247	402753	25
546347	971303	575044	424956	565995	968379	597616	402384	24
546683	971256	575427	424573	566314	968329	597985	402015	23
547019	971208	575810	424190	566632	968278	598354	401646	22
547354	971161	576193	423807	566951	968228	598722	401278	21
547689	971113	576576	423424	567269	968178	599091	400909	20
9.548024	9.971066	9.576959	10.423041	9.567587	9.968128	9.599459	10.400541	19
548359	971018	577341	422659	567904	968078	599827	400173	18
548693	970970	577723	422277	568222	968027	600194	399806	17
549027	970922	578104	421896	568539	967977	600562	399438	16
549360	970874	578486	421514	568856	967927	600929	399071	15
549693	970827	578867	421133	569172	967876	601296	398704	14
550026	970779	579248	420752	569488	967826	601662	398338	13
550359	970731	579629	420371	569804	967775	602029	397971	12
550692	970683	580009	419991	570120	967725	602395	397605	11
551024	970635	580389	419611	570435	967674	602761	397239	10
9.551356	9.970586	9.580769	10.419231	9.570751	9.967624	9.603127	10.396873	9
551687	970538	581149	418851	571066	967573	603493	396507	8
552018	970490	581528	418472	571380	967522	603858	396142	7
552349	970442	581907	418093	571695	967471	604223	395777	6
552680	970394	582286	417714	572009	967421	604588	395412	5
553010	970345	582665	417335	572323	967370	604953	395047	4
553341	970297	583043	416957	572636	967319	605317	394683	3
553670	970249	583422	416578	572950	967268	605682	394318	2
554000	970200	583800	416200	573263	967217	606046	393954	1
554329	970152	584177	415823	573575	967166	606410	393590	0
Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
69 Degrees.				68 Degrees.				

	22 Degrees.				23 Degrees.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
0	9.573575	9.967166	9.606410	10.393590	9.591878	9.964026	9.627852	10.372148	60
1	573888	967115	606773	393227	592176	963972	628203	371797	59
2	574200	967064	607137	392863	592473	963919	628554	371446	58
3	574512	967013	607500	392500	592770	963865	628905	371095	57
4	574824	966961	607863	392137	593067	963811	629255	370745	56
5	575136	966910	608225	391775	593363	963757	629606	370394	55
6	575447	966859	608588	391412	593659	963704	629956	370044	54
7	575758	966808	608950	391050	593955	963650	630306	369694	53
8	576069	966756	609312	390688	594251	963596	630656	369344	52
9	576379	966705	609674	390326	594547	963542	631005	368995	51
10	576689	966653	610036	389964	594842	963488	631355	368645	50
11	9.576999	9.966602	9.610397	10.389603	9.595137	9.963434	9.631704	10.368296	49
12	577309	966550	610759	389241	595432	963379	632053	367947	48
13	577618	966499	611120	388880	595727	963325	632401	367599	47
14	577927	966447	611480	388520	596021	963271	632750	367250	46
15	578236	966395	611841	388159	596315	963217	633098	366902	45
16	578545	966344	612201	387799	596609	963163	633447	366553	44
17	578853	966292	612561	387439	596903	963108	633795	366205	43
18	579162	966240	612921	387079	597196	963054	634143	365857	42
19	579470	966188	613281	386719	597490	962999	634490	365510	41
20	579777	966136	613641	386359	597783	962945	634838	365162	40
21	9.580085	9.966085	9.614000	10.386000	9.598075	9.962890	9.635185	10.364815	39
22	580392	966033	614359	385641	598368	962836	635532	364468	38
23	580699	965981	614718	385282	598660	962781	635879	364121	37
24	581005	965929	615077	384923	598952	962727	636226	363774	36
25	581312	965876	615435	384565	599244	962672	636572	363428	35
26	581618	965824	615793	384207	599536	962617	636919	363081	34
27	581924	965772	616151	383849	599827	962562	637265	362735	33
28	582229	965720	616509	383491	600118	962508	637611	362389	32
29	582535	965668	616867	383133	600409	962453	637956	362044	31
30	582840	965615	617224	382776	600700	962398	638302	361698	30
31	9.583145	9.965563	9.617582	10.382418	9.600990	9.962343	9.638647	10.361353	29
32	583449	965511	617939	382061	601280	962288	638992	361008	28
33	583754	965458	618295	381705	601570	962233	639337	360663	27
34	584058	965406	618652	381348	601860	962178	639682	360318	26
35	584361	965353	619008	380992	602150	962123	640027	359973	25
36	584665	965301	619364	380636	602439	962067	640371	359629	24
37	584968	965248	619721	380279	602728	962012	640716	359284	23
38	585272	965195	620076	379924	603017	961957	641060	358940	22
39	585574	965143	620432	379568	603305	961902	641404	358596	21
40	585877	965090	620787	379213	603594	961846	641747	358253	20
41	9.586179	9.965037	9.621142	10.378858	9.603882	9.961791	9.642091	10.357909	19
42	586482	964984	621497	378503	604170	961735	642434	357566	18
43	586783	964931	621852	378148	604457	961680	642777	357223	17
44	587085	964879	622207	377793	604745	961624	643120	356880	16
45	587386	964826	622561	377439	605032	961569	643463	356537	15
46	587688	964773	622915	377085	605319	961513	643806	356194	14
47	587989	964720	623269	376731	605606	961458	644148	355852	13
48	588289	964666	623623	376377	605892	961402	644490	355510	12
49	588590	964613	623976	376024	606179	961346	644832	355168	11
50	588890	964560	624330	375670	606465	961290	645174	354826	10
51	9.589190	9.964507	9.624683	10.375317	9.606751	9.961235	9.645516	10.354484	9
52	589489	964454	625036	374964	607036	961179	645857	354143	8
53	589789	964400	625388	374612	607322	961123	646199	353801	7
54	590088	964347	625741	374259	607607	961067	646540	353460	6
55	590387	964294	626093	373907	607892	961011	646881	353119	5
56	590686	964240	626445	373555	608177	960955	647222	352778	4
57	590984	964187	626797	373203	608461	960899	647562	352438	3
58	591282	964133	627149	372851	608745	960843	647903	352097	2
59	591580	964080	627501	372499	609029	960786	648243	351757	1
60	591878	964026	627852	372148	609313	960730	648583	351417	0
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	67 Degrees.				66 Degrees.				

24 Degrees.				25 Degrees.				
Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
9.609313	9.960730	9.648583	10.351417	9.625948	9.957276	9.668673	10.331327	60
609597	960674	648923	351077	626219	957217	669002	330998	59
609880	960618	649263	350737	626490	957158	669332	330668	58
610164	960561	649602	350398	626760	957099	669661	330339	57
610447	960505	649942	350058	627030	957040	669991	330009	56
610729	960448	650281	349719	627300	956981	670320	329680	55
611012	960392	650620	349380	627570	956921	670649	329351	54
611294	960335	650959	349041	627840	956862	670977	329023	53
611576	960279	651297	348703	628109	956803	671306	328694	52
611858	960222	651636	348364	628378	956744	671634	328366	51
612140	960165	651974	348026	628647	956684	671963	328037	50
9.612421	9.960109	9.652312	10.347688	9.628916	9.956625	9.672291	10.327709	49
612702	960052	652650	347350	629185	956566	672619	327381	48
612983	959995	652988	347012	629453	956506	672947	327053	47
613264	959938	653326	346674	629721	956447	673274	326726	46
613545	959882	653663	346337	629989	956387	673602	326398	45
613825	959825	654000	346000	630257	956327	673929	326071	44
614105	959768	654337	345663	630524	956268	674257	325743	43
614385	959711	654674	345326	630792	956208	674584	325416	42
614665	959654	655011	344989	631059	956148	674910	325090	41
614944	959596	655348	344652	631326	956089	675237	324763	40
9.615223	9.959539	9.655684	10.344316	9.631593	9.956029	9.675564	10.324436	39
615502	959482	656020	343980	631859	955969	675890	324110	38
615781	959425	656356	343644	632125	955909	676217	323783	37
616060	959368	656692	343308	632392	955849	676543	323457	36
616338	959310	657028	342972	632658	955789	676869	323131	35
616616	959253	657364	342636	632923	955729	677194	322806	34
616894	959195	657699	342301	633189	955669	677520	322480	33
617172	959138	658034	341966	633454	955609	677846	322154	32
617450	959080	658369	341631	633719	955548	678171	321829	31
617727	959023	658704	341296	633984	955488	678496	321504	30
9.618004	9.958965	9.659039	10.340961	9.634249	9.955428	9.678821	10.321179	29
618281	958908	659373	340627	634514	955368	679146	320854	28
618558	958850	659708	340292	634778	955307	679471	320529	27
618834	958792	660042	339958	635042	955247	679795	320205	26
619110	958734	660376	339624	635306	955186	680120	319880	25
619386	958677	660710	339290	635570	955126	680444	319556	24
619662	958619	661043	338957	635834	955065	680768	319232	23
619938	958561	661377	338623	636097	955005	681092	318908	22
620213	958503	661710	338290	636360	954944	681416	318584	21
620488	958445	662043	337957	636623	954883	681740	318260	20
9.620763	9.958387	9.662376	10.337624	9.636886	9.954823	9.682063	10.317937	19
621038	958329	662709	337291	637148	954762	682387	317613	18
621313	958271	663042	336958	637411	954701	682710	317290	17
621587	958213	663375	336625	637673	954640	683033	316967	16
621861	958154	663707	336293	637935	954579	683356	316644	15
622135	958096	664039	335961	638197	954518	683679	316321	14
622409	958038	664371	335629	638458	954457	684001	315999	13
622682	957979	664703	335297	638720	954396	684324	315676	12
622956	957921	665035	334965	638981	954335	684646	315354	11
623229	957863	665366	334634	639242	954274	684968	315032	10
9.623502	9.957804	9.665697	10.334303	9.639503	9.954213	9.685290	10.314710	9
623774	957746	666029	333971	639764	954152	685612	314388	8
624047	957687	666360	333640	640024	954090	685934	314066	7
624319	957628	666691	333309	640284	954029	686255	313745	6
624591	957570	667021	332979	640544	953968	686577	313423	5
624863	957511	667352	332648	640804	953906	686898	313102	4
625135	957452	667682	332318	641064	953845	687219	312781	3
625406	957393	668013	331987	641324	953783	687540	312460	2
625677	957335	668343	331657	641583	953722	687861	312139	1
625948	957276	668672	331328	641842	953660	688182	311818	0
Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
65 Degrees.				64 Degrees.				

	26 Degrees.				27 Degrees.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
0	9.641842	9.953660	9.688182	10.311818	9.657047	9.949881	9.707166	10.292834	60
1	642101	953599	688502	311498	657295	949816	707478	292522	59
2	642360	953537	688823	311177	657542	949752	707790	292210	58
3	642618	953475	689143	310857	657790	949688	708102	291898	57
4	642877	953413	689463	310537	658037	949623	708414	291586	56
5	643135	953352	689783	310217	658284	949558	708726	291274	55
6	643393	953290	690103	309897	658531	949494	709037	290963	54
7	643650	953228	690423	309577	658778	949429	709349	290651	53
8	643908	953166	690742	309258	659025	949364	709660	290340	52
9	644165	953104	691062	308938	659271	949300	709971	290029	51
10	644423	953042	691381	308619	659517	949235	710282	289718	50
11	9.644680	9.952980	9.691700	10.308300	9.659763	9.949170	9.710593	10.289407	49
12	644936	952918	692019	307981	660009	949105	710904	289096	48
13	645193	952855	692338	307662	660255	949040	711215	288785	47
14	645450	952793	692656	307344	660501	948975	711525	288475	46
15	645706	952731	692975	307025	660746	948910	711836	288164	45
16	645962	952669	693293	306707	660991	948845	712146	287854	44
17	646218	952606	693612	306388	661236	948780	712456	287544	43
18	646474	952544	693930	306070	661481	948715	712766	287234	42
19	646729	952481	694248	305752	661726	948650	713076	286924	41
20	646984	952419	694566	305434	661970	948584	713386	286614	40
21	9.647240	9.952356	9.694883	10.305117	9.662214	9.948519	9.713696	10.286304	39
22	647494	952294	695201	304799	662459	948454	714005	285995	38
23	647749	952231	695518	304482	662703	948388	714314	285686	37
24	648004	952168	695836	304164	662946	948323	714624	285376	36
25	648258	952106	696153	303847	663190	948257	714933	285067	35
26	648512	952043	696470	303530	663433	948192	715242	284758	34
27	648766	951980	696787	303213	663677	948126	715551	284449	33
28	649020	951917	697103	302897	663920	948060	715860	284140	32
29	649274	951854	697420	302580	664163	947995	716168	283832	31
30	649527	951791	697736	302264	664406	947929	716477	283523	30
31	9.649781	9.951728	9.698053	10.301947	9.664648	9.947863	9.716785	10.283215	29
32	650034	951665	698369	301631	664891	947797	717093	282907	28
33	650287	951602	698685	301315	665133	947731	717401	282599	27
34	650539	951539	699001	300999	665375	947665	717709	282291	26
35	650792	951476	699316	300684	665617	947600	718017	281983	25
36	651044	951412	699632	300368	665859	947533	718325	281675	24
37	651297	951349	699947	300053	666100	947467	718633	281367	23
38	651549	951286	700263	299737	666342	947401	718940	281060	22
39	651800	951222	700578	299422	666583	947335	719248	280752	21
40	652052	951159	700893	299107	666824	947269	719555	280445	20
41	9.652304	9.951096	9.701208	10.298792	9.667065	9.947203	9.719862	10.280138	19
42	652555	951032	701523	298477	667305	947136	720169	279831	18
43	652806	950968	701837	298163	667546	947070	720476	279524	17
44	653057	950905	702152	297848	667786	947004	720783	279217	16
45	653308	950841	702466	297534	668027	946937	721089	278911	15
46	653558	950778	702780	297220	668267	946871	721396	278604	14
47	653808	950714	703095	296905	668506	946804	721702	278298	13
48	654059	950650	703409	296591	668746	946738	722009	277991	12
49	654309	950586	703723	296277	668986	946671	722315	277685	11
50	654558	950522	704036	295964	669225	946604	722621	277379	10
51	9.654808	9.950458	9.704350	10.295650	9.669464	9.946538	9.722927	10.277073	9
52	655058	950394	704663	295337	669703	946471	723232	276768	8
53	655307	950330	704977	295023	669942	946404	723538	276462	7
54	655556	950266	705290	294710	670181	946337	723844	276156	6
55	655805	950202	705603	294397	670419	946270	724149	275851	5
56	656054	950138	705916	294084	670658	946203	724454	275546	4
57	656302	950074	706228	293772	670896	946136	724759	275241	3
58	656551	950010	706541	293459	671134	946069	725065	274935	2
59	656799	949945	706854	293146	671372	946002	725369	274631	1
60	657047	949881	707166	292834	671609	945935	725674	274326	0
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	63 Degrees.				62 Degrees.				

28 Degrees.				29 Degrees.				
Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
9.671609	9.945935	9.725674	10.274326	9.685571	9.941819	9.743752	10.256248	60
671847	945868	725979	274021	685799	941749	744050	255950	59
672084	945800	726284	273716	686027	941679	744348	255652	58
672321	945733	726588	273412	686254	941609	744645	255355	57
672558	945666	726892	273108	686482	941539	744943	255057	56
672795	945598	727197	272803	686709	941469	745240	254760	55
673032	945531	727501	272499	686936	941398	745538	254462	54
673268	945464	727805	272195	687163	941328	745835	254165	53
673505	945396	728109	271891	687389	941258	746132	253868	52
673741	945328	728412	271588	687616	941187	746429	253571	51
673977	945261	728716	271284	687843	941117	746726	253274	50
9.674213	9.945193	9.729020	10.270980	9.688069	9.941046	9.747023	10.252977	49
674448	945125	729323	270677	688295	940975	747319	252681	48
674684	945058	729626	270374	688521	940905	747616	252384	47
674919	944990	729929	270071	688747	940834	747913	252087	46
675155	944922	730233	269767	688972	940763	748209	251791	45
675390	944854	730535	269465	689198	940693	748505	251495	44
675624	944786	730838	269162	689423	940622	748801	251199	43
675859	944718	731141	268859	689648	940551	749097	250903	42
676094	944650	731444	268556	689873	940480	749393	250607	41
676328	944582	731746	268254	690098	940409	749689	250311	40
9.676562	9.944514	9.732048	10.267952	9.690323	9.940338	9.749985	10.250015	39
676796	944446	732351	267649	690548	940267	750281	249719	38
677030	944377	732653	267347	690772	940196	750576	249424	37
677264	944309	732955	267045	690996	940125	750872	249128	36
677498	944241	733257	266743	691220	940054	751167	248833	35
677731	944172	733558	266442	691444	939982	751462	248538	34
677964	944104	733860	266140	691668	939911	751757	248243	33
678197	944036	734162	265838	691892	939840	752052	247948	32
678430	943967	734463	265537	692115	939768	752347	247653	31
678663	943899	734764	265236	692339	939697	752642	247358	30
9.678895	9.943830	9.735066	10.264934	9.692562	9.939625	9.752937	10.247063	29
679128	943761	735367	264633	692785	939554	753231	246769	28
679360	943693	735668	264332	693008	939482	753526	246474	27
679592	943624	735969	264031	693231	939410	753820	246180	26
679824	943555	736269	263731	693453	939339	754115	245885	25
680056	943486	736570	263430	693676	939267	754409	245591	24
680288	943417	736871	263129	693898	939195	754703	245297	23
680519	943348	737171	262829	694120	939123	754997	245003	22
680750	943279	737471	262529	694342	939052	755291	244709	21
680982	943210	737771	262229	694564	938980	755585	244415	20
9.681213	9.943141	9.738071	10.261929	9.694786	9.938908	9.755878	10.244122	19
681443	943072	738371	261629	695007	938836	756172	243828	18
681674	943003	738671	261329	695229	938763	756465	243535	17
681905	942934	738971	261029	695450	938691	756759	243241	16
682135	942864	739271	260729	695671	938619	757052	242948	15
682365	942795	739570	260430	695892	938547	757345	242655	14
682595	942726	739870	260130	696113	938475	757638	242362	13
682825	942656	740169	259831	696334	938402	757931	242069	12
683055	942587	740468	259532	696554	938330	758224	241776	11
683284	942517	740767	259233	696775	938258	758517	241483	10
9.683514	9.942448	9.741066	10.258934	9.696995	9.938185	9.758810	10.241190	9
683743	942378	741365	258635	697215	938113	759102	240898	8
683972	942308	741664	258336	697435	938040	759395	240605	7
684201	942239	741962	258038	697654	937967	759687	240313	6
684430	942169	742261	257739	697874	937895	759979	240021	5
684658	942099	742559	257441	698094	937822	760272	239728	4
684887	942029	742858	257142	698313	937749	760564	239436	3
685115	941959	743156	256844	698532	937676	760856	239144	2
685343	941889	743454	256546	698751	937604	761148	238852	1
685571	941819	743752	256248	698970	937531	761439	238561	0
Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
61 Degrees.				60 Degrees.				

