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ON THE

GENEALOGY OF MODERN NUMERALS.

PART II.

BY

SIR E. CLIVE BAYLEY, K.C.S.I., C.I.E.

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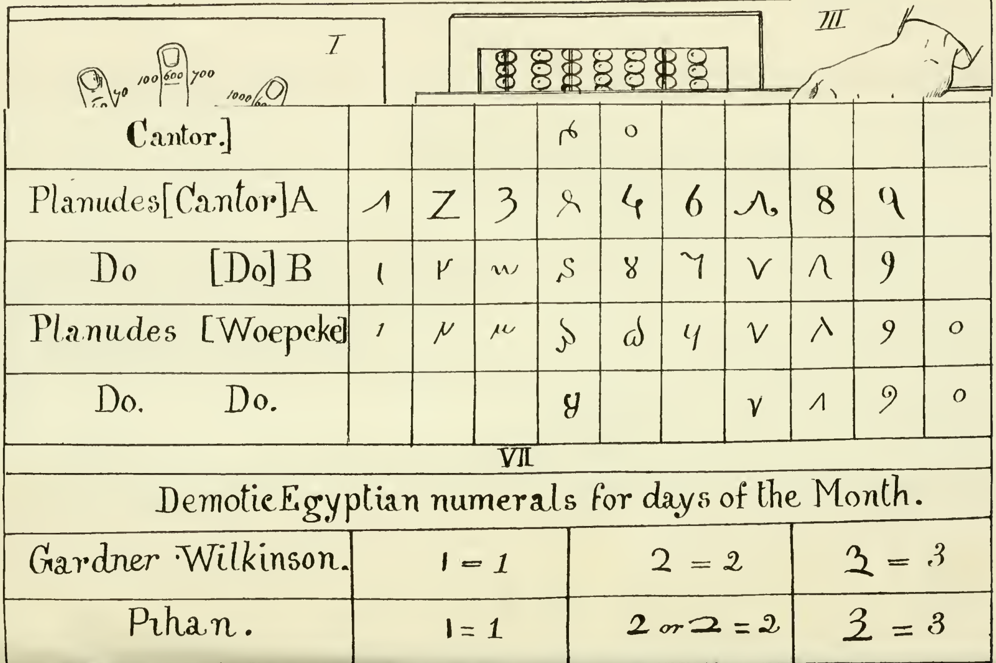
I
100
600
700
1000
III
Cantor.]
Planudes [Cantor] A
Do [Do] B
Planudes [Woepcke]
Do. Do.
VII
Demotic Egyptian numerals for days of the Month.
Gardner Wilkinson.
Pihan.

Plate II

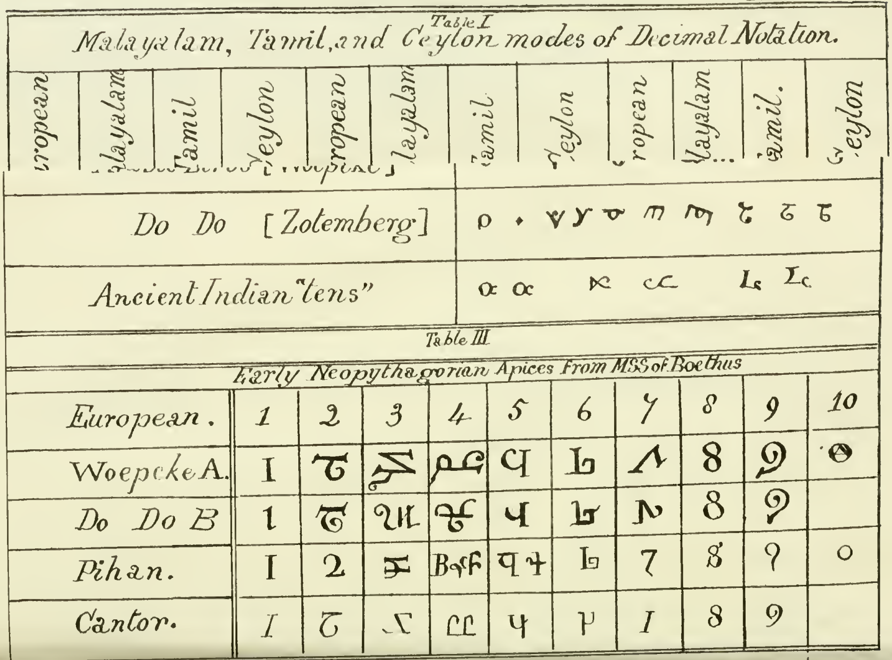
Table I
Malayalam, Tamil, and Ceylon modes of Decimal Notation.
European
Malayalam
Tamil
Ceylon
Do Do [Zotemberg]
Ancient Indian "tens"
Table III
Early Neopythagorian Apices From MSS of Boethus
European.
Woepcke A.
Do Do B
Pihan.
Cantor.
1
2
3
4
5
6
7
8
9
10

ON

THE GENEALOGY OF MODERN NUMERALS.