	30 Degrees.				31 Degrees.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
0	9.698970	9.937531	9.761439	10.238561	9.711839	9.933066	9.778774	10.221226	60
1	699189	937458	761731	238269	712050	932990	779060	220940	59
2	699407	937385	762023	237977	712260	932914	779346	220654	58
3	699626	937312	762314	237686	712469	932838	779632	220368	57
4	699844	937238	762606	237394	712679	932762	779918	220082	56
5	700062	937165	762897	237103	712889	932685	780203	219797	55
6	700280	937092	763188	236812	713098	932609	780489	219511	54
7	700498	937019	763479	236521	713308	932533	780775	219225	53
8	700716	936946	763770	236230	713517	932457	781060	218940	52
9	700933	936872	764061	235939	713726	932380	781346	218654	51
10	701151	936799	764352	235648	713935	932304	781631	218369	50
11	9.701368	9.936725	9.764643	10.235357	9.714144	9.932228	9.781916	10.218084	49
12	701585	936652	764933	235067	714352	932151	782201	217799	48
13	701802	936578	765224	234776	714561	932075	782486	217514	47
14	702019	936505	765514	234486	714769	931998	782771	217229	46
15	702236	936431	765805	234195	714978	931921	783056	216944	45
16	702452	936357	766095	233905	715186	931845	783341	216659	44
17	702669	936284	766385	233615	715394	931768	783626	216374	43
18	702885	936210	766675	233325	715602	931691	783910	216090	42
19	703101	936136	766965	233035	715809	931614	784195	215805	41
20	703317	936062	767255	232745	716017	931537	784479	215521	40
21	9.703533	9.935988	9.767545	10.232455	9.716224	9.931460	9.784764	10.215236	39
22	703749	935914	767834	232166	716432	931383	785048	214952	38
23	703964	935840	768124	231876	716639	931306	785332	214668	37
24	704179	935766	768414	231586	716846	931229	785616	214384	36
25	704395	935692	768703	231297	717053	931152	785900	214100	35
26	704610	935618	768992	231008	717259	931075	786184	213816	34
27	704825	935543	769281	230719	717466	930998	786468	213532	33
28	705040	935469	769570	230430	717673	930921	786752	213248	32
29	705254	935395	769860	230140	717879	930843	787036	212964	31
30	705469	935320	770148	229852	718085	930766	787319	212681	30
31	9.705683	9.935246	9.770437	10.229563	9.718291	9.930688	9.787603	10.212397	29
32	705898	935171	770726	229274	718497	930611	787886	212114	28
33	706112	935097	771015	228985	718703	930533	788170	211830	27
34	706326	935022	771303	228697	718909	930456	788453	211547	26
35	706539	934948	771592	228408	719114	930378	788736	211264	25
36	706753	934873	771880	228120	719320	930300	789019	210981	24
37	706967	934798	772168	227832	719525	930223	789302	210698	23
38	707180	934723	772457	227543	719730	930145	789585	210415	22
39	707393	934649	772745	227255	719935	930067	789868	210132	21
40	707606	934574	773033	226967	720140	929989	790151	209849	20
41	9.707819	9.934499	9.773321	10.226679	9.720345	9.929911	9.790433	10.209567	19
42	708032	934424	773608	226392	720549	929833	790716	209284	18
43	708245	934349	773896	226104	720754	929755	790999	209001	17
44	708458	934274	774184	225816	720958	929677	791281	208719	16
45	708670	934199	774471	225529	721162	929599	791563	208437	15
46	708882	934123	774759	225241	721366	929521	791846	208154	14
47	709094	934048	775046	224954	721570	929442	792128	207872	13
48	709306	933973	775333	224667	721774	929364	792410	207590	12
49	709518	933898	775621	224379	721978	929286	792692	207308	11
50	709730	933822	775908	224092	722181	929207	792974	207026	10
51	9.709941	9.933747	9.776195	10.223805	9.722385	9.929129	9.793256	10.206744	9
52	710153	933671	776482	223518	722588	929050	793538	206462	8
53	710364	933596	776769	223231	722791	928972	793819	206181	7
54	710575	933520	777055	222945	722994	928893	794101	205899	6
55	710786	933445	777342	222658	723197	928815	794383	205617	5
56	710997	933369	777628	222372	723400	928736	794664	205336	4
57	711208	933293	777915	222085	723603	928657	794945	205055	3
58	711419	933217	778201	221799	723805	928578	795227	204773	2
59	711629	933141	778487	221513	724007	928499	795508	204492	1
60	711839	933066	778774	221226	724210	928420	795789	204211	0
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	59 Degrees.				58 Degrees.				

32 Degrees.				33 Degrees.				
Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
9.724210	9.928420	9.795789	10.204211	9.736109	9.923591	9.812517	10.187483	60
724412	928342	796070	203930	736303	923509	812794	187206	59
724614	928263	796351	203649	736498	923427	813070	186930	58
724816	928183	796632	203368	736692	923345	813347	186653	57
725017	928104	796913	203087	736886	923263	813623	186377	56
725219	928025	797194	202806	737080	923181	813899	186101	55
725420	927946	797475	202525	737274	923098	814175	185825	54
725622	927867	797755	202245	737467	923016	814452	185548	53
725823	927787	798036	201964	737661	922933	814728	185272	52
726024	927708	798316	201684	737855	922851	815004	184996	51
726225	927629	798596	201404	738048	922768	815279	184721	50
9.726426	9.927549	9.798877	10.201123	9.738241	9.922686	9.815555	10.184445	49
726626	927470	799157	200843	738434	922603	815831	184169	48
726827	927390	799437	200563	738627	922520	816107	183893	47
727027	927310	799717	200283	738820	922438	816382	183618	46
727228	927231	799997	200003	739013	922355	816658	183342	45
727428	927151	800277	199723	739206	922272	816933	183067	44
727628	927071	800557	199443	739398	922189	817209	182791	43
727828	926991	800836	199164	739590	922106	817484	182516	42
728027	926911	861116	198884	739783	922023	817759	182241	41
728227	926831	801396	198604	739975	921940	818035	181965	40
9.728427	9.926751	9.801675	10.198325	9.740167	9.921857	9.818310	10.181690	39
728626	926671	801955	198045	740359	921774	818585	181415	38
728825	926591	802234	197766	740550	921691	818860	181140	37
729024	926511	802513	197487	740742	921607	819135	180865	36
729223	926431	802792	197208	740934	921524	819410	180590	35
729422	926351	803072	196928	741125	921441	819684	180316	34
729621	926270	803351	196649	741316	921357	819959	180041	33
729820	926190	803630	196370	741508	921274	820234	179766	32
730018	926110	803908	196092	741699	921190	820508	179492	31
730217	926029	804187	195813	741889	921107	820783	179217	30
9.730415	9.925949	9.804466	10.195534	9.742080	9.921023	9.821057	10.178943	29
730613	925868	804745	195255	742271	920939	821332	178668	28
730811	925788	805023	194977	742462	920856	821606	178394	27
731009	925707	805302	194698	742652	920772	821880	178120	26
731206	925626	805580	194420	742842	920688	822154	177846	25
731404	925545	805859	194141	743033	920604	822429	177571	24
731602	925465	806137	193863	743223	920520	822703	177297	23
731799	925384	806415	193585	743413	920436	822977	177023	22
731996	925303	806693	193307	743602	920352	823250	176750	21
732193	925222	806971	193029	743792	920268	823524	176476	20
9.732390	9.925141	9.807249	10.192751	9.743982	9.920184	9.823798	10.176202	19
732587	925060	807527	192473	744171	920099	824072	175928	18
732784	924979	807805	192195	744361	920015	824345	175655	17
732980	924897	808083	191917	744550	919931	824619	175381	16
733177	924816	808361	191639	744739	919846	824893	175107	15
733373	924735	808638	191362	744928	919762	825166	174834	14
733569	924654	808916	191084	745117	919677	825439	174561	13
733765	924572	809193	190807	745306	919593	825713	174287	12
733961	924491	809471	190529	745494	919508	825986	174014	11
734157	924409	809748	190252	745683	919424	826259	173741	10
9.734353	9.924328	9.810025	10.189975	9.745871	9.919339	9.826532	10.173468	9
734549	924246	810302	189698	746060	919254	826805	173195	8
734744	924164	810580	189420	746248	919169	827078	172922	7
734939	924083	810857	189143	746436	919085	827351	172649	6
735135	924001	811134	188866	746624	919000	827624	172376	5
735330	923919	811410	188590	746812	918915	827897	172103	4
735525	923837	811687	188313	746999	918830	828170	171830	3
735719	923755	811964	188036	747187	918745	828442	171558	2
735914	923673	812241	187759	747374	918659	828715	171285	1
736109	923591	812517	187483	747562	918574	828987	171013	0
Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
57 Degrees.				56 Degrees.				

	34 Degrees.				35 Degrees.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
0	9.747562	9.918574	9.828987	10.171013	9.758591	9.913365	9.845227	10.154773	60
1	747749	918489	829260	170740	758772	913276	845496	154504	59
2	747936	918404	829532	170468	758952	913187	845764	154236	58
3	748123	918318	829805	170195	759132	913099	846033	153967	57
4	748310	918233	830077	169923	759312	913010	846302	153698	56
5	748497	918147	830349	169651	759492	912922	846570	153430	55
6	748683	918062	830621	169379	759672	912833	846839	153161	54
7	748870	917976	830893	169107	759852	912744	847107	152893	53
8	749056	917891	831165	168835	760031	912655	847376	152624	52
9	749243	917805	831437	168563	760211	912566	847644	152356	51
10	749429	917719	831709	168291	760390	912477	847913	152087	50
11	9.749615	9.917634	9.831981	10.168019	9.760569	9.912388	9.848181	10.151819	49
12	749801	917548	832253	167747	760748	912299	848449	151551	48
13	749987	917462	832525	167475	760927	912210	848717	151283	47
14	750172	917376	832796	167204	761106	912121	848986	151014	46
15	750358	917290	833068	166932	761285	912031	849254	150746	45
16	750543	917204	833339	166661	761464	911942	849522	150478	44
17	750729	917118	833611	166389	761642	911853	849790	150210	43
18	750914	917032	833882	166118	761821	911763	850058	149942	42
19	751099	916946	834154	165846	761999	911674	850325	149675	41
20	751284	916859	834425	165575	762177	911584	850593	149407	40
21	9.751469	9.916773	9.834696	10.165304	9.762356	9.911495	9.850861	10.149139	39
22	751654	916687	834967	165033	762534	911405	851129	148871	38
23	751839	916600	835238	164762	762712	911315	851396	148604	37
24	752023	916514	835509	164491	762889	911226	851664	148336	36
25	752208	916427	835780	164220	763067	911136	851931	148069	35
26	752392	916341	836051	163949	763245	911046	852199	147801	34
27	752576	916254	836322	163678	763422	910956	852466	147534	33
28	752760	916167	836593	163407	763600	910866	852733	147267	32
29	752944	916081	836864	163136	763777	910776	853001	146999	31
30	753128	915994	837134	162866	763954	910686	853268	146732	30
31	9.753312	9.915907	9.837405	10.162595	9.764131	9.910596	9.853535	10.146465	29
32	753495	915820	837675	162325	764308	910506	853802	146198	28
33	753679	915733	837946	162054	764485	910415	854069	145931	27
34	753862	915646	838216	161784	764662	910325	854336	145664	26
35	754046	915559	838487	161513	764838	910235	854603	145397	25
36	754229	915472	838757	161243	765015	910144	854870	145130	24
37	754412	915385	839027	160973	765191	910054	855137	144863	23
38	754595	915297	839297	160703	765367	909963	855404	144596	22
39	754778	915210	839568	160432	765544	909873	855671	144329	21
40	754960	915123	839838	160162	765720	909782	855938	144062	20
41	9.755143	9.915035	9.840108	10.159892	9.765896	9.909691	9.856204	10.143796	19
42	755326	914948	840378	159622	766072	909601	856471	143529	18
43	755508	914860	840647	159353	766247	909510	856737	143263	17
44	755690	914773	840917	159083	766423	909419	857004	142996	16
45	755872	914685	841187	158813	766598	909328	857270	142730	15
46	756054	914598	841457	158543	766774	909237	857537	142463	14
47	756236	914510	841726	158274	766949	909146	857803	142197	13
48	756418	914422	841996	158004	767124	909055	858069	141931	12
49	756600	914334	842266	157734	767300	908964	858336	141664	11
50	756782	914246	842535	157465	767475	908873	858602	141398	10
51	9.756963	9.914158	9.842805	10.157195	9.767649	9.908781	9.858868	10.141132	9
52	757144	914070	843074	156926	767824	908690	859134	140866	8
53	757326	913982	843343	156657	767999	908599	859400	140600	7
54	757507	913894	843612	156388	768173	908507	859666	140334	6
55	757688	913806	843882	156118	768348	908416	859932	140068	5
56	757869	913718	844151	155849	768522	908324	860198	139802	4
57	758050	913630	844420	155580	768697	908233	860464	139536	3
58	758230	913541	844689	155311	768871	908141	860730	139270	2
59	758411	913453	844958	155042	769045	908049	860995	139005	1
60	758591	913365	845227	154773	769219	907958	861261	138739	0
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	55 Degrees.				54 Degrees.				

36 Degrees.				37 Degrees.				
Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
9.769219	9.907958	9.861261	10.138739	9.779463	9.902349	9.877114	10.122886	60
769393	907866	861527	138473	779631	902253	877377	122623	59
769566	907774	861792	138208	779798	902158	877640	122360	58
769740	907682	862058	137942	779966	902063	877903	122097	57
769913	907590	862323	137677	780133	901967	878165	121835	56
770087	907498	862589	137411	780300	901872	878428	121572	55
770260	907406	862854	137146	780467	901776	878691	121309	54
770433	907314	863119	136881	780634	901681	878953	121047	53
770606	907222	863385	136615	780801	901585	879216	120784	52
770779	907129	863650	136350	780968	901490	879478	120522	51
770952	907037	863915	136085	781134	901394	879741	120259	50
9.771125	9.906945	9.864180	10.135820	9.781301	9.901298	9.880003	10.119997	49
771298	906852	864445	135555	781468	901202	880265	119735	48
771470	906760	864710	135290	781634	901106	880528	119472	47
771643	906667	864975	135025	781800	901010	880790	119210	46
771815	906575	865240	134760	781966	900914	881052	118948	45
771987	906482	865505	134495	782132	900818	881314	118686	44
772159	906389	865770	134230	782298	900722	881576	118424	43
772331	906296	866035	133965	782464	900626	881839	118161	42
772503	906204	866300	133700	782630	900529	882101	117899	41
772675	906111	866564	133436	782796	900433	882363	117637	40
9.772847	9.906018	9.866829	10.133171	9.782961	9.900337	9.882625	10.117375	39
773018	905925	867094	132906	783127	900240	882887	117113	38
773190	905832	867358	132642	783292	900144	883148	116852	37
773361	905739	867623	132377	783458	900047	883410	116590	36
773533	905645	867887	132113	783623	899951	883672	116328	35
773704	905552	868152	131848	783788	899854	883934	116066	34
773875	905459	868416	131584	783953	899757	884196	115804	33
774046	905366	868680	131320	784118	899660	884457	115543	32
774217	905272	868945	131055	784282	899564	884719	115281	31
774388	905179	869209	130791	784447	899467	884980	115020	30
9.774558	9.905085	9.869473	10.130527	9.784612	9.899370	9.885242	10.114758	29
774729	904992	869737	130263	784776	899273	885503	114497	28
774899	904898	870001	129999	784941	899176	885765	114235	27
775070	904804	870265	129735	785105	899078	886026	113974	26
775240	904711	870529	129471	785269	898981	886288	113712	25
775410	904617	870793	129207	785433	898884	886549	113451	24
775580	904523	871057	128943	785597	898787	886810	113190	23
775750	904429	871321	128679	785761	898689	887072	112928	22
775920	904335	871585	128415	785925	898592	887333	112667	21
776090	904241	871849	128151	786089	898494	887594	112406	20
9.776259	9.904147	9.872112	10.127888	9.786252	9.898397	9.887855	10.112145	19
776429	904053	872376	127624	786416	898299	888116	111884	18
776598	903959	872640	127360	786579	898202	888377	111623	17
776768	903864	872903	127097	786742	898104	888639	111361	16
776937	903770	873167	126833	786906	898006	888900	111100	15
777106	903676	873430	126570	787069	897908	889160	110840	14
777275	903581	873694	126306	787232	897810	889421	110579	13
777444	903487	873957	126043	787395	897712	889682	110318	12
777613	903392	874220	125780	787557	897614	889943	110057	11
777781	903298	874484	125516	787720	897516	890204	109796	10
9.777950	9.903203	9.874747	10.125253	9.787883	9.897418	9.890465	10.109535	9
778119	903108	875010	124990	788045	897320	890725	109275	8
778287	903014	875273	124727	788208	897222	890986	109014	7
778455	902919	875536	124464	788370	897123	891247	108753	6
778624	902824	875800	124200	788532	897025	891507	108493	5
778792	902729	876063	123937	788694	896926	891768	108232	4
778960	902634	876326	123674	788856	896828	892028	107972	3
779128	902539	876589	123411	789018	896729	892289	107711	2
779295	902444	876851	123149	789180	896631	892549	107451	1
779463	902349	877114	122886	789342	896532	892810	107190	0
Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
53 Degrees.				52 Degrees.				

	38 Degrees.				39 Degrees.			
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.
0	9.789342	9.896532	9.892810	10.107190	9.798872	9.890503	9.908369	10.091631
1	789504	896433	893070	106930	799028	890400	908628	091372
2	789665	896335	893331	106669	799184	890298	908886	091114
3	789827	896236	893591	106409	799339	890195	909144	090856
4	789988	896137	893851	106149	799495	890093	909402	090598
5	790149	896038	894111	105889	799651	889990	909660	090340
6	790310	895939	894371	105629	799806	889888	909918	090082
7	790471	895840	894632	105368	799962	889785	910177	089823
8	790632	895741	894892	105108	800117	889682	910435	089565
9	790793	895641	895152	104848	800272	889579	910693	089307
10	790954	895542	895412	104588	800427	889477	910951	089049
11	9.791115	9.895443	9.895672	10.104328	9.800582	9.889374	9.911209	10.088791
12	791275	895343	895932	104068	800737	889271	911467	088533
13	791436	895244	896192	103808	800892	889168	911724	088276
14	791596	895145	896452	103548	801047	889064	911982	088018
15	791757	895045	896712	103288	801201	888961	912240	087760
16	791917	894945	896971	103029	801356	888858	912498	087502
17	792077	894846	897231	102769	801511	888755	912756	087244
18	792237	894746	897491	102509	801665	888651	913014	086986
19	792397	894646	897751	102249	801819	888548	913271	086729
20	792557	894546	898010	101990	801973	888444	913529	086471
21	9.792716	9.894446	9.898270	10.101730	9.802128	9.888341	9.913787	10.086213
22	792876	894346	898530	101470	802282	888237	914044	085956
23	793035	894246	898789	101211	802436	888134	914302	085698
24	793195	894146	899049	100951	802589	888030	914560	085440
25	793354	894046	899308	100692	802743	887926	914817	085183
26	793514	893946	899568	100432	802897	887822	915075	084925
27	793673	893846	899827	100173	803050	887718	915332	084668
28	793832	893745	900086	099914	803204	887614	915590	084410
29	793991	893645	900346	099654	803357	887510	915847	084153
30	794150	893544	900605	099395	803511	887406	916104	083896
31	9.794308	9.893444	9.900864	10.099136	9.803664	9.887302	9.916362	10.083638
32	794467	893343	901124	098876	803817	887198	916619	083381
33	794626	893243	901383	098617	803970	887093	916877	083123
34	794784	893142	901642	098358	804123	886989	917134	082866
35	794942	893041	901901	098099	804276	886885	917391	082609
36	795101	892940	902160	097840	804428	886780	917648	082352
37	795259	892839	902419	097581	804581	886676	917905	082095
38	795417	892739	902679	097321	804734	886571	918163	081837
39	795575	892638	902938	097062	804886	886466	918420	081580
40	795733	892536	903197	096803	805039	886362	918677	081323
41	9.795891	9.892435	9.903455	10.096545	9.805191	9.886257	9.918934	10.081066
42	796049	892334	903714	096286	805343	886152	919191	080809
43	796206	892233	903973	096027	805495	886047	919448	080552
44	796364	892132	904232	095768	805647	885942	919705	080295
45	796521	892030	904491	095509	805799	885837	919962	080038
46	796679	891929	904750	095250	805951	885732	920219	079781
47	296836	891827	905008	094992	806103	885627	920476	079524
48	796993	891726	905267	094733	806254	885522	920733	079267
49	797150	891624	905526	094474	806406	885416	920990	079010
50	797307	891523	905784	094216	806557	885311	921247	078753
51	9.797464	9.891421	9.906043	10.093957	9.806709	9.885205	9.921503	10.078497
52	797621	891319	906302	093698	806860	885100	921760	078240
53	797777	891217	906560	093440	807011	884994	922017	077983
54	797934	891115	906819	093181	807163	884889	922274	077726
55	798091	891013	907077	092923	807314	884783	922530	077470
56	798247	890911	907336	092664	807465	884677	922787	077213
57	798403	890809	907594	092406	807615	884572	923044	076956
58	798560	890707	907852	092148	807766	884466	923300	076700
59	798716	890605	908111	091889	807917	884360	923557	076443
60	798872	890503	908369	091631	808067	884254	923813	076187
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.
	51 Degrees.				50 Degrees.			

′	40 Degrees.				41 Degrees.				′
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
0	9.808067	9.884254	9.923813	10.076187	9.816943	9.877780	9.939163	10.060837	60
1	808218	884148	924070	075930	817088	877670	939418	060582	59
2	808368	884042	924327	075673	817233	877560	939673	060327	58
3	808519	883936	924583	075417	817379	877450	939928	060072	57
4	808669	883829	924840	075160	817524	877340	940183	059817	56
5	808819	883723	925096	074904	817668	877230	940438	059562	55
6	808969	883617	925352	074648	817813	877120	940694	059306	54
7	809119	883510	925609	074391	817958	877010	940949	059051	53
8	809269	883404	925865	074135	818103	876899	941204	058796	52
9	809419	883297	926122	073878	818247	876789	941458	058542	51
10	809569	883191	926378	073622	818392	876678	941714	058286	50
11	9.809718	9.883084	9.926634	10.073366	9.818536	9.876568	9.941968	10.058032	49
12	809868	882977	926890	073110	818681	876457	942223	057777	48
13	810017	882871	927147	072853	818825	876347	942478	057522	47
14	810167	882764	927403	072597	818969	876236	942733	057267	46
15	810316	882657	927659	072341	819113	876125	942988	057012	45
16	810465	882550	927915	072085	819257	876014	943243	056757	44
17	810614	882443	928171	071829	819401	875904	943498	056502	43
18	810763	882336	928427	071573	819545	875793	943752	056248	42
19	810912	882229	928683	071317	819689	875682	944007	055993	41
20	811061	882121	928940	071060	819832	875571	944262	055738	40
21	9.811210	9.882014	9.929196	10.070804	9.819976	9.875459	9.944517	10.055483	39
22	811358	881907	929452	070548	820120	875348	944771	055229	38
23	811507	881799	929708	070292	820263	875237	945026	054974	37
24	811655	881692	929964	070036	820406	875126	945281	054719	36
25	811804	881584	930220	069780	820550	875014	945535	054465	35
26	811952	881477	930475	069525	820693	874903	945790	054210	34
27	812100	881369	930731	069269	820836	874791	946045	053955	33
28	812248	881261	930987	069013	820979	874680	946299	053701	32
29	812396	881153	931243	068757	821122	874568	946554	053446	31
30	812544	881046	931499	068501	821265	874456	946808	053192	30
31	9.812692	9.880938	9.931755	10.068245	9.821407	9.874344	9.947063	10.052937	29
32	812840	880830	932010	067990	821550	874232	947318	052682	28
33	812988	880722	932266	067734	821693	874121	947572	052428	27
34	813135	880613	932522	067478	821835	874009	947826	052174	26
35	813283	880505	932778	067222	821977	873896	948081	051919	25
36	813430	880397	933033	066967	822120	873784	948336	051664	24
37	813578	880289	933289	066711	822262	873672	948590	051410	23
38	813725	880180	933545	066455	822404	873560	948844	051156	22
39	813872	880072	933800	066200	822546	873448	949099	050901	21
40	814019	879963	934056	065944	822688	873335	949353	050647	20
41	9.814166	9.879855	9.934311	10.065689	9.822830	9.873223	9.949607	10.050393	19
42	814313	879746	934567	065433	822972	873110	949862	050138	18
43	814460	879637	934823	065177	823114	872998	950116	049884	17
44	814607	879529	935078	064922	823255	872885	950370	049630	16
45	814753	879420	935333	064667	823397	872772	950625	049375	15
46	814900	879311	935589	064411	823539	872659	950879	049121	14
47	815046	879202	935844	064156	823680	872547	951133	048867	13
48	815193	879093	936100	063900	823821	872434	951388	048612	12
49	815339	878984	936355	063645	823963	872321	951642	048358	11
50	815485	878875	936610	063390	824104	872208	951896	048104	10
51	9.815632	9.878766	9.936866	10.063134	9.824245	9.872095	9.952150	10.047850	9
52	815778	878656	937121	062879	824386	871981	952405	047595	8
53	815924	878547	937376	062624	824527	871868	952659	047341	7
54	816069	878438	937632	062368	824668	871755	952913	047087	6
55	816215	878328	937887	062113	824808	871641	953167	046833	5
56	816361	878219	938142	061858	824949	871528	953421	046579	4
57	816507	878109	938398	061602	825090	871414	953675	046325	3
58	816652	877999	938653	061347	825230	871301	953929	046071	2
59	816798	877890	938908	061092	825371	871187	954183	045817	1
60	816943	877780	939163	060837	825511	871073	954437	045563	0
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	49 Degrees.				48 Degrees.				

	42 Degrees.				43 Degrees.			
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.
0	9.825511	9.871073	9.954437	10.045563	9.833783	9.864127	9.969656	10.030344
1	825651	870960	954691	045309	833919	864010	969909	030091
2	825791	870846	954945	045055	834054	863892	970162	029838
3	825931	870732	955200	044800	834189	863774	970416	029584
4	826071	870618	955454	044546	834325	863656	970669	029331
5	826211	870504	955707	044293	834460	863538	970922	029078
6	826351	870390	955961	044039	834595	863419	971175	028825
7	826491	870276	956215	043785	834730	863301	971429	028571
8	826631	870161	956469	043531	834865	863183	971682	028318
9	826770	870047	956723	043277	834999	863064	971935	028065
10	826910	869933	956977	043023	835134	862946	972188	027812
11	9.827049	9.869818	9.957231	10.042769	9.835269	9.862827	9.972441	10.027559
12	827189	869704	957485	042515	835403	862709	972694	027306
13	827328	869589	957739	042261	835538	862590	972948	027052
14	827467	869474	957993	042007	835672	862471	973201	026799
15	827606	869360	958246	041754	835807	862353	973454	026546
16	827745	869245	958500	041500	835941	862234	973707	026293
17	827884	869130	958754	041246	836075	862115	973960	026040
18	828023	869015	959008	040992	836209	861996	974213	025787
19	828162	868900	959262	040738	836343	861877	974466	025534
20	828301	868785	959516	040484	836477	861758	974719	025281
21	9.828439	9.868670	9.959769	10.040231	9.836611	9.861638	9.974973	10.025027
22	828578	868555	960023	039977	836745	861519	975226	024774
23	828716	868440	960277	039723	836878	861400	975479	024521
24	828855	868324	960531	039469	837012	861280	975732	024268
25	828993	868209	960784	039216	837146	861161	975985	024015
26	829131	868093	961038	038962	837279	861041	976238	023762
27	829269	867978	961291	038709	837412	860922	976491	023509
28	829407	867862	961545	038455	837546	860802	976744	023256
29	829545	867747	961799	038201	837679	860682	976997	023003
30	829683	867631	962052	037948	837812	860562	977250	022750
31	9.829821	9.867515	9.962306	10.037694	9.837945	9.860442	9.977503	10.022497
32	829959	867399	962560	037440	838078	860322	977756	022244
33	830097	867283	962813	037187	838211	860202	978009	021991
34	830234	867167	963067	036933	838344	860082	978262	021738
35	830372	867051	963320	036680	838477	859962	978515	021485
36	830509	866935	963574	036426	838610	859842	978768	021232
37	830646	866819	963827	036173	838742	859721	979021	020979
38	830784	866703	964081	035919	838875	859601	979274	020726
39	830921	866586	964335	035665	839007	859480	979527	020473
40	831058	866470	964588	035412	839140	859360	979780	020220
41	9.831195	9.866353	9.964842	10.035158	9.839272	9.859239	9.980033	10.019967
42	831332	866237	965095	034905	839404	859119	980286	019714
43	831469	866120	965349	034651	839536	858998	980538	019462
44	831606	866004	965602	034398	839668	858877	980791	019209
45	831742	865887	965855	034145	839800	858756	981044	018956
46	831879	865770	966109	033891	839932	858635	981297	018703
47	832015	865653	966362	033638	840064	858514	981550	018450
48	832152	865536	966616	033384	840196	858393	981803	018197
49	832288	865419	966869	033131	840328	858272	982056	017944
50	832425	865302	967123	032877	840459	858151	982309	017691
51	9.832561	9.865185	9.967376	10.032624	9.840591	9.858029	9.982562	10.017438
52	832697	865068	967629	032371	840722	857908	982814	017186
53	832833	864950	967883	032117	840854	857786	983067	016933
54	832969	864833	968136	031864	840985	857665	983320	016680
55	833105	864716	968389	031611	841116	857543	983573	016427
56	833241	864598	968643	031357	841247	857422	983826	016174
57	833377	864481	968896	031104	841378	857300	984079	015921
58	833512	864363	969149	030851	841509	857178	984331	015669
59	833648	864245	969403	030597	841640	857056	984584	015416
60	833783	864127	969656	030344	841771	856934	984837	015163
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.
	47 Degrees.				46 Degrees.			

44 Degrees.					
′	Sine.	Cosine.	Tang.	Cotang.	′
0	9.841771	9.856934	9.984837	10.015163	60
1	841902	856812	985090	014910	59
2	842033	856690	985343	014657	58
3	842163	856568	985596	014404	57
4	842294	856446	985848	014152	56
5	842424	856323	986101	013899	55
6	842555	856201	986354	013646	54
7	842685	856078	986607	013393	53
8	842815	855956	986860	013140	52
9	842946	855833	987112	012888	51
10	843076	855711	987365	012635	50
11	9.843206	9.855588	9.987618	10.012382	49
12	843336	855465	987871	012129	48
13	843466	855342	988123	011877	47
14	843595	855219	988376	011624	46
15	843725	855096	988629	011371	45
16	843855	854973	988882	011118	44
17	843984	854850	989134	010866	43
18	844114	854727	989387	010613	42
19	844243	854603	989640	010360	41
20	844372	854480	989893	010107	40
21	9.844502	9.854356	9.990145	10.009855	39
22	844631	854233	990398	009602	38
23	844760	864109	990651	009349	37
24	844889	853986	990903	009097	36
25	845018	853862	991156	008844	35
26	845147	853738	991409	008591	34
27	845276	853614	991662	008338	33
28	845405	853490	991914	008086	32
29	845533	853366	992167	007833	31
30	845662	853242	992420	007580	30
31	9.845790	9.853118	9.992672	10.007328	29
32	845919	852994	992925	007075	28
33	846047	852869	993178	006822	27
34	846175	852745	993430	006570	26
35	846304	852620	993683	006317	25
36	846432	852496	993936	006064	24
37	846560	852371	994189	005811	23
38	846688	852247	994441	005559	22
39	846816	852122	994694	005306	21
40	846944	851997	994947	005053	20
41	9.847071	9.851872	9.995199	10.004801	19
42	847199	851747	995452	004548	18
43	847327	851622	995705	004295	17
44	847454	851497	995957	004043	16
45	847582	851372	996210	003790	15
46	847709	851246	996463	003537	14
47	847836	851121	996715	003285	13
48	847964	850996	996968	003032	12
49	848091	850870	997221	002779	11
50	848218	850745	997473	002527	10
51	9.848345	9.850619	9.997726	10.002274	9
52	848472	850493	997979	002021	8
53	848599	850368	998231	001769	7
54	848726	850242	998484	001516	6
55	848852	850116	998737	001263	5
56	848979	849990	998989	001011	4
57	849106	849864	999242	000758	3
58	849232	849738	999495	000505	2
59	849359	849611	999747	000253	1
60	849485	849485	10.00000	000000	0
′	Cosine.	Sine.	Cotang.	Tang.	′
45 Degrees.					

RULES FOR FINDING LOGARITHMIC SECANTS, VERSED SINES, &c.

I. To find the Secant.—Subtract the Log. Cosine from 20.
II. To find the Cosecant.—Subtract the Log. Sine from 20.
III. To find the Versed Sine.—Add 0.301030 to twice the Log. Sine of half the arc, and diminish the index of the sum by 10.
IV. To find the Coversed Sine.—Add 0.301030 to twice the Log. Sine of half the complement of the arc, and diminish the index of the sum by 10.

RULES FOR FINDING NATURAL SECANTS, VERSED SINES, &c.

I. To find the Secant.—Divide 1 by the Natural Cosine.
II. To find the Cosecant.—Divide 1 by the Natural Sine.
III. To find the Versed Sine.—Subtract the Natural Cosine from 1.
IV. To find the Coversed Sine.—Subtract the Natural Sine from 1.

NOTE. In France the circumference of the circle has lately been divided into 400 degrees, the degree into 100 minutes, and the minute into 100 seconds, &c. which is called the centesimal division, and is to the sexagesimal in the ratio of 9 to 10; hence, to reduce centesimal into sexagesimal degrees, &c. subtract one-tenth; and to reduce sexagesimal into centesimal degrees, add one-ninth of the arc to itself.

′	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	′
0	000000	017452	034899	052336	069756	087156	104528	121869	139173	156434	60
5	1454	8907	6353	3788	071207	8605	5975	3313	140613	7871	55
10	2909	020361	7806	5241	2658	090053	7421	4756	2053	9307	50
15	4363	1815	9260	6693	4108	1502	8867	6199	3493	160743	45
20	5818	3269	040713	8145	5559	2950	110313	7642	4932	2178	40
25	7272	4723	2166	9597	7009	4398	1758	9084	6371	3613	35
30	8727	6177	3619	061049	8459	5846	3203	130526	7809	5048	30
35	010181	7631	5072	2500	9909	7293	4648	1968	9248	6482	25
40	1635	9085	6525	3952	081359	8741	6093	3410	150686	7916	20
45	3090	030539	7978	5403	2808	100188	7537	4851	2123	9350	15
50	4544	1992	9431	6854	4258	1635	8982	6292	3561	170783	10
55	5998	3446	050883	8306	5707	3082	120426	7733	4998	2216	5
60	7452	4899	2336	9756	7156	4528	1869	9173	6434	3648	0
Cos.	89°	88°	87°	86°	85°	84°	83°	82°	81°	80°	
Sin.	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	
0	173648	190809	207912	224951	241922	258819	275637	292372	309017	325568	60
5	5080	2237	9334	6368	3333	260224	7035	3762	310400	6943	55
10	6512	3664	210756	7784	4743	1628	8432	5152	1782	8317	50
15	7944	5090	2178	9200	6153	3031	9829	6542	3164	9691	45
20	9375	6517	3599	230616	7563	4434	281225	7930	4545	331063	40
25	180805	7942	5019	2031	8972	5837	2620	9318	5925	2435	35
30	2236	9368	6440	3445	250380	7238	4015	300706	7305	3807	30
35	3665	200793	7859	4859	1788	8640	5410	2093	8684	5178	25
40	5095	2218	9279	6273	3195	270040	6803	3479	320062	6547	20
45	6524	3642	220697	7686	4602	1440	8196	4864	1439	7917	15
50	7953	5065	2116	9098	6008	2840	9589	6249	2816	9285	10
55	9381	6489	3534	240510	7414	4239	290981	7633	4193	340653	5
60	190809	7912	4951	1922	8819	5637	2372	9017	5568	2020	0
Cos.	79°	78°	77°	76°	75°	74°	73°	72°	71°	70°	
Sin.	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	
0	342020	358368	374607	390731	406737	422618	438371	453990	469472	484810	60
5	3387	9725	5955	2070	8065	3936	9678	5286	470755	6081	55
10	4752	361082	7302	3407	9392	5253	440984	6580	2038	7352	50
15	6117	2438	8649	4744	410719	6569	2289	7874	3320	8621	45
20	7481	3793	9994	6080	2045	7884	3593	9166	4600	9890	40
25	8845	5148	381339	7415	3369	9198	4896	460458	5880	491157	35
30	350207	6501	2683	8749	4693	430511	6198	1749	7159	2424	30
35	1569	7854	4027	400082	6016	1823	7499	3038	8436	3689	25
40	2931	9206	5369	1415	7338	3135	8799	4327	9713	4953	20
45	4291	370557	6711	2747	8660	4445	450098	5615	480989	6217	15
50	5651	1908	8052	4078	9980	5755	1397	6901	2263	7479	10
55	7010	3258	9392	5408	421300	7063	2694	8187	3537	8740	5
60	8368	4607	390731	6737	2618	8371	3990	9472	4810	500000	0
Cos.	69°	68°	67°	66°	65°	64°	63°	62°	61°	60°	
Sin.	30°	31°	32°	33°	34°	35°	36°	37°	38°	39°	
0	500000	515038	529919	544639	559193	573576	587785	601815	615661	629320	60
5	1259	6284	531152	5858	560398	4767	8961	2976	6807	630450	55
10	2517	7529	2384	7076	1602	5957	590136	4136	7951	1578	50
15	3774	8773	3615	8293	2805	7145	1310	5294	9094	2705	45
20	5030	520016	4844	9509	4007	8332	2482	6451	620235	3831	40
25	6285	1258	6072	550724	5207	9518	3653	7607	1376	4955	35
30	7538	2499	7300	1937	6406	580703	4823	8761	2515	6078	30
35	8791	3738	8526	3149	7604	1886	5991	9915	3652	7200	25
40	510043	4977	9751	4360	8801	3069	7159	611067	4789	8320	20
45	1293	6214	540974	5570	9997	4250	8325	2217	5923	9439	15
50	2543	7450	2197	6779	571191	5429	9489	3367	7057	640557	10
55	3791	8685	3419	7987	2384	6608	600653	4515	8189	1673	5
60	5038	9919	4639	9193	3576	7785	1815	5661	9320	2788	0
′	59°	58°	57°	56°	55°	54°	53°	52°	51°	50°	′

NATURAL COSINES.

′	40°	41°	42°	43°	44°	45°	46°	47°	48°	49°	′
0	642788	656059	669131	681998	694658	707107	719340	731354	743145	754710	60
5	3901	7156	670211	3061	5704	8134	720349	2345	4117	5663	55
10	5013	8252	1289	4123	6748	9161	1357	3334	5088	6615	50
15	6124	9346	2367	5183	7790	710185	2364	4323	6057	7565	45
20	7233	660439	3443	6242	8832	1209	3369	5309	7025	8514	40
25	8341	1530	4517	7299	9871	2230	4372	6294	7991	9461	35
30	9448	2620	5590	8355	700909	3250	5374	7277	8956	760406	30
35	650553	3709	6662	9409	1946	4269	6375	8259	9919	1350	25
40	1657	4796	7732	690462	2981	5286	7374	9239	750880	2292	20
45	2760	5882	8801	1513	4015	6302	8371	740218	1840	3232	15
50	3861	6966	9868	2563	5047	7316	9367	1195	2798	4171	10
55	4961	8049	680934	3611	6078	8329	730361	2171	3755	5109	5
60	6059	9131	1998	4658	7107	9340	1354	3145	4710	6044	0
Cos.	49°	48°	47°	46°	45°	44°	43°	42°	41°	40°	
Sin.	50°	51°	52°	53°	54°	55°	56°	57°	58°	59°	
0	766044	777146	788011	798636	809017	819152	829038	838671	848048	857167	60
5	6979	8060	8905	9510	9871	9985	9850	9462	8818	7915	55
10	7911	8973	9798	800383	810723	820817	830661	840251	9586	8662	50
15	8842	9884	790690	1254	1574	1647	1470	1039	850352	9406	45
20	9771	780794	1579	2123	2423	2475	2277	1825	1117	860149	40
25	770699	1702	2467	2991	3270	3302	3082	2609	1879	0890	35
30	1625	2608	3353	3857	4116	4126	3886	3391	2640	1629	30
35	2549	3513	4238	4721	4959	4949	4688	4172	3399	2366	25
40	3472	4416	5121	5584	5801	5770	5488	4951	4156	3102	20
45	4393	5317	6002	6445	6642	6590	6286	5728	4912	3836	15
50	5312	6217	6882	7304	7480	7407	7083	6503	5665	4567	10
55	6230	7114	7759	8161	8317	8223	7878	7277	6417	5297	5
60	7146	8011	8636	9017	9152	9038	8671	8048	7167	6025	0
Cos.	39°	38°	37°	36°	35°	34°	33°	32°	31°	30°	
Sin.	60°	61°	62°	63°	64°	65°	66°	67°	68°	69°	
0	866025	874620	882948	891007	898794	906308	913545	920505	927184	933580	60
5	6752	5324	3629	1666	9431	6922	4136	1072	7728	4101	55
10	7476	6026	4309	2323	900065	7533	4725	1638	8270	4619	50
15	8199	6727	4988	2979	0698	8143	5311	2201	8810	5135	45
20	8920	7425	5664	3633	1329	8751	5896	2762	9348	5650	40
25	9639	8122	6338	4284	1958	9357	6479	3322	9884	6162	35
30	870356	8817	7011	4934	2585	9961	7060	3880	930418	6672	30
35	1071	9510	7681	5582	3210	910563	7639	4435	0950	7181	25
40	1784	880201	8350	6229	3834	1164	8216	4989	1480	7687	20
45	2496	0891	9017	6873	4455	1762	8791	5541	2008	8191	15
50	3206	1578	9682	7515	5075	2358	9364	6090	2534	8694	10
55	3914	2264	890345	8156	5692	2953	9936	6638	3058	9194	5
60	4620	2948	1007	8794	6308	3545	920505	7184	3580	9693	0
Cos.	29°	28°	27°	26°	25°	24°	23°	22°	21°	20°	
Sin.	70°	71°	72°	73°	74°	75°	76°	77°	78°	79°	
0	939693	945519	951057	956305	961262	965926	970296	974370	978148	981627	60
5	940189	5991	1505	6729	1662	6301	0647	4696	8449	1904	55
10	0684	6462	1951	7151	2059	6675	0995	5020	8748	2178	50
15	1176	6930	2396	7571	2455	7046	1342	5342	9045	2450	45
20	1666	7397	2838	7990	2849	7415	1687	5662	9341	2721	40
25	2155	7861	3279	8406	3241	7782	2029	5980	9634	2989	35
30	2641	8324	3717	8820	3630	8148	2370	6296	9925	3255	30
35	3126	8784	4153	9232	4018	8511	2708	6610	980214	3519	25
40	3609	9243	4588	9642	4404	8872	3045	6921	0500	3781	20
45	4089	9699	5020	960050	4787	9231	3379	7231	0785	4041	15
50	4568	950154	5450	0456	5169	9588	3712	7539	1068	4298	10
55	5044	0606	5879	0860	5548	9943	4042	7844	1349	4554	5
60	5519	1057	6305	1262	5926	970296	4370	8148	1627	4808	0
′	19°	18°	17°	16°	15°	14°	13°	12°	11°	10°	′

′	80°	81°	82°	83°	84°	85°	86°	87°	88°	89°	′
0	984808	987688	990268	992546	994522	996195	997564	998630	999391	999848	60
5	5059	7915	0469	2722	4673	6320	7664	8705	9441	9872	55
10	5309	8139	0669	2896	4822	6444	7763	8778	9488	9894	50
15	5556	8362	0866	3068	4969	6566	7859	8848	9534	9914	45
20	5801	8582	1061	3238	5113	6685	7953	8917	9577	9932	40
25	6045	8800	1254	3406	5256	6802	8045	8984	9618	9948	35
30	6286	9016	1445	3572	5396	6917	8135	9048	9657	9962	30
35	6525	9230	1634	3735	5535	7030	8223	9111	9694	9974	25
40	6762	9442	1820	3897	5671	7141	8308	9171	9729	9983	20
45	6996	9651	2005	4056	5805	7250	8392	9229	9762	9990	15
50	7229	9859	2187	4214	5937	7357	8473	9285	9793	9996	10
55	7460	990065	2368	4369	6067	7462	8552	9339	9821	9999	5
60	7688	0268	2546	4522	6195	7564	8630	9391	9848	1.00000	0
Cos.	9°	8°	7°	6°	5°	4°	3°	2°	1°	0°	

NATURAL COSINES.

NATURAL TANGENTS.

′	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	′
0	000000	017455	034921	052408	069927	087489	105104	122785	140541	158384	60
5	1454	8910	6377	3866	071389	8954	6575	4261	2024	9876	55
10	2909	020365	7834	5325	2851	090421	8046	5738	3508	161368	50
15	4363	1820	9290	6784	4313	1887	9518	7216	4993	2860	45
20	5818	3275	040747	8243	5775	3354	110990	8694	6478	4354	40
25	7272	4731	2204	9703	7238	4821	2463	130173	7964	5848	35
30	8727	6186	3661	061163	8702	6289	3936	1652	9451	7343	30
35	010181	7641	5118	2623	080165	7757	5409	3132	150938	8838	25
40	1636	9097	6576	4083	1629	9226	6883	4613	2426	170334	20
45	3091	030553	8033	5543	3094	100695	8358	6094	3915	1831	15
50	4545	2009	9491	7004	4558	2164	9833	7576	5404	3329	10
55	6000	3465	050949	8465	6023	3634	121309	9058	6894	4828	5
60	7455	4921	2408	9927	7489	5104	2785	140541	8384	6327	0
Cot.	89°	88°	87°	86°	85°	84°	83°	82°	81°	80°	
Tan.	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	
0	176327	194380	212557	230868	249328	267949	286745	305731	324920	344328	60
5	7827	5890	4077	2401	250873	9509	8320	7322	6528	5955	55
10	9328	7401	5599	3934	2420	271069	9896	8914	8139	7585	50
15	180830	8912	7121	5469	3968	2631	291473	310508	9751	9216	45
20	2332	200425	8645	7004	5517	4194	3052	2104	331364	350848	40
25	3835	1938	220169	8541	7066	5759	4632	3701	2979	2483	35
30	5339	3452	1695	240079	8618	7325	6214	5299	4595	4119	30
35	6844	4967	3221	1618	260170	8892	7796	6899	6213	5756	25
40	8350	6483	4749	3157	1723	280460	9380	8500	7833	7396	20
45	9856	8000	6277	4698	3278	2029	300966	320103	9454	9037	15
50	191363	9518	7806	6241	4834	3600	2553	1707	341077	360680	10
55	2871	211037	9337	7784	6391	5172	4141	3313	2702	2324	5
60	4380	2557	230868	9328	7949	6745	5731	4920	4328	3970	0
Cot.	79°	78°	77°	76°	75°	74°	73°	72°	71°	70°	
Tan.	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	
0	363970	383864	404026	424475	445229	466308	487733	509525	531709	554309	60
5	5618	5534	5719	6192	6973	8080	9534	511359	3577	6212	55
10	7268	7205	7414	7912	8719	9854	491339	3195	5447	8118	50
15	8920	8879	9111	9634	450467	471631	3145	5034	7319	560027	45
20	370573	390554	410810	431358	2218	3410	4955	6876	9195	1939	40
25	2228	2231	2511	3084	3971	5191	6767	8720	541074	3854	35
30	3885	3910	4214	4812	5726	6976	8582	520567	2956	5773	30
35	5543	5592	5919	6543	7484	8762	500399	2417	4840	7694	25
40	7204	7275	7626	8276	9244	480551	2219	4270	6728	9619	20
45	8866	8960	9335	440011	461006	2343	4042	6126	8619	571547	15
50	380530	400646	421046	1748	2771	4137	5867	7984	550513	3478	10
55	2196	2335	2759	3487	4538	5933	7695	9845	2409	5413	5
60	3864	4026	4475	5229	6308	7733	9525	531709	4309	7350	0
′	69°	68°	67°	66°	65°	64°	63°	62°	61°	60°	′

NATURAL COTANGENTS.

′	30°	31°	32°	33°	34°	35°	36°	37°	′
0	577350	600861	624869	649408	674509	700208	726543	753554	60
5	9291	2842	6894	651477	6627	2377	8767	5837	55
10	581235	4827	8921	3551	8749	4551	730996	8125	50
15	3183	6815	630953	5629	680876	6730	3230	760418	45
20	5134	8807	2988	7710	3007	8913	5469	2716	40
25	7088	610802	5027	9796	5142	711101	7713	5019	35
30	9045	2801	7070	661886	7281	3293	9961	7327	30
35	591006	4803	9117	3979	9425	5430	742214	9640	25
40	2970	6809	641167	6077	691572	7691	4472	771959	20
45	4938	8819	3222	8179	3725	9897	6735	4283	15
50	6908	620832	5280	670285	5881	722108	9003	6612	10
55	8883	2849	7342	2394	8042	4323	751276	8946	5
60	600861	4869	9408	4509	700208	6543	3554	781286	0
Cot.	59°	58°	57°	56°	55°	54°	53°	52°	
Tan.	38°	39°	40°	41°	42°	43°	44°	45°	
0	781286	809784	839100	869287	900404	932515	965689	1.000000	60
5	3631	812195	841581	871844	3041	5238	8504	002913	55
10	5981	4612	4069	4407	5635	7968	971326	005835	50
15	8336	7034	6563	6977	8336	940706	4157	008765	45
20	790698	9463	9062	9553	910994	3451	6996	011704	40
25	3064	821897	851568	882136	3659	6204	9842	014651	35
30	5436	4336	4081	4725	6331	8965	982697	017607	30
35	7813	6782	6599	7322	9010	951733	5560	020572	25
40	800196	9234	9124	9924	921697	4508	8432	023546	20
45	2585	831691	861655	892534	4391	7292	991311	026529	15
50	4979	4155	4193	5151	7091	960083	4199	029520	10
55	7379	6624	6736	7774	9800	2882	7095	032521	5
60	9784	9100	9287	900404	932515	5689	1.000000	035530	0
Cot.	51°	50°	49°	48°	47°	46°	45°	44°	
Tan.	46°	47°	48°	49°	50°	51°	52°	53°	
0	1.035530	1.072369	1.110613	1.150368	1.191754	1.234897	1.279942	1.327045	60
5	038549	075501	113866	153753	195280	238576	283786	331068	55
10	041577	078642	117131	157150	198818	242269	287645	335108	50
15	044614	081794	120405	160557	202360	245974	291518	339162	45
20	047660	084955	123691	163076	205933	249693	295406	343233	40
25	050715	088127	126987	167407	209509	253426	299308	347320	35
30	053780	091309	130294	170850	213097	257172	303225	351422	30
35	056854	094500	133612	174304	216698	260932	307158	355541	25
40	059938	097702	136941	177770	220312	264706	311105	359676	20
45	063031	100914	140282	181248	223939	268494	315067	363828	15
50	066134	104137	143633	184738	227579	272296	319044	367996	10
55	069247	107369	146995	188240	231231	276112	323037	372181	5
60	072369	110613	150368	191754	234897	279942	327045	376382	0
Cot.	43°	42°	41°	40°	39°	38°	37°	36°	
Tan.	54°	55°	56°	57°	58°	59°	60°	61°	
0	1.376382	1.428148	1.482561	1.539865	1.600335	1.664280	1.732051	1.804048	60
5	380600	432578	487222	544779	605526	669776	737883	810252	55
10	384835	437027	491904	549716	610742	675299	743745	816489	50
15	389088	441494	496606	554674	615982	680849	749637	822759	45
20	393357	445980	501328	559655	621247	686426	755559	829063	40
25	397644	450485	506071	564659	626537	692031	761511	835400	35
30	401948	455009	510835	569686	631852	697663	767494	841771	30
35	406270	459552	515620	574735	637192	703323	773508	848176	25
40	410610	464115	520426	579808	642558	709012	779552	854616	20
45	414967	468697	525254	584904	647949	714728	785629	861091	15
50	419343	473298	530102	590024	653366	720474	791736	867600	10
55	423736	477920	534973	595167	658810	726248	797876	874146	5
60	428148	482561	539865	600335	664280	732051	804048	880727	0
′	35°	34°	33°	32°	31°	30°	29°	28°	′

NATURAL COTANGENTS.

′	62°	63°	64°	65°	66°	67°	68°	69°	′
0	1.880727	1.962611	2.050304	2.144507	2.246037	2.355852	2.475087	2.605089	60
5	887344	969687	057895	152676	254857	365412	485489	616457	55
10	893997	976805	065532	160896	263736	375037	495966	627912	50
15	900687	983964	073215	169168	272673	384729	506520	639455	45
20	907415	991164	080944	177492	281669	394489	517151	651087	40
25	914179	998406	088720	185869	290726	404317	527860	662809	35
30	920982	2.005690	096544	194300	299843	414214	538648	674622	30
35	927823	013016	104415	202784	309021	424180	549516	686527	25
40	934702	020386	112335	211323	318261	434217	560465	698525	20
45	941620	027799	120303	219918	327563	444326	571496	710619	15
50	948577	035257	128321	228568	336929	454506	582609	722808	10
55	955574	042758	136389	237274	346358	464760	593807	735093	5
60	962611	050304	144507	246037	355852	475087	605089	747477	0
Cot.	27°	26°	25°	24°	23°	22°	21°	20°	
Tan.	70°	71°	72°	73°	74°	75°	76°	77°	
0	2.747477	2.904211	3.077684	3.270853	3.487414	3.732051	4.010781	4.331476	60
5	759961	917991	092983	287949	506656	753882	035778	360400	55
10	772545	931889	108421	305209	526094	775952	061070	389694	50
15	785231	945905	123999	322636	545733	798266	086663	419364	45
20	798020	960042	139719	340233	565575	820828	112561	449418	40
25	810913	974302	155584	358001	585624	843642	138772	479864	35
30	823913	988685	171595	375943	605884	866713	165300	510709	30
35	837020	3.003194	187754	394063	626357	890045	192151	541961	25
40	850235	017830	204064	412363	647047	913642	219332	573629	20
45	863560	032595	220526	430845	667958	937509	246848	605721	15
50	876997	047492	237144	449512	689093	961652	274707	638246	10
55	890547	062520	253918	468368	710456	986074	302914	671212	5
60	904211	077684	270853	487414	732051	4.010781	331476	704630	0
Cot.	19°	18°	17°	16°	15°	14°	13°	12°	
Tan.	78°	79°	80°	81°	82°	83°	84°	85°	
0	4.704630	5.144554	5.671282	6.313752	7.115370	8.144346	9.514365	11.43005	60
5	738508	184804	719917	373736	191246	243449	649348	62476	55
10	772857	225665	769369	434843	268726	344956	788173	82617	50
15	807685	267152	819657	497104	347861	448957	931009	12.03462	45
20	843005	309279	870804	560554	428706	555547	10.07803	25051	40
25	878825	352063	922832	625226	511318	664822	22943	47422	35
30	915157	395517	975764	691156	595754	776887	38540	70620	30
35	952013	439659	6.029625	758383	682077	891851	54615	94692	25
40	989403	484505	084438	826944	770351	9.009826	71191	13.19688	20
45	5.027340	530072	140230	896880	860642	130935	88292	45663	15
50	065835	576379	197028	968234	953022	255304	11.05943	72674	10
55	104902	623442	254859	7.041048	8.047565	383066	24171	14.00786	5
60	144554	671282	313752	115370	144346	514365	43005	30067	0
Cot.	11°	10°	9°	8°	7°	6°	5°	4°	
Tan.	86°	Diff.	87°	Diff.	88°	Diff.	89°	Diff.	
0	14.30067	30525	19.08114	54616	28.63625	1.24605	57.28996	5.20919	60
5	60592	31850	62730	57825	29.88230	1.35928	62.49915	6.25094	55
10	92442	33263	20.20555	61328	31.24158	1.48868	68.75009	7.63992	50
15	15.25705	34773	81883	65157	32.73026	1.63751	76.39001	9.54978	45
20	60478	36389	21.47040	69358	34.36777	1.80983	85.93979	12.2782	40
25	96867	38119	22.16398	73979	36.17760	2.01086	98.21794	16.3708	35
30	16.34986	39975	90377	79077	38.18846	2.24738	114.5887	22.9188	30
35	74961	41973	23.69454	84722	40.43584	2.52824	137.5075	34.3779	25
40	17.16934	44122	24.54176	90994	42.96408	2.86527	171.8854	57.2963	20
45	61056	46442	25.45170	97990	45.82935	3.27453	229.1817	114.5920	15
50	18.07498	48949	26.43160	1.05825	49.10388	3.77823	343.7737	343.7752	10
55	56447	51667	27.48985	1.14640	52.88211	4.40785	687.5489	Infinite.	5
60	19.08114		28.63625		57.28996		Infinite.		0
′	3°	Diff.	2°	Diff.	1°	Diff.	0°	Diff.	′

NATURAL COTANGENTS.

A TABLE

OF THE

AREAS OF CIRCULAR SEGMENTS.

Height.	Area.	Height.	Area.	Height.	Area.	Height.	Area.	Height.	Area.
.001	.000042	.051	.015119	.101	.041476	.151	.074589	.201	.112624
.002	.000119	.052	.015561	.102	.042080	.152	.075306	.202	.113426
.003	.000219	.053	.016007	.103	.042687	.153	.076026	.203	.114230
.004	.000337	.054	.016457	.104	.043296	.154	.076747	.204	.115035
.005	.000470	.055	.016911	.105	.043908	.155	.077469	.205	.115842
.006	.000618	.056	.017369	.106	.044522	.156	.078194	.206	.116650
.007	.000779	.057	.017831	.107	.045139	.157	.078921	.207	.117460
.008	.000951	.058	.018296	.108	.045759	.158	.079649	.208	.118271
.009	.001135	.059	.018766	.109	.046381	.159	.080380	.209	.119084
.010	.001329	.060	.019239	.110	.047005	.160	.081112	.210	.119897
.011	.001533	.061	.019716	.111	.047632	.161	.081846	.211	.120712
.012	.001746	.062	.020196	.112	.048262	.162	.082582	.212	.121529
.013	.001969	.063	.020691	.113	.048894	.163	.083320	.213	.122347
.014	.002199	.064	.021168	.114	.049528	.164	.084059	.214	.123167
.015	.002438	.065	.021659	.115	.050165	.165	.084801	.215	.123988
.016	.002685	.066	.022154	.116	.050804	.166	.085544	.216	.124810
.017	.002940	.067	.022652	.117	.051446	.167	.086289	.217	.125634
.018	.003202	.068	.023154	.118	.052090	.168	.087036	.218	.126459
.019	.003471	.069	.023659	.119	.052736	.169	.087785	.219	.127285
.020	.003748	.070	.024168	.120	.053385	.170	.088535	.220	.128113
.021	.004031	.071	.024680	.121	.054036	.171	.089287	.221	.128942
.022	.004322	.072	.025195	.122	.054689	.172	.090041	.222	.129773
.023	.004618	.073	.025714	.123	.055345	.173	.090797	.223	.130605
.024	.004921	.074	.026236	.124	.056003	.174	.091554	.224	.131438
.025	.005230	.075	.026761	.125	.056663	.175	.092313	.225	.132272
.026	.005546	.076	.027289	.126	.057326	.176	.093074	.226	.133108
.027	.005867	.077	.027821	.127	.057991	.177	.093836	.227	.133945
.028	.006194	.078	.028356	.128	.058658	.178	.094601	.228	.134784
.029	.006527	.079	.028894	.129	.059327	.179	.095366	.229	.135624
.030	.006865	.080	.029435	.130	.059999	.180	.096134	.230	.136465
.031	.007209	.081	.029979	.131	.060672	.181	.096904	.231	.137307
.032	.007558	.082	.030526	.132	.061348	.182	.097674	.232	.138150
.033	.007913	.083	.031076	.133	.062026	.183	.098447	.233	.138995
.034	.008273	.084	.031629	.134	.062707	.184	.099221	.234	.139841
.035	.008638	.085	.032186	.135	.063389	.185	.099997	.235	.140688
.036	.009008	.086	.032745	.136	.064074	.186	.100774	.236	.141537
.037	.009383	.087	.033307	.137	.064760	.187	.101553	.237	.142387
.038	.009763	.088	.033872	.138	.065449	.188	.102334	.238	.143238
.039	.010148	.089	.034441	.139	.066140	.189	.103116	.239	.144091
.040	.010537	.090	.035011	.140	.066833	.190	.103900	.240	.144944
.041	.010931	.091	.035585	.141	.067528	.191	.104685	.241	.145799
.042	.011330	.092	.036162	.142	.068225	.192	.105472	.242	.146655
.043	.011734	.093	.036741	.143	.068924	.193	.106261	.243	.147512
.044	.012142	.094	.037323	.144	.069625	.194	.107051	.244	.148371
.045	.012554	.095	.037909	.145	.070328	.195	.107842	.245	.149230
.046	.012971	.096	.038497	.146	.071033	.196	.108636	.246	.150091
.047	.013392	.097	.039087	.147	.071741	.197	.109431	.247	.150953
.048	.013818	.098	.039680	.148	.072450	.198	.110226	.248	.151816
.049	.014247	.099	.040276	.149	.073161	.199	.111025	.249	.152680
.050	.014681	.100	.040875	.150	.073874	.200	.111823	.250	.153546

Height.	Area.	Height.	Area.	Height.	Area.	Height.	Area.	Height.	Area.
.251	.154412	.301	.199085	.351	.245934	.401	.294349	.451	.343777
.252	.155280	.302	.200003	.352	.246889	.402	.295330	.452	.344772
.253	.156149	.303	.200922	.353	.247845	.403	.296311	.453	.345768
.254	.157019	.304	.201841	.354	.248801	.404	.297292	.454	.346764
.255	.157890	.305	.202761	.355	.249757	.405	.298273	.455	.347759
.256	.158762	.306	.203683	.356	.250715	.406	.299255	.456	.348755
.257	.159636	.307	.204605	.357	.251673	.407	.300238	.457	.349752
.258	.160510	.308	.205527	.358	.252631	.408	.301220	.458	.350748
.259	.161386	.309	.206451	.359	.253590	.409	.302203	.459	.351745
.260	.162263	.310	.207376	.360	.254550	.410	.303187	.460	.352742
.261	.163140	.311	.208301	.361	.255510	.411	.304171	.461	.353739
.262	.164019	.312	.209227	.362	.256471	.412	.305155	.462	.354736
.263	.164899	.313	.210154	.363	.257433	.413	.306140	.463	.355732
.264	.165780	.314	.211083	.364	.258395	.414	.307125	.464	.356730
.265	.166663	.315	.212011	.365	.259357	.415	.308110	.465	.357727
.266	.167546	.316	.212940	.366	.260320	.416	.309095	.466	.358725
.267	.168430	.317	.213871	.367	.261284	.417	.310081	.467	.359723
.268	.169316	.318	.214802	.368	.262248	.418	.311068	.468	.360721
.269	.170202	.319	.215733	.369	.263213	.419	.312054	.469	.361719
.270	.171089	.320	.216666	.370	.264178	.420	.313041	.470	.362717
.271	.171978	.321	.217599	.371	.265144	.421	.314029	.471	.363715
.272	.172867	.322	.218533	.372	.266111	.422	.315016	.472	.364713
.273	.173758	.323	.219468	.373	.267078	.423	.316004	.473	.365712
.274	.174649	.324	.220404	.374	.268045	.424	.316992	.474	.366710
.275	.175542	.325	.221341	.375	.269013	.425	.317981	.475	.367709
.276	.176435	.326	.222277	.376	.269982	.426	.318970	.476	.368708
.277	.177330	.327	.223215	.377	.270951	.427	.319959	.477	.369707
.278	.178225	.328	.224154	.378	.271920	.428	.320948	.478	.370706
.279	.179122	.329	.225093	.379	.272890	.429	.321938	.479	.371705
.280	.180019	.330	.226033	.380	.273861	.430	.322928	.480	.372704
.281	.180918	.331	.226974	.381	.274832	.431	.323918	.481	.373703
.282	.181818	.332	.227915	.382	.275803	.432	.324909	.482	.374702
.283	.182718	.333	.228858	.383	.276775	.433	.325900	.483	.375702
.284	.183619	.334	.229801	.384	.277748	.434	.326892	.484	.376702
.285	.184521	.335	.230745	.385	.278721	.435	.327882	.485	.377701
.286	.185425	.336	.231689	.386	.279694	.436	.328874	.486	.378701
.287	.186329	.337	.232634	.387	.280668	.437	.329866	.487	.379700
.288	.187234	.338	.233580	.388	.281642	.438	.330858	.488	.380700
.289	.188140	.339	.234526	.389	.282617	.439	.331850	.489	.381699
.290	.189047	.340	.235473	.390	.283592	.440	.332843	.490	.382699
.291	.189955	.341	.236421	.391	.284568	.441	.333836	.491	.383699
.292	.190864	.342	.237369	.392	.285544	.442	.334829	.492	.384699
.293	.191775	.343	.238318	.393	.286521	.443	.335822	.493	.385699
.294	.192684	.344	.239268	.394	.287498	.444	.336816	.494	.386699
.295	.193596	.345	.240218	.395	.288476	.445	.337810	.495	.387699
.296	.194509	.346	.241169	.396	.289454	.446	.338804	.496	.388699
.297	.195422	.347	.242121	.397	.290432	.447	.339798	.497	.389699
298	.196337	.348	.243074	.398	.291411	.448	.340793	.498	.390699
.299	.197252	.349	.244026	.399	.292390	.449	.341787	.499	.391699
.300	.198168	.350	.244980	.400	.293369	.450	.342782	.500	.392699

A

TABLE

OF

SQUARES, CUBES, SQUARE ROOTS, AND CUBE ROOTS.

No.	Square.	Cube.	Square Root.	Cube Root.	No.	Square.	Cube.	Square Root.	Cube Root.
1	1	1	1·000000	1·000000	53	2809	148877	7·280110	3·756286
2	4	8	1·414214	1·259921	54	2916	157464	7·348469	3·779763
3	9	27	1·732051	1·442250	55	3025	166375	7·416198	3·802953
4	16	64	2·000000	1·587401	56	3136	175616	7·483315	3·825862
5	25	125	2·236068	1·709976	57	3249	185193	7·549834	3·848501
6	36	216	2·449490	1·817121	58	3364	195112	7·615773	3·870877
7	49	343	2·645751	1·912931	59	3481	205379	7·681146	3·892996
8	64	512	2·828427	2·000000	60	3600	216000	7·745967	3·914868
9	81	729	3·000000	2·080084	61	3721	226981	7·810250	3·936497
10	100	1000	3·162278	2·154435	62	3844	238328	7·874008	3·957892
11	121	1331	3·316625	2·223980	63	3969	250047	7·937254	3·979057
12	144	1728	3·464102	2·289429	64	4096	262144	8·000000	4·000000
13	169	2197	3·605551	2·351335	65	4225	274625	8·062258	4·020726
14	196	2744	3·741657	2·410142	66	4356	287496	8·124038	4·041240
15	225	3375	3·872983	2·466212	67	4489	300763	8·185353	4·061548
16	256	4096	4·000000	2·519842	68	4624	314432	8·246211	4·081655
17	289	4913	4·123106	2·571282	69	4761	328509	8·306624	4·101566
18	324	5832	4·242641	2·620741	70	4900	343000	8·366600	4·121285
19	361	6859	4·358899	2·668402	71	5041	357911	8·426150	4·140818
20	400	8000	4·472136	2·714418	72	5184	373248	8·485281	4·160168
21	441	9261	4·582576	2·758924	73	5329	389017	8·544004	4·179339
22	484	10648	4·690416	2·802039	74	5476	405224	8·602325	4·198336
23	529	12167	4·795832	2·843867	75	5625	421875	8·660254	4·217163
24	576	13824	4·898979	2·884499	76	5776	438976	8·717798	4·235824
25	625	15625	5·000000	2·924018	77	5929	456533	8·774964	4·254321
26	676	17576	5·099020	2·962496	78	6084	474552	8·831761	4·272659
27	729	19683	5·196152	3·000000	79	6241	493039	8·888194	4·290840
28	784	21952	5·291503	3·036589	80	6400	512000	8·944272	4·308870
29	841	24389	5·385165	3·072317	81	6561	531441	9·000000	4·326749
30	900	27000	5·477226	3·107232	82	6724	551368	9·055385	4·344481
31	961	29791	5·567764	3·141381	83	6889	571787	9·110434	4·362071
32	1024	32768	5·656854	3·174802	84	7056	592704	9·165151	4·379519
33	1089	35937	5·744563	3·207534	85	7225	614125	9·219544	4·396830
34	1156	39304	5·830952	3·239612	86	7396	636056	9·273618	4·414005
35	1225	42875	5·916080	3·271066	87	7569	658503	9·327379	4·431048
36	1296	46656	6·000000	3·301927	88	7744	681472	9·380832	4·447960
37	1369	50653	6·082763	3·332222	89	7921	704969	9·433981	4·464745
38	1444	54872	6·164414	3·361975	90	8100	729000	9·486833	4·481405
39	1521	59319	6·244998	3·391211	91	8281	753571	9·539392	4·497941
40	1600	64000	6·324555	3·419952	92	8464	778688	9·591663	4·514357
41	1681	68921	6·403124	3·448217	93	8649	804357	9·643651	4·530655
42	1764	74088	6·480741	3·476027	94	8836	830584	9·695360	4·546836
43	1849	79507	6·557439	3·503398	95	9025	857375	9·746794	4·562903
44	1936	85184	6·633250	3·530348	96	9216	884736	9·797959	4·578857
45	2025	91125	6·708204	3·556893	97	9409	912673	9·848858	4·594701
46	2116	97336	6·782330	3·583048	98	9604	941192	9·899495	4·610436
47	2209	103823	6·855655	3·608826	99	9801	970299	9·949874	4·626065
48	2304	110592	6·928203	3·634241	100	10000	1000000	10·000000	4·641589
49	2401	117649	7·000000	3·659306	101	10201	1030301	10·049876	4·657010
50	2500	125000	7·071068	3·684031	102	10404	1061208	10·099505	4·672329
51	2601	132651	7·141428	3·708430	103	10609	1092727	10·148892	4·687548
52	2704	140608	7·211103	3·732511	104	10816	1124864	10·198039	4·702669

No.	Square.	Cube.	Square Root.	Cube Root.	No.	Square.	Cube.	Square Root.	Cube Root.
105	11025	1157625	10·246951	4·717694	169	28561	4826809	13·000000	5·52877
106	11236	1191016	10·295630	4·732624	170	28900	4913000	13·038405	5·539658
107	11449	1225043	10·344080	4·747459	171	29241	5000211	13·076697	5·550499
108	11664	1259712	10·392305	4·762203	172	29584	5088448	13·114877	5·561298
109	11881	1295029	10·440306	4·776856	173	29929	5177717	13·152946	5·57205
110	12100	1331000	10·488088	4·791420	174	30276	5268024	13·190906	5·58277
111	12321	1367631	10·535654	4·805896	175	30625	5359375	13·228757	5·59344
112	12544	1404928	10·583005	4·820284	176	30976	5451776	13·266499	5·60407
113	12769	1442897	10·630146	4·834588	177	31329	5545233	13·304135	5·61467
114	12996	1481544	10·677078	4·848808	178	31684	5639752	13·341664	5·62522
115	13225	1520875	10·723805	4·862944	179	32041	5735339	13·379088	5·635741
116	13456	1560896	10·770330	4·876999	180	32400	5832000	13·416408	5·64621
117	13689	1601613	10·816654	4·890973	181	32761	5929741	13·453624	5·65665
118	13924	1643032	10·862780	4·904868	182	33124	6028568	13·490738	5·66705
119	14161	1685159	10·908712	4·918685	183	33489	6128487	13·527749	5·67741
120	14400	1728000	10·954451	4·932424	184	33856	6229504	13·564660	5·68773
121	14641	1771561	11·000000	4·946087	185	34225	6331625	13·601470	5·69801
122	14884	1815848	11·045361	4·959676	186	34596	6434856	13·638182	5·70826
123	15129	1860867	11·090536	4·973190	187	34969	6539203	13·674794	5·71847
124	15376	1906624	11·135529	4·986631	188	35344	6644672	13·711309	5·72865
125	15625	1953125	11·180340	5·000000	189	35721	6751269	13·747727	5·73879
126	15876	2000376	11·224972	5·013298	190	36100	6859000	13·784049	5·74889
127	16129	2048383	11·269428	5·026526	191	36481	6967871	13·820275	5·75896
128	16384	2097152	11·313708	5·039684	192	36864	7077888	13·856406	5·76899
129	16641	2146689	11·357817	5·052774	193	37249	7189057	13·892444	5·77899
130	16900	2197000	11·401754	5·065797	194	37636	7301384	13·928388	5·78896
131	17161	2248091	11·445523	5·078753	195	38025	7414875	13·964240	5·79889
132	17424	2299968	11·489125	5·091643	196	38416	7529536	14·000000	5·80878
133	17689	2352637	11·532563	5·104469	197	38809	7645373	14·035669	5·81864
134	17956	2406104	11·575837	5·117230	198	39204	7762392	14·071247	5·82847
135	18225	2460375	11·618950	5·129928	199	39601	7880599	14·106736	5·83827
136	18496	2515456	11·661904	5·142563	200	40000	8000000	14·142136	5·84803
137	18769	2571353	11·704700	5·155137	201	40401	8120601	14·177447	5·85776
138	19044	2628072	11·747344	5·167649	202	40804	8242408	14·212670	5·86746
139	19321	2685619	11·789826	5·180101	203	41209	8365427	14·247807	5·87713
140	19600	2744000	11·832160	5·192494	204	41616	8489664	14·282857	5·88676
141	19881	2803221	11·874342	5·204828	205	42025	8615125	14·317821	5·89636
142	20164	2863288	11·916375	5·217103	206	42436	8741816	14·352700	5·90594
143	20449	2924207	11·958261	5·229321	207	42849	8869743	14·387495	5·91548
144	20736	2985984	12·000000	5·241483	208	43264	8998912	14·422205	5·92499
145	21025	3048625	12·041595	5·253588	209	43681	9129329	14·456832	5·93447
146	21316	3112136	12·083046	5·265637	210	44100	9261000	14·491377	5·94392
147	21609	3176523	12·124356	5·277632	211	44521	9393931	14·525839	5·95334
148	21904	3241792	12·165525	5·289572	212	44944	9528128	14·560220	5·962731
149	22201	3307949	12·206556	5·301459	213	45369	9663597	14·594520	5·97209
150	22500	3375000	12·247449	5·313293	214	45796	9800344	14·628739	5·98142
151	22801	3442951	12·288206	5·335074	215	46225	9938375	14·662878	5·99072
152	23104	3511808	12·328828	5·336803	216	46656	10077696	14·696939	6·00000
153	23409	3581577	12·369317	5·348481	217	47089	10218313	14·730920	6·009244
154	23716	3652264	12·409674	5·360108	218	47524	10360232	14·764823	6·01846
155	24025	3723875	12·449900	5·371685	219	47961	10503459	14·798649	6·02765
156	24336	3796416	12·489996	5·383213	220	48400	10648000	14·832397	6·03681
157	24649	3869893	12·529964	5·394691	221	48841	10793861	14·866069	6·04594
158	24964	3944312	12·569805	5·406120	222	49284	10941048	14·899664	6·05504
159	25281	4019679	12·609520	5·417501	223	49729	11089567	14·933185	6·06412
160	25600	4096000	12·649111	5·428835	224	50176	11239424	14·966630	6·07317
161	25921	4173281	12·688577	5·440122	225	50625	11390625	15·000000	6·082201
162	26244	4251528	12·727922	5·451362	226	51076	11543176	15·033296	6·09119
163	26569	4330747	12·767145	5·462556	227	51529	11697083	15·066519	6·10017
164	26896	4410944	12·806248	5·473704	228	51984	11852352	15·099669	6·109115
165	27225	4492125	12·845233	5·484807	229	52441	12008989	15·132746	6·11803
166	27556	4574296	12·884099	5·495865	230	52900	12167000	15·165751	6·12692
167	27889	4657463	12·922848	5·506878	231	53361	12326391	15·198684	6·13579
168	28224	4741632	12·961481	5·517848	232	53824	12487168	15·231546	6·144634

No.	Square.	Cube.	Square Root.	Cube Root.	No.	Square.	Cube.	Square Root.	Cube Root.
233	54289	12649337	15·264338	6·153449	297	88209	26198073	17·233688	6·671940
234	54756	12812904	15·297059	6·162240	298	88804	26463592	17·262676	6·679420
235	55225	12977875	15·329710	6·171006	299	89401	26730899	17·291617	6·686883
236	55696	13144256	15·362292	6·179747	300	90000	27000000	17·320508	6·694329
237	56169	13312053	15·394804	6·188463	301	90601	27270901	17·349352	6·701759
238	56644	13481272	15·427249	6·197154	302	91204	27543608	17·378147	6·709173
239	57121	13651919	15·459625	6·205822	303	91809	27818127	17·406895	6·716570
240	57600	13824000	15·491933	6·214465	304	92416	28094464	17·435596	6·723951
241	58081	13997521	15·524175	6·223084	305	93025	28372625	17·464249	6·731316
242	58564	14172488	15·556349	6·231680	306	93636	28652616	17·492856	6·738664
243	59049	14348907	15·588457	6·240251	307	94249	28934443	17·521416	6·745997
244	59536	14526784	15·620499	6·248800	308	94864	29218112	17·549929	6·753313
245	60025	14706125	15·652476	6·257325	309	95481	29503629	17·578396	6·760614
246	60516	14886936	15·684387	6·265827	310	96100	29791000	17·606817	6·767899
247	61009	15069223	15·716234	6·274305	311	96721	30080231	17·635192	6·775169
248	61504	15252992	15·748016	6·282761	312	97344	30371328	17·663522	6·782423
249	62001	15438249	15·779734	6·291195	313	97969	30664297	17·691806	6·789661
250	62500	15625000	15·811388	6·299605	314	98596	30959144	17·720045	6·796884
251	63001	15813251	15·842980	6·307994	315	99225	31255875	17·748239	6·804092
252	63504	16003008	15·874508	6·316360	316	99856	31554496	17·776389	6·811285
253	64009	16194277	15·905974	6·324704	317	100489	31855013	17·804494	6·818462
254	64516	16387064	15·937378	6·333026	318	101124	32157432	17·832555	6·825624
255	65025	16581375	15·968719	6·341326	319	101761	32461759	17·860571	6·832771
256	65536	16777216	16·000000	6·349604	320	102400	32768000	17·888544	6·839904
257	66049	16974593	16·031220	6·357861	321	103041	33076161	17·916473	6·847021
258	66564	17173512	16·062378	6·366097	322	103684	33386248	17·944358	6·854124
259	67081	17373979	16·093477	6·374311	323	104329	33698267	17·972201	6·861212
260	67600	17576000	16·124516	6·382504	324	104976	34012224	18·000000	6·868285
261	68121	17779581	16·155494	6·390676	325	105625	34328125	18·027756	6·875344
262	68644	17984728	16·186414	6·398828	326	106276	34645976	18·055470	6·882389
263	69169	18191447	16·217275	6·406958	327	106929	34965783	18·083141	6·889419
264	69696	18399744	16·248077	6·415069	328	107584	35287552	18·110770	6·896435
265	70225	18609625	16·278821	6·423158	329	108241	35611289	18·138357	6·903436
266	70756	18821096	16·309506	6·431228	330	108900	35937000	18·165902	6·910423
267	71289	19034163	16·340135	6·439277	331	109561	36264691	18·193405	6·917396
268	71824	19248832	16·370706	6·447306	332	110224	36594368	18·220867	6·924356
269	72361	19465109	16·401220	6·455315	333	110889	36926037	18·248288	6·931301
270	72900	19683000	16·431677	6·463304	334	111556	37259704	18·275667	6·938232
271	73441	19902511	16·462078	6·471274	335	112225	37595375	18·303005	6·945150
272	73984	20123648	16·492423	6·479224	336	112896	37933056	18·330303	6·952053
273	74529	20346417	16·522712	6·487154	337	113569	38272753	18·357560	6·958943
274	75076	20570824	16·552945	6·495065	338	114244	38614472	18·384776	6·965820
275	75625	20796875	16·583124	6·502957	339	114921	38958219	18·411953	6·972683
276	76176	21024576	16·613248	6·510830	340	115600	39304000	18·439089	6·979532
277	76729	21253933	16·643317	6·518684	341	116281	39651821	18·466185	6·986368
278	77284	21484952	16·673332	6·526519	342	116964	40001688	18·493242	6·993491
279	77841	21717639	16·703293	6·534335	343	117649	40353607	18·520259	7·000000
280	78400	21952000	16·733201	6·542133	344	118336	40707584	18·547237	7·006796
281	78961	22188041	16·763055	6·549911	345	119025	41063625	18·574176	7·013579
282	79524	22425768	16·792856	6·557672	346	119716	41421736	18·601075	7·020349
283	80089	22665187	16·822604	6·565414	347	120409	41781923	18·627936	7·027106
284	80656	22906304	16·852300	6·573139	348	121104	42144192	18·654758	7·033850
285	81225	23149125	16·881943	6·580844	349	121801	42508549	18·681542	7·040581
286	81796	23393656	16·911535	6·588532	350	122500	42875000	18·708287	7·047299
287	82369	23639903	16·941074	6·596202	351	123201	43243551	18·734994	7·054004
288	82944	23887872	16·970563	6·603854	352	123904	43614208	18·761663	7·060697
289	83521	24137569	17·000000	6·611489	353	124609	43986977	18·788294	7·067377
290	84100	24389000	17·029386	6·619106	354	125316	44361864	18·814888	7·074044
291	84681	24642171	17·058722	6·626705	355	126025	44738875	18·841444	7·080699
292	85264	24897088	17·088008	6·634287	356	126736	45118016	18·867962	7·087341
293	85849	25153757	17·117243	6·641852	357	127449	45499293	18·894444	7·093971
294	86436	25412184	17·146428	6·649400	358	128164	45882712	18·920888	7·100588
295	87025	25672375	17·175564	6·656930	359	128881	46268279	18·947295	7·107194
296	87616	25934336	17·204651	6·664444	360	129600	46656000	18·973666	7·113787

No.	Square.	Cube.	Square Root.	Cube Root.	No.	Square.	Cube.	Square Root.	Cube Root.
361	130321	47045881	19·000000	7·120367	425	180625	76765625	20·615528	7·51847
362	131044	47437928	19·026298	7·126936	426	181476	77308776	20·639767	7·52436
363	131769	47832147	19·052559	7·133492	427	182329	77854483	20·663978	7·53024
364	132496	48228544	19·078784	7·140037	428	183184	78402752	20·688161	7·53612
365	133225	48627125	19·104973	7·146569	429	184041	78953589	20·712315	7·54198
366	133956	49027896	19·131127	7·153090	430	184900	79507000	20·736441	7·54784
367	134689	49430863	19·157244	7·159599	431	185761	80062991	20·760540	7·55368
368	135424	49836032	19·183326	7·166096	432	186624	80621568	20·784610	7·55952
369	136161	50243409	19·209373	7·172581	433	187489	81182737	20·808652	7·56535
370	136900	50653000	19·235384	7·179054	434	188356	81746504	20·832667	7·57117
371	137641	51064811	19·261360	7·185516	435	189225	82312875	20·856654	7·57698
372	138384	51478848	19·287302	7·191966	436	190096	82881856	20·880613	7·58278
373	139129	51895117	19·313208	7·198405	437	190969	83453453	20·904545	7·58857
374	139876	52313624	19·339080	7·204832	438	191844	84027672	20·928450	7·59436
375	140625	52734375	19·364917	7·211248	439	192721	84604519	20·952327	7·60013
376	141376	53157376	19·390719	7·217652	440	193600	85184000	20·976177	7·60590
377	142129	53582633	19·416488	7·224045	441	194481	85766121	21·000000	7·61166
378	142884	54010152	19·442222	7·230427	442	195364	86350888	21·023796	7·61741
379	143641	54439939	19·467922	7·236797	443	196249	86938307	21·047565	7·62315
380	144400	54872000	19·493589	7·243156	444	197136	87528384	21·071308	7·62888
381	145161	55306341	19·519221	7·249504	445	198025	88121125	21·095023	7·63460
382	145924	55742968	19·544820	7·255841	446	198916	88716536	21·118712	7·64032
383	146689	56181887	19·570386	7·262167	447	199809	89314623	21·142375	7·64602
384	147456	56623104	19·595918	7·268482	448	200704	89915392	21·166011	7·65172
385	148225	57066625	19·621417	7·274786	449	201601	90518849	21·189620	7·65741
386	148996	57512456	19·646883	7·281079	450	202500	91125000	21·213203	7·66309
387	149769	57960603	19·672316	7·287362	451	203401	91733851	21·236761	7·66876
388	150544	58411072	19·697716	7·293633	452	204304	92345408	21·260292	7·67443
389	151321	58863869	19·723083	7·299894	453	205209	92959677	21·283797	7·68008
390	152100	59319000	19·748418	7·306144	454	206116	93576664	21·307276	7·68573
391	152881	59776471	19·773720	7·312383	455	207025	94196375	21·330729	7·69137
392	153664	60236288	19·798990	7·318611	456	207936	94818816	21·354157	7·69700
393	154449	60698457	19·824228	7·324829	457	208849	95443993	21·377558	7·70262
394	155236	61162984	19·849433	7·331037	458	209764	96071912	21·400935	7·70823
395	156025	61629875	19·874607	7·337234	459	210681	96702579	21·424285	7·71384
396	156816	62099136	19·899749	7·343420	460	211600	97336000	21·447611	7·71944
397	157609	62570773	19·924859	7·349597	461	212521	97972181	21·470911	7·72503
398	158404	63044792	19·949937	7·355762	462	213444	98611128	21·494185	7·73061
399	159201	63521199	19·974984	7·361918	463	214369	99252847	21·517435	7·73618
400	160000	64000000	20·000000	7·368063	464	215296	99897344	21·540659	7·74175
401	160801	64481201	20·024984	7·374198	465	216225	100544625	21·563859	7·74731
402	161604	64964808	20·049938	7·380323	466	217156	101194696	21·587033	7·75286
403	162409	65450827	20·074860	7·386437	467	218089	101847563	21·610183	7·75840
404	163216	65939264	20·099751	7·392542	468	219024	102503232	21·633308	7·76393
405	164025	66430125	20·124612	7·398636	469	219961	103161709	21·656408	7·76946
406	164836	66923416	20·149442	7·404721	470	220900	103823000	21·679483	7·77498
407	165649	67419143	20·174241	7·410795	471	221841	104487111	21·702534	7·78049
408	166464	67917312	20·199010	7·416859	472	222784	105154048	21·725561	7·785993
409	167281	68417929	20·223748	7·422914	473	223729	105823817	21·748563	7·791487
410	168100	68921000	20·248457	7·428959	474	224676	106496424	21·771541	7·79697
411	168921	69426531	20·273135	7·434994	475	225625	107171875	21·794495	7·802454
412	169744	69934528	20·297783	7·441019	476	226576	107850176	21·817424	7·807925
413	170569	70444997	20·322401	7·447034	477	227529	108531333	21·840330	7·813389
414	171396	70957944	20·346990	7·453040	478	228484	109215352	21·863211	7·81884
415	172225	71473375	20·371549	7·459036	479	229441	109902239	21·886069	7·824294
416	173056	71991296	20·396078	7·465022	480	230400	110592000	21·908902	7·829735
417	173889	72511713	20·420578	7·470999	481	231361	111284641	21·931712	7·83516
418	174724	73034632	20·445048	7·476966	482	232324	111980168	21·954498	7·840595
419	175561	73560059	20·469490	7·482924	483	233289	112678587	21·977261	7·84601
420	176400	74088000	20·493902	7·488872	484	234256	113379904	22·000000	7·851424
421	177241	74618461	20·518285	7·494811	485	235225	114084125	22·022716	7·85682
422	178084	75151448	20·542639	7·500741	486	236196	114791256	22·045408	7·862224
423	178929	75686967	20·566964	7·506661	487	237169	115501303	22·068077	7·867613
424	179776	76225024	20·591260	7·512571	488	238144	116214272	22·090722	7·872994

No.	Square.	Cube.	Square Root.	Cube Root.	No.	Square.	Cube.	Square Root.	Cube Root.
489	239121	116930169	22·113344	7·878368	553	305809	169112377	23·515952	8·208082
490	240100	117649000	22·135944	7·883735	554	306916	170031464	23·537205	8·213027
491	241081	118370771	22·158520	7·889095	555	308025	170953875	23·558438	8·217966
492	242064	119095488	22·181073	7·894447	556	309136	171879616	23·579652	8·222898
493	243049	119823157	22·203603	7·899792	557	310249	172808693	23·600847	8·227825
494	244036	120553784	22·226111	7·905129	558	311364	173741112	23·622024	8·232746
495	245025	121287375	22·248596	7·910460	559	312481	174676879	23·643181	8·237661
496	246016	122023936	22·271058	7·915783	560	313600	175616000	23·664319	8·242571
497	247009	122763473	22·293497	7·921099	561	314721	176558481	23·685439	8·247474
498	248004	123505992	22·315914	7·926408	562	315844	177504328	23·706539	8·252371
499	249001	124251499	22·338308	7·931710	563	316969	178453547	23·727621	8·257263
500	250000	125000000	22·360680	7·937005	564	318096	179406144	23·748684	8·262149
501	251001	125751501	22·383029	7·942293	565	319225	180362125	23·769729	8·267029
502	252004	126506008	22·405357	7·947574	566	320356	181321496	23·790755	8·271904
503	253009	127263527	22·427662	7·952848	567	321489	182284263	23·811762	8·276773
504	254016	128024064	22·449944	7·958114	568	322624	183250432	23·832751	8·281635
505	255025	128787625	22·472205	7·963374	569	323761	184220009	23·853721	8·286493
506	256036	129554216	22·494444	7·968627	570	324900	185193000	23·874673	8·291344
507	257049	130323843	22·516661	7·973873	571	326041	186169411	23·895606	8·296190
508	258064	131096512	22·538855	7·979112	572	327184	187149248	23·916522	8·301030
509	259081	131872229	22·561028	7·984344	573	328329	188132517	23·937418	8·305865
510	260100	132651000	22·583180	7·989570	574	329476	189119224	23·958297	8·310694
511	261121	133432831	22·605309	7·994788	575	330625	190109375	23·979158	8·315517
512	262144	134217728	22·627417	8·000000	576	331776	191102976	24·000000	8·320335
513	263169	135005697	22·649503	8·005205	577	332929	192100033	24·020824	8·325147
514	264196	135796744	22·671568	8·010403	578	334084	193100552	24·041631	8·329954
515	265225	136590875	22·693611	8·015595	579	335241	194104539	24·062419	8·334755
516	266256	137388096	22·715633	8·020779	580	336400	195112000	24·083189	8·339551
517	267289	138188413	22·737634	8·025957	581	337561	196122941	24·103942	8·344341
518	268324	138991832	22·759613	8·031129	582	338724	197137368	24·124676	8·349126
519	269361	139798359	22·781572	8·036293	583	339889	198155287	24·145393	8·353905
520	270400	140608000	22·803509	8·041451	584	341056	199176704	24·166092	8·358678
521	271441	141420761	22·825424	8·046603	585	342225	200201625	24·186773	8·363447
522	272484	142236648	22·847319	8·051748	586	343396	201230056	24·207437	8·368209
523	273529	143055667	22·869193	8·056886	587	344569	202262003	24·228083	8·372967
524	274576	143877824	22·891046	8·062018	588	345744	203297472	24·248711	8·377719
525	275625	144703125	22·912879	8·067143	589	346921	204336469	24·269322	8·382465
526	276676	145531576	22·934690	8·072262	590	348100	205379000	24·289916	8·387206
527	277729	146363183	22·956481	8·077374	591	349281	206425071	24·310492	8·391942
528	278784	147197952	22·978251	8·082480	592	350464	207474688	24·331050	8·396673
529	279841	148035889	23·000000	8·087579	593	351649	208527857	24·351591	8·401398
530	280900	148877000	23·021729	8·092672	594	352836	209584584	24·372115	8·406118
531	281961	149721291	23·043437	8·097759	595	354025	210644875	24·392622	8·410833
532	283024	150568768	23·065125	8·102839	596	355216	211708736	24·413111	8·415542
533	284089	151419437	23·086793	8·107913	597	356409	212776173	24·433583	8·420246
534	285156	152273304	23·108440	8·112980	598	357604	213847192	24·454039	8·424945
535	286225	153130375	23·130067	8·118041	599	358801	214921799	24·474477	8·429638
536	287296	153990656	23·151674	8·123096	600	360000	216000000	24·494897	8·434327
537	288369	154854153	23·173261	8·128145	601	361201	217081801	24·515301	8·439010
538	289444	155720872	23·194827	8·133187	602	362404	218167208	24·535688	8·443688
539	290521	156590819	23·216374	8·138223	603	363609	219256227	24·556058	8·448360
540	291600	157464000	23·237900	8·143253	604	364816	220348864	24·576412	8·453028
541	292681	158340421	23·259407	8·148276	605	366025	221445125	24·596748	8·457691
542	293764	159220088	23·280894	8·153294	606	367236	222545016	24·617067	8·462348
543	294849	160103007	23·302360	8·158305	607	368449	223648543	24·637370	8·467000
544	295936	160989184	23·323808	8·163310	608	369664	224755712	24·657656	8·471647
545	297025	161878625	23·345235	8·168309	609	370881	225866529	24·677925	8·476289
546	298116	162771336	23·366643	8·173302	610	372100	226981000	24·698178	8·480926
547	299209	163667323	23·288031	8·178289	611	373321	228099131	24·718414	8·485558
548	300304	164566592	23·409400	8·183269	612	374544	229220928	24·738634	8·490185
549	301401	165469149	23·430749	8·188244	613	375769	230346397	24·758837	8·494806
550	302500	166375000	23·452079	8·193213	614	376996	231475544	24·779023	8·499423
551	303601	167284151	23·473389	8·198175	615	378225	232608375	24·799194	8·504035
552	304704	168196608	23·494680	8·203132	616	379456	233744896	24·819347	8·508642

No.	Square.	Cube.	Square Root.	Cube Root.	No.	Square.	Cube.	Square Root.	Cube Root.
617	380689	234885113	24·839485	8·513243	681	463761	315821241	26·095977	8·79796
618	381924	236029032	24·859606	8·517840	682	465124	317214568	26·115130	8·80227
619	383161	237176659	24·879711	8·522432	683	466489	318611987	26·134269	8·80657
620	384400	238328000	24·899799	8·527019	684	467856	320013504	26·153394	8·810868
621	385641	239483061	24·919872	8·531601	685	469225	321419125	26·172505	8·81516
622	386884	240641848	24·939928	8·536178	686	470596	322828856	26·191602	8·819447
623	388129	241804367	24·959968	8·540750	687	471969	324242703	26·210685	8·82373
624	389376	242970624	24·979992	8·545317	688	473344	325660672	26·229754	8·82801
625	390625	244140625	25·000000	8·549880	689	474721	327082769	26·248810	8·83228
626	391876	245314376	25·019992	8·554437	690	476100	328509000	26·267851	8·83655
627	393129	246491883	25·039968	8·558990	691	477481	329939371	26·286879	8·84082
628	394384	247673152	25·059928	8·563538	692	478864	331373888	26·305893	8·84508
629	395641	248858189	25·079872	8·568081	693	480249	332812557	26·324893	8·84934
630	396900	250047000	25·099801	8·572619	694	481636	334255384	26·343880	8·85359
631	398161	251239591	25·119713	8·577152	695	483025	335702375	26·362853	8·85784
632	399424	252435968	25·139610	8·581681	696	484416	337153536	26·381812	8·86209
633	400689	253636137	25·159491	8·586205	697	485809	338608873	26·400758	8·86633
634	401956	254840104	25·179357	8·590724	698	487204	340068392	26·419690	8·87057
635	403225	256047875	25·199206	8·595238	699	488601	341532099	26·438608	8·87481
636	404496	257259456	25·219040	8·599748	700	490000	343000000	26·457513	8·87904
637	405769	258474853	25·238859	8·604252	701	491401	344472101	26·476405	8·88326
638	407044	259694072	25·258662	8·608753	702	492804	345948408	26·495283	8·88748
639	408321	260917119	25·278449	8·613248	703	494209	347428927	26·514147	8·89170
640	409600	262144000	25·298221	8·617739	704	495616	348913664	26·532998	8·89592
641	410881	263374721	25·317978	8·622225	705	497025	350402625	26·551836	8·90013
642	412164	264609288	25·337719	8·626706	706	498436	351895816	26·570661	8·904337
643	413449	265847707	25·357445	8·631183	707	499849	353393243	26·589472	8·908539
644	414736	267089984	25·377155	8·635655	708	501264	354894912	26·608269	8·912737
645	416025	268336125	25·396850	8·640123	709	502681	356400829	26·627054	8·91693
646	417316	269586136	25·416530	8·644585	710	504100	357911000	26·645825	8·92112
647	418609	270840023	25·436195	8·649044	711	505521	359425431	26·664583	8·92530
648	419904	272097792	25·455844	8·653497	712	506944	360944128	26·683328	8·92949
649	421201	273359449	25·475478	8·657946	713	508369	362467097	26·702060	8·93366
650	422500	274625000	25·495098	8·662391	714	509796	363994344	26·720778	8·937843
651	423801	275894451	25·514702	8·666831	715	511225	365525875	26·739484	8·94201
652	425104	277167808	25·534291	8·671266	716	512656	367061696	26·758176	8·94618
653	426409	278445077	25·553865	8·675697	717	514089	368601813	26·776856	8·95034
654	427716	279726264	25·573424	8·680124	718	515524	370146232	26·795522	8·954503
655	429025	281011375	25·592968	8·684546	719	516961	371694959	26·814175	8·958658
656	430336	282300416	25·612497	8·688963	720	518400	373248000	26·832816	8·962809
657	431649	283593393	25·632011	8·693376	721	519841	374805361	26·851443	8·966957
658	432964	284890312	25·651511	8·697784	722	521284	376367048	26·870058	8·97110
659	434281	286191179	25·670995	8·702188	723	522729	377933067	26·888659	8·975241
660	435600	287496000	25·690465	8·706588	724	524176	379503424	26·907248	8·979377
661	436921	288804781	25·709920	8·710983	725	525625	381078125	26·925824	8·983509
662	438244	290117528	25·729361	8·715373	726	527076	382657176	26·944387	8·987637
663	439569	291434247	25·748786	8·719760	727	528529	384240583	26·962938	8·991762
664	440896	292754944	25·768198	8·724141	728	529984	385828352	26·981475	8·995883
665	442225	294079625	25·787594	8·728519	729	531441	387420489	27·000000	9·000000
666	443556	295408296	25·806976	8·732892	730	532900	389017000	27·018512	9·00411
667	444889	296740963	25·826343	8·737260	731	534361	390617891	27·037012	9·00822
668	446224	298077632	25·845696	8·741625	732	535824	392223168	27·055499	9·012329
669	447561	299418309	25·865034	8·745985	733	537289	393832837	27·073973	9·016431
670	448900	300763000	25·884358	8·750340	734	538756	395446904	27·092434	9·020529
671	450241	302111711	25·903668	8·754691	735	540225	397065375	27·110883	9·024624
672	451584	303464448	25·922963	8·759038	736	541696	398688256	27·129320	9·028715
673	452929	304821217	25·942244	8·763381	737	543169	400315553	27·147744	9·032802
674	454276	306182024	25·961510	8·767719	738	544644	401947272	27·166155	9·036886
675	455625	307546875	25·980762	8·772053	739	546121	403583419	27·184554	9·040965
676	456976	308915776	26·000000	8·776383	740	547600	405224000	27·202941	9·045042
677	458329	310288733	26·019224	8·780708	741	549081	406869021	27·221315	9·049114
678	459684	311665752	26·038433	8·785030	742	550564	408518488	27·239677	9·053183
679	461041	313046839	26·057628	8·789347	743	552049	410172407	27 258026	9·057248
680	462400	314432000	26·076810	8·793659	744	553536	411830784	27·276363	9·061310

No.	Square.	Cube.	Square Root.	Cube Root.	No.	Square.	Cube.	Square Root.	Cube Root.
745	555025	413493625	27·294688	9·065368	809	654481	529475129	28·442925	9·317860
746	556516	415160936	27·313001	9·069422	810	656100	531441000	28·460499	9·321697
747	558009	416832723	27·331301	9·073473	811	657721	533411731	28·478062	9·325532
748	559504	418508992	27·349589	9·077520	812	659344	535387328	28·495614	9·329363
749	561001	420189749	27·367864	9·081563	813	660969	537367797	28·513155	9·333192
750	562500	421875000	27·386128	9·085603	814	662596	539353144	28·530685	9·337017
751	564001	423564751	27·404379	9·089639	815	664225	541343375	28·548205	9·340839
752	565504	425259008	27·422618	9·093672	816	665856	543338496	28·565714	9·344657
753	567009	426957777	27·440846	9·097701	817	667489	545338513	28·583212	9·348473
754	568516	428661064	27·459060	9·101726	818	669124	547343432	28·600699	9·352286
755	570025	430368875	27·477263	9·105748	819	670761	549353259	28·618176	9·356095
756	571536	432081216	27·495454	9·109767	820	672400	551368000	28·635642	9·359902
757	573049	433798093	27·513633	9·113782	821	674041	553387661	28·653098	9·363705
758	574564	435519512	27·531800	9·117793	822	675684	555412248	28·670542	9·367505
759	576081	437245479	27·549955	9·121801	823	677329	557441767	28·687977	9·371302
760	577600	438976000	27·568098	9·125805	824	678976	559476224	28·705400	9·375096
761	579121	440711081	27·586228	9·129806	825	680625	561515625	28·722813	9·378887
762	580644	442450728	27·604348	9·133803	826	682276	563559976	28·740216	9·382675
763	582169	444194947	27·622455	9·137797	827	683929	565609283	28·757608	9·386460
764	583696	445943744	27·640550	9·141788	828	685584	567663552	28·774989	9·390242
765	585225	447697125	27·658633	9·145774	829	687241	569722789	28·792360	9·394021
766	586756	449455096	27·676705	9·149758	830	688900	571787000	28·809721	9·397796
767	588289	451217663	27·694765	9·153737	831	690561	573856191	28·827071	9·401569
768	589824	452984832	27·712813	9·157714	832	692224	575930368	28·844410	9·405339
769	591361	454756609	27·730849	9·161687	833	693889	578009537	28·861739	9·409105
770	592900	456533000	27·748874	9·165656	834	695556	580093704	28·879058	9·412869
771	594441	458314011	27·766887	9·169622	835	697225	582182875	28·896367	9·416630
772	595984	460099648	27·784888	9·173585	836	698896	584277056	28·913665	9·420387
773	597529	461889917	27·802878	9·177544	837	700569	586376253	28·930952	9·424142
774	599076	463684824	27·820856	9·181500	838	702244	588480472	28·948230	9·427894
775	600625	465484375	27·838822	9·185453	839	703921	590589719	28·965497	9·431642
776	602176	467288576	27·856777	9·189402	840	705600	592704000	28·982754	9·435388
777	603729	469097433	27·874720	9·193347	841	707281	594823321	29·000000	9·439131
778	605284	470910952	27·892651	9·197290	842	708964	596947688	29·017236	9·442870
779	606841	472729139	27·910572	9·201229	843	710649	599077107	29·034462	9·446607
780	608400	474552000	27·928480	9·205164	844	712336	601211584	29·051678	9·450341
781	609961	476379541	27·946377	9·209096	845	714025	603351125	29·068884	9·454072
782	611524	478211768	27·964263	9·213025	846	715716	605495736	29·086079	9·457800
783	613089	480048687	27·982137	9·216950	847	717409	607645423	29·103264	9·461525
784	614656	481890304	28·000000	9·220873	848	719104	609800192	29·120440	9·465247
785	616225	483736625	28·017852	9·224791	849	720801	611960049	29·137605	9·468966
786	617796	485587656	28·035692	9·228707	850	722500	614125000	29·154760	9·472682
787	619369	487443403	28·053520	9·232619	851	724201	616295051	29·171904	9·476396
788	620944	489303872	28·071338	9·236528	852	725904	618470208	29·189039	9·480106
789	622521	491169069	28·089144	9·240433	853	727609	620650477	29·206164	9·483814
790	624100	493039000	28·106939	9·244335	854	729316	622835864	29·223278	9·487518
791	625681	494913671	28·124722	9·248234	855	731025	625026375	29·240383	9·491220
792	627264	496793088	28·142495	9·252130	856	732736	627222016	29·257478	9·494919
793	628849	498677257	28·160256	9·256022	857	734449	629422793	29·274562	9·498615
794	630436	500566184	28·178006	9·259911	858	736164	631628712	29·291637	9·502308
795	632025	502459875	28·195744	9·263797	859	737881	633839779	29·308702	9·505998
796	633616	504358336	28·213472	9·267680	860	739600	636056000	29·325757	9·509685
797	635209	506261573	28·231188	9·271559	861	741321	638277381	29·342802	9·513370
798	636804	508169592	28·248894	9·275435	862	743044	640503928	29·359837	9·517051
799	638401	510082399	28·266588	9·279308	863	744769	642735647	29·376862	9·520730
800	640000	512000000	28·284271	9·283178	864	746496	644972544	29·393877	9·524406
801	641601	513922401	28·301943	9·287044	865	748225	647214625	29·410882	9·528079
802	643204	515849608	28·319605	9·290907	866	749956	649461896	29·427878	9·531750
803	644809	517781627	28·337255	9·294767	867	751689	651714363	29·444864	9·535417
804	646416	519718464	28·354894	9·298624	868	753424	653972032	29·461840	9·539082
805	648025	521660125	28·372522	9·302477	869	755161	656234909	29·478806	9·542744
806	649636	523606616	28·390139	9·306328	870	756900	658503000	29·495762	9·546403
807	651249	525557943	28·407745	9·310175	871	758641	660776311	29·512709	9·550059
808	652864	527514112	28·425341	9·314019	872	760384	663054848	29·529646	9·553712

No.	Square.	Cube.	Square Root.	Cube Root.	No.	Square.	Cube.	Square Root.	Cube Root.
873	762129	665338617	29·546573	9·557363	937	877969	822656953	30·610456	9·785429
874	763876	667627624	29·563491	9·561011	938	879844	825293672	30·626786	9·788909
875	765625	669921875	29·580399	9·564656	939	881721	827936019	30·643107	9·792386
876	767376	672221376	29·597297	9·568298	940	883600	830584000	30·659419	9·795861
877	769129	674526133	29·614186	9·571938	941	885481	833237621	30·675723	9·799334
878	770884	676836152	29·631065	9·575574	942	887364	835896888	30·692019	9·802804
879	772641	679151439	29·647934	9·579208	943	889249	838561807	30·708305	9·806271
880	774400	681472000	29·664794	9·582840	944	891136	841232384	30·724583	9·809736
881	776161	683797841	29·681644	9·586468	945	893025	843908625	30·740852	9·813199
882	777924	686128968	29·698485	9·590094	946	894916	846590536	30·757113	9·816659
883	779689	688465387	29·715316	9·593717	947	896809	849278123	30·773365	9·820117
884	781456	690807104	29·732138	9·597337	948	898704	851971392	30·789609	9·823572
885	783225	693154125	29·748950	9·600955	949	900601	854670349	30·805844	9·827025
886	784996	695506456	29·765752	9·604570	950	902500	857375000	30·822070	9·830476
887	786769	697864103	29·782545	9·608182	951	904401	860085351	30·838288	9·833924
888	788544	700227072	29·799329	9·611791	952	906304	862801408	30·854497	9·837369
889	790321	702595369	29·816103	9·615398	953	908209	865523177	30·870698	9·840813
890	792100	704969000	29·832868	9·619002	954	910116	868250664	30·886890	9·844254
891	793881	707347971	29·849623	9·622603	955	912025	870983875	30·903074	9·847692
892	795664	709732288	29·866369	9·626202	956	913936	873722816	30·919250	9·851128
893	797449	712121957	29·883106	9·629797	957	915849	876467493	30·935417	9·854562
894	799236	714516984	29·899833	9·633391	958	917764	879217912	30·951575	9·857993
895	801025	716917375	29·916551	9·636981	959	919681	881974079	30·967725	9·861422
896	802816	719323136	29·933259	9·640569	960	921600	884736000	30·983867	9·864848
897	804609	721734273	29·949958	9·644154	961	923521	887503681	31·000000	9·868272
898	806404	724150792	29·966648	9·647737	962	925444	890277128	31·016125	9·871694
899	808201	726572699	29·983329	9·651317	963	927369	893056347	31·032241	9·875113
900	810000	729000000	30·000000	9·654894	964	929296	895841344	31·048349	9·878530
901	811801	731432701	30·016662	9·658468	965	931225	898632125	31·064449	9·881945
902	813604	733870808	30·033315	9·662040	966	933156	901428696	31·080541	9·885357
903	815409	736314327	30·049958	9·665610	967	935089	904231063	31·096624	9·888767
904	817216	738763264	30·066593	9·669176	968	937024	907039232	31·112698	9·892175
905	819025	741217625	30·083218	9·672740	969	938961	909853209	31·128765	9·895580
906	820836	743677416	30·099834	9·676302	970	940900	912673000	31·144823	9·898983
907	822649	746142643	30·116441	9·679860	971	942841	915498611	31·160873	9·902383
908	824464	748613312	30·133038	9·683417	972	944784	918330048	31·176915	9·905782
909	826281	751089429	30·149627	9·686970	973	946729	921167317	31·192948	9·909178
910	828100	753571000	30·163206	9·690521	974	948676	924010424	31·208973	9·912571
911	829921	756058031	30·182777	9·694069	975	950625	926859375	31·224990	9·915962
912	831744	758550528	30·199338	9·697615	976	952576	929714176	31·240999	9·919351
913	833569	761048497	30·215890	9·701158	977	954529	932574833	31·256999	9·922738
914	835396	763551944	30·232433	9·704699	978	956484	935441352	31·272992	9·926122
915	837225	766060875	30·248967	9·708237	979	958441	938313739	31·288976	9·929504
916	839056	768575296	30·265492	9·711772	980	960400	941192000	31·304952	9·932884
917	840889	771095213	30·282008	9·715305	981	962361	944076141	31·320920	9·936261
918	842724	773620632	30·298515	9·718835	982	964324	946966168	31·336879	9·939636
919	844561	776151559	30·315013	9·722363	983	966289	949862087	31·352831	9·943009
920	846400	778688000	30·331502	9·725888	984	968256	952763904	31·368774	9·946380
921	848241	781229961	30·347982	9·729411	985	970225	955671625	31·384710	9·949748
922	850084	783777448	30·364453	9·732931	986	972196	958585256	31·400637	9·953114
923	851929	786330467	30·380915	9·736448	987	974169	961504803	31·416556	9·956477
924	853776	788889024	30·397368	9·739963	988	976144	964430272	31·432467	9·959839
925	855625	791453125	30·413813	9·743476	989	978121	967361669	31·448370	9·963198
926	857476	794022776	30·430248	9·746986	990	980100	970299000	31·464265	9·966555
927	859329	796597983	30·446675	9·750493	991	982081	973242271	31·480153	9·969909
928	861184	799178752	30·463092	9·753998	992	984064	976191488	31·496032	9·973262
929	863041	801765089	30·479501	9·757500	993	986049	979146657	31·511903	9·976612
930	864900	804357000	30·495901	9·761000	994	988036	982107784	31·527766	9·979960
931	866761	806954491	30·512293	9·764497	995	990025	985074875	31·543621	9·983305
932	868624	809557568	30·528675	9·767992	996	992016	988047936	31·559468	9·986649
933	870489	812166237	30·545049	9·771484	997	994009	991026973	31·575307	9·989990
934	872356	814780504	30·561414	9·774974	998	996004	994011992	31·591138	9·993329
935	874225	817400375	30·577770	9·778462	999	998001	997002999	31·606961	9·996666
936	876096	820025856	30·594117	9·782947					

A TABLE

SHOWING THE CUBIC CONTENT OF TIMBER IN A SQUARE YARD OF JOISTING, OR OTHER SCANTLINGS OF VARIOUS SIZES; AND FROM TWENTY-FOUR INCHES TO THIRTEEN INCHES FROM CENTRE TO CENTRE, UPON THE SUPPOSITION THAT AN EXTRA JOIST IS REQUIRED AT EVERY EIGHTEEN FEET SPACE.

Size of Scantling.	Central Distance.	Cubic Timber in a Square Yard.				Size of Scantling.	Cubic Timber in a Square Yard.				Size of Scantling.	Central Distance.	Cubic Timber in a Square Yard.				Size of Scantling.	Cubic Timber in a Square Yard.			
	in.	ft.	in.	pts.	sec.		ft.	in.	pts.	sec.		in.	ft.	in.	pts.	sec.		ft.	in.	pts.	sec.
12 inches by 3 inches.	24	1	3	0	0	11 inches by 3 inches.	1	1	9	0	10 inches by 3 inches.	24	1	0	6	0	9 inches by 3 inches.	0	11	3	0
	23	1	3	7	0		1	2	3	5		23	1	0	11	10		0	11	8	3
	22	1	4	2	8		1	2	10	6		22	1	1	6	3		1	0	2	0
	21	1	4	11	1		1	3	6	2		21	1	2	1	3		1	0	8	4
	20	1	5	8	4		1	4	2	8		20	1	2	9	0		1	1	3	3
	19	1	6	6	7		1	5	0	1		19	1	3	5	6		1	1	10	11
	18	1	7	6	0		1	5	10	6		18	1	4	3	0		1	2	7	6
	17	1	8	6	8		1	6	10	2		17	1	5	1	7		1	3	5	0
	16	1	9	9	0		1	7	11	3		16	1	6	1	6		1	4	3	9
	15	1	11	1	2		1	9	2	1		15	1	7	3	0		1	5	3	11
	14	2	0	7	8		1	10	7	1		14	1	8	6	5		1	6	5	9
	13	2	2	5	1		2	0	2	8		13	1	10	0	3		1	7	9	10
8 inches by 3 inches.	24	0	10	0	0	7 inches by 3 inches.	0	8	9	0	6 inches by 3 inches.	24	0	7	6	0	5 inches by 3 inches.	0	6	3	0
	23	0	10	4	8		0	9	1	1		23	0	7	9	6		0	6	5	11
	22	0	10	9	9		0	9	5	7		22	0	8	1	4		0	6	9	1
	21	0	11	3	5		0	9	10	6		21	0	8	5	6		0	7	0	7
	20	0	11	9	7		0	10	3	10		20	0	8	10	2		0	7	4	6
	19	1	0	4	5		0	10	9	10		19	0	9	3	3		0	7	8	9
	18	1	1	0	0		0	11	4	6		18	0	9	9	0		0	8	1	6
	17	1	1	8	5		0	11	11	11		17	0	10	3	4		0	8	6	9
	16	1	2	6	0		1	0	8	3		16	0	10	10	6		0	9	0	9
	15	1	3	4	9		1	1	5	8		15	0	11	6	7		0	9	7	6
	14	1	4	5	1		1	2	4	6		14	1	0	3	10		0	10	3	2
	13	1	5	7	5		1	3	5	0		13	1	1	2	6		0	11	0	1
4 inches by 3 inches.	24	0	5	0	0	3 inches by 3 inches.	0	3	9	0	12 inches by 2½ inches.	24	1	0	6	0	11 inches by 2½ inches.	0	11	5	6
	23	0	5	2	4		0	3	10	9		23	1	0	11	10		0	11	10	10
	22	0	5	4	11		0	4	0	8		22	1	1	6	3		1	0	4	9
	21	0	5	7	8		0	4	2	9		21	1	2	1	3		1	0	11	2
	20	0	5	10	9		0	4	5	1		20	1	2	9	0		1	1	6	3
	19	0	6	2	2		0	4	7	8		19	1	3	5	6		1	2	2	1
	18	0	6	6	0		0	4	10	6		18	1	4	3	0		1	2	10	9
	17	0	6	10	3		0	5	1	8		17	1	5	1	7		1	3	8	5
	16	0	7	3	0		0	5	5	3		16	1	6	1	6		1	4	7	5
	15	0	7	8	5		0	5	9	3		15	1	7	3	0		1	5	7	9
	14	0	8	2	7		0	6	1	11		14	1	8	6	5		1	6	9	11
	13	0	8	9	8		0	6	7	3		13	1	10	0	3		1	8	2	3
10 inches by 2½ inches.	24	0	10	5	0	9 inches by 2½ inches.	0	9	4	6	8 inches by 2½ inches.	24	0	8	4	0	7 inches by 2½ inches.	0	7	3	6
	23	0	10	9	11		0	9	8	11		23	0	8	7	11		0	7	6	11
	22	0	11	3	3		0	10	1	8		22	0	9	0	2		0	7	10	8
	21	0	11	9	1		0	10	6	11		21	0	9	4	10		0	8	2	9
	20	1	0	3	6		0	11	0	9		20	0	9	10	0		0	8	7	3
	19	1	0	10	7		0	11	7	2		19	0	10	3	8		0	9	0	3
	18	1	1	6	6		1	0	2	3		18	0	10	10	0		0	9	5	9
	17	1	2	3	4		1	0	10	2		17	0	11	5	1		0	9	11	11
	16	1	3	1	3		1	1	7	2		16	1	0	1	0		0	10	6	11
	15	1	4	0	6		1	2	5	3		15	1	0	10	0		0	11	2	9
	14	1	5	1	4		1	3	4	10		14	1	1	8	3		0	11	11	9
	13	1	6	4	3		1	4	6	2		13	1	2	8	2		1	0	10	2

Size of Scantling.	Central Distance.	Cubic Timber in a Square Yard.				Size of Scantling.	Cubic Timber in a Square Yard.				Size of Scantling.	Central Distance.	Cubic Timber in a Square Yard.				Size of Scantling.	Cubic Timber in a Square Yard.			
	in.	ft.	in.	pts.	sec.		ft.	in.	pts.	sec.		in.	ft.	in.	pts.	sec.		ft.	in.	pts.	sec.
6 inches by 2½ inches.	24	0	6	3	0	5 inches by 2½ inches.	0	5	2	6	4 inches by 2½ inches.	24	0	4	2	0	3 inches by 2½ inches.	0	3	1	6
	23	0	6	5	11		0	5	4	11		23	0	4	3	11		0	3	3	0
	22	0	6	9	1		0	5	7	7		22	0	4	6	1		0	3	4	7
	21	0	7	0	7		0	5	10	6		21	0	4	8	5		0	3	6	4
	20	0	7	4	6		0	6	1	9		20	0	4	11	0		0	3	8	3
	19	0	7	8	9		0	6	5	3		19	0	5	1	10		0	3	10	4
	18	0	8	1	6		0	6	9	3		18	0	5	5	0		0	4	0	9
	17	0	8	6	9		0	7	1	8		17	0	5	8	6		0	4	3	5
	16	0	9	0	9		0	7	6	8		16	0	6	0	6		0	4	6	5
	15	0	9	7	6		0	8	0	3		15	0	6	5	0		0	4	9	9
	14	0	10	3	3		0	8	6	8		14	0	6	10	2		0	5	1	7
	13	0	11	0	2		0	9	2	1		13	0	7	4	1		0	5	6	1
2 inches by 2½ inches.	24	0	2	1	0	12 inches by 2 inches.	0	10	0	0	11 inches by 2 inches.	24	0	9	2	0	10 inches by 2 inches.	0	8	4	0
	23	0	2	2	0		0	10	4	8		23	0	9	6	3		0	8	7	11
	22	0	2	3	1		0	10	9	9		22	0	9	11	0		0	9	0	2
	21	0	2	4	3		0	11	3	5		21	0	10	4	1		0	9	4	10
	20	0	2	5	6		0	11	9	7		20	0	10	9	9		0	9	10	0
	19	0	2	6	11		1	0	4	5		19	0	11	4	1		0	10	3	8
	18	0	2	8	6		1	1	0	0		18	0	11	11	0		0	10	10	0
	17	0	2	10	3		1	1	3	5		17	1	0	6	9		0	11	5	1
	16	0	3	0	3		1	2	6	0		16	1	1	3	6		1	0	1	0
	15	0	3	2	6		1	3	4	9		15	1	2	1	4		1	0	10	0
	14	0	3	5	1		1	4	5	1		14	1	3	0	9		1	1	8	4
	13	0	3	8	1		1	5	7	4		13	1	4	1	9		1	2	8	2
9 inches by 2 inches.	24	0	7	6	0	8 inches by 2 inches.	0	6	8	0	7 inches by 2 inches.	24	0	5	10	0	6 inches by 2 inches.	0	5	0	0
	23	0	7	9	6		0	6	11	1		23	0	6	0	9		0	5	2	4
	22	0	8	1	4		0	7	2	6		22	0	6	3	9		0	5	4	11
	21	0	8	5	7		0	7	6	4		21	0	6	7	0		0	5	7	9
	20	0	8	10	2		0	7	10	5		20	0	6	10	7		0	5	10	10
	19	0	9	3	3		0	8	2	11		19	0	7	2	7		0	6	2	3
	18	0	9	9	0		0	8	8	0		18	0	7	7	0		0	6	6	0
	17	0	10	3	4		0	9	1	7		17	0	8	0	0		0	6	10	3
	16	0	10	10	6		0	9	8	0		16	0	8	5	6		0	7	3	0
	15	0	11	6	7		0	10	3	2		15	0	8	11	9		0	7	8	5
	14	1	0	3	10		0	10	11	5		14	0	9	7	0		0	8	2	7
	13	1	1	2	7		0	11	8	11		13	0	10	3	4		0	8	9	8
5 inches by 2 inches.	24	0	4	2	0	4 inches by 2 inches.	0	3	4	0	3 inches by 2 inches.	24	0	2	6	0	2 inches by 2 inches.	0	1	8	0
	23	0	4	4	0		0	3	5	7		23	0	2	7	2		0	1	8	9
	22	0	4	6	1		0	3	7	3		22	0	2	8	6		0	1	9	7
	21	0	4	8	5		0	3	9	2		21	0	2	9	11		0	1	10	7
	20	0	4	11	0		0	3	11	3		20	0	2	11	5		0	1	11	7
	19	0	5	1	10		0	4	1	6		19	0	3	1	1		0	2	0	9
	18	0	5	5	0		0	4	4	0		18	0	3	3	0		0	2	2	0
	17	0	5	8	7		0	4	6	10		17	0	3	5	2		0	2	3	5
	16	0	6	0	6		0	4	10	0		16	0	3	7	6		0	2	5	0
	15	0	6	5	0		0	5	1	7		15	0	3	10	3		0	2	6	9
	14	0	6	10	2		0	5	5	9		14	0	4	1	3		0	2	8	11
	13	0	7	4	1		0	5	10	6		13	0	4	4	11		0	2	11	3
2½ inches by 2 inches.	24	0	2	1	0	3 inches by 1½ inch.	0	1	10	6	2½ inches by 1½ inch.	24	0	1	6	9	2 inches by 1½ inch.	0	1	3	0
	23	0	2	2	0		0	1	11	5		23	0	1	7	6		0	1	3	7
	22	0	2	3	1		0	2	0	4		22	0	1	8	3		0	1	4	3
	21	0	2	4	2		0	2	1	5		21	0	1	9	2		0	1	5	0
	20	0	2	5	6		0	2	2	7		20	0	1	10	2		0	1	5	9
	19	0	2	6	11		0	2	3	10		19	0	1	11	2		0	1	6	7
	18	0	2	8	6		0	2	5	3		18	0	2	0	4		0	1	7	6
	17	0	2	10	3		0	2	6	10		17	0	2	1	5		0	1	8	7
	16	0	3	0	1		0	2	8	8		16	0	2	3	2		0	1	9	9
	15	0	3	2	6		0	2	10	8		15	0	2	4	11		0	1	11	2
	14	0	3	5	1		0	3	1	0		14	0	2	6	10		0	2	0	8
	13	0	3	8	1		0	3	3	8		13	0	2	9	0		0	2	2	6

A TABLE

SHOWING THE LENGTHS OF THE SIDES OF INSCRIBED AND CIRCUMSCRIBED POLYGONS, THE DIAMETER OF THE CIRCLE BEING UNITY.

No. of Sides.	Side of Inscribed Polygon.	Side of Circumscribed Polygon.	No. of Sides.	Side of Inscribed Polygon.	Side of Circumscribed Polygon.
3	·8660254	1·7320508	21	·1490422	·1507257
4	·7071068	1·0000000	22	·1423148	·1437783
5	·5877853	·7265425	23	·1361666	·1374468
6	·5000000	·5773503	24	·1305262	·1316525
7	·4338837	·4815745	25	·1253332	·1263294
8	·3826834	·4142136	26	·1205366	·1214219
9	·3420201	·3639702	27	·1160929	·1168832
10	·3090170	·3249197	28	·1119644	·1126729
11	·2817325	·2936264	29	·1081190	·1087566
12	·2588190	·2679492	30	·1045285	·1051042
13	·2393156	·2464778	31	·1011682	·1016900
14	·2225209	·2282434	32	·0980171	·0984914
15	·2079117	·2125566	33	·0950560	·0954884
16	·1950903	·1989124	34	·0922684	·0926637
17	·1837492	·1869321	35	·0896393	·0900016
18	·1736482	·1763270	36	·0871557	·0874887
19	·1645945	·1668704	37	·0848059	·0851124
20	·1564345	·1583844			

HYPERBOLIC LOGARITHMS OF NUMBERS,

FROM 1 to 100.

No.	Logarithm.	No.	Logarithm.	No.	Logarithm.
2	0·69314718	35	3·55534806	68	4·21950771
3	1·09861229	36	3·58351694	69	4·23410650
4	1·38629436	37	3·61091791	70	4·24849524
5	1·60943791	38	3·63758616	71	4·26267988
6	1·79175947	39	3·66356165	72	4·27666612
7	1·94591015	40	3·68887945	73	4·29045944
8	2·07944154	41	3·71357207	74	4·30406509
9	2·19722458	42	3·73766962	75	4·31748811
10	2·30258509	43	3·76120012	76	4·33073334
11	2·39789527	44	3·78418963	77	4·34380542
12	2·48490665	45	3·80666249	78	4·35670883
13	2·56494936	46	3·82864140	79	4·36944785
14	2·63905733	47	3·85014760	80	4·38202663
15	2·70805020	48	3·87120101	81	4·39444915
16	2·77258872	49	3·89182030	82	4·40671925
17	2·83321334	50	3·91202301	83	4·41884061
18	2·89037176	51	3·93182563	84	4·43081680
19	2·94443898	52	3·95124372	85	4·44265126
20	2·99573227	53	3·97029191	86	4·45434730
21	3·04452244	54	3·98898405	87	4·46590812
22	3·09104245	55	4·00733319	88	4·47733681
23	3·13549422	56	4·02535169	89	4·48863637
24	3·17805383	57	4·04305127	90	4·49980967
25	3·21887582	58	4·06044301	91	4·51085951
26	3·25809654	59	4·07753744	92	4·52178858
27	3·29583687	60	4·09434456	93	4·53259949
28	3·33220451	61	4·11087386	94	4·54329478
29	3·36729583	62	4·12713439	95	4·55387689
30	3·40119738	63	4·14313473	96	4·56434819
31	3·43398720	64	4·15888308	97	4·57471098
32	3·46573590	65	4·17438727	98	4·58496748
33	3·49650756	66	4·18965474	99	4·59511985
34	3·52636052	67	4·20469262	100	4·60517019

Numbers frequently used in Calculation.	Logarithms.	Arithmetical Complemts.
Hyperbolic logarithm of 10 = 2·302585092994045684	0·362216	9·637784
Reciprocal of ditto, or Modulus of common logs. = M } = ·434294481903251828	9·637784	0·362216
Circumf. of a circle to diameter 1, Surface of a sphere to diameter 1, Area of circle to radius 1 } = 3·141592653589793	0·497150	9·502850
Square of 3·14159265359 = 9·869604399639..........	0·994300	9·005700
Area of a circle to diameter 1 = ·785398163397448......	9·895090	0·104910
Surface of a sphere to radius 1 = 12·566370614..............	1·099210	8·900790
Solidity of a sphere to radius 1 = 4·188790205..............	0·622089	9·377911
Square of circumference of circle × ·07957747 = area.......	8·900790	1·099210
Solidity of sphere to diameter 1 = ·523598775598298........	9·718999	0·281001
Radius equal to the arc of 57·295779513 degrees..............	1·758123	8·241877
Ditto ditto 3437·74677 minutes..................	3·536274	6·463726
Ditto ditto 206264·81 seconds..................	5·314425	4·685575
Length of an arc of 1 second = ·0000048481368.............	4·685575	5·314425
Ditto of 1 minute = ·000290888208..............	6·463726	3·536274
Ditto of 1 degree = ·01745329248..................	8·241877	1·758123
Sine of an arc of 1 second = ·0000048481368.............	4·685575	5·314425
Ditto of 2 seconds = ·0000096962736.............	4·986605	5·013395
Ditto of 3 seconds = ·0000145444104.............	5·162696	4·837304
A circle = 360 degrees..	2·556303	7·443697
Ditto = 21600 minutes..	4·334454	5·665546
Ditto = 1296000 seconds..	6·112605	3·887395
One hour = 3600 seconds..	3·556303	6·443697
Twelve hours = 43200 seconds..	4·635484	5·364516
Twenty-four hours = 86400 seconds..	4·936514	5·063486
Mean diameter of the earth = 7912 miles........................	3·898286	6·101714
Mean radius of the earth = 20887680 feet....................	7·319890	2·680110
Radius of the equator = 20921180 feet....................	7·320586	2·679414
A degree on the equator = 365144 feet........................	5·562464	4·437536
Earth's polar axis = 41706360 feet....................	7·620202	2·379798
English mile = 5280 feet..........................	3·722634	6·277366
Geographical or nautical mile = 6075·6 feet.....................	3·783589	6·216411
Time of the diurnal rotation of the earth = 86164·0908 sec.	4·935326	5·064674
Length of the solar or tropical year = 365·24223 days.......	2·562581	7·437419
Length of the seconds pendulum at Edinburgh = 39·1555 in.	1·592793	8·407207
Force of gravity at Edinburgh = 32·2041 feet..................	1·507911	8·492089
Length of seconds pendulum at London = 39·1393 inches...	1·592613	8·407387
Force of gravity at London = 32·1908 feet........................	1·507732	8·492268
Length of the French toise = 6·39495 imperial feet.........	0·805837	9·194163
Ditto ditto pied = 1·065825 imperial foot.......	0·027686	9·972314
Ditto ditto metre = 3·2808992 imperial feet......	0·515993	9·484007
French hectare = 2·47114 imperial acres.......................	0·392898	9·607102
Cubic metre = 35·3166 imperial cubic feet..................	1·547979	8·452021
French gramme = 15·434 imperial Troy grains................	1·188479	8·811521

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