PART II.

SIMPLIFICATION OF THE ANCIENT INDIAN NUMERATION.

By Sir E. Clive Bayley, K.C.S.I., C.I.E.

The second part of this paper will be occupied by an attempt to show how the ancient Indian system of numeral signs, described in Part I., was simplified. In other words, it will be attempted to show how this old system became the parent of that now used in India, which employs only nine units and a zero,—indeed of that system as used not in India alone, but now almost universally both in eastern and western countries.

Since this simplification of the signs was the outcome of a reform in the system of numeration itself, it becomes necessary to deal to a large extent with the latter also. In entering upon this question however, it is necessary to premise that, as it has already been the subject of long and learned discussions by writers of the highest ability, it cannot be pretended in the present paper to examine it with any degree of completeness. Indeed, the literature of the question is in itself so extensive that it would be impossible, except after years of study and in the compass of a very considerable volume, to attempt an analysis of it,—much less to discuss completely the conflicting views held by many competent authorities. To those who desire to go more deeply into the subject, Dr. Moritz Cantor's Mathematische Beiträge (Halle, 1869) will afford a good view of it. From the information contained in this work it will be seen that I have borrowed largely, although I have not, in some material points, been able to accept the writer's views.

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At any rate, from the authorities cited in this work and in M. Woepcke's Traité sur l'Introduction de l'Arithmétique Indienne en Occident, some notion may be formed of the extent of the field over which a full and complete inquiry should extend.

All now attempted will be a sketch of the leading and more important facts, so arranged as to sustain a consistent theory for their explanation. It will be endeavoured to do this within the compass of an ordinary paper in the Society's Journal, without the omission of anything really material, but without entering upon any controversy. The conclusions formed will be submitted with the data on which they are based, to be accepted or rejected on their own merits.

It may be said that on arriving at this point in the history of numerals, we are no longer under the necessity of depending almost entirely on inference and conjecture. Much positive evidence exists, though unfortunately on some points of a conflicting nature. The task to be accomplished is to rearrange and reconcile it. Some of the direct testimony with which we have to deal is that of the early Arabic historians. This, with the indirect and undesigned proofs derived from the writings of the Arabic and Sanskrit mathematicians, forms by far the most important and trustworthy material available. Other information, obtained from European sources, both ancient and mediæval, will be also used, though some caution has to be used in dealing with the latter.

The ground, as has been said, has long since been occupied by writers of the highest ability and most profound learning, such as Humboldt and Chasles, and by a writer whose acquaintance with Oriental mathematics is probably still unrivalled—the late M. Woepcke. Indeed, it may be admitted at once that the lines of the present paper follow closely those on which M. Woepcke has written his two papers on the subject, viz., the Traité sur l'Introduction de l'Arithmétique Indienne en Occident (Rome, 1859), and Sur la propagation des Chiffres Indiennes (Journal Asiatique, ser. 6, tom. i.). The question has since been carried somewhat further by M. Leon Rodet in his papers on the writings of

Aryabháta in the Journal Asiatique. Little more will be here attempted than to bring together the main items of the already existing evidence, and to use them together with some little recently discovered matter, in enforcing and carrying out to their legitimate conclusion the views of these two latter writers.

Before, however, dealing with the subject as one of history, it is necessary to clearly understand the principle of the great reform to which it refers. It has been shown in Part I. that the old Indian system, as eventually established, employed twenty "self-contained" signs which, by the aid of a system of differentiation, were in fact capable of expressing any series of numbers—those at least likely to be used in the ordinary concerns of life.[1] These were used without any reference,

[1] It is not necessary to explain here the methods by which the still higher numbers used for mathematical calculations were expressed. It is sufficient for the present inquiry to take note of the early, and it may be said universal employment of the decimal arrangement. Nor is it necessary to dwell on the much wider question of the causes which led to its adoption. It is possible that there was a stage in the very early history of civilization, when mankind were more restricted in their power of numeration, as is the case to this day with some of the savage races on the Andamanese Islands, who cannot count beyond three,—indeed indications may, perhaps, still be traced that such a condition once existed among the most highly civilized nations, and that even when this was exceeded they continued to count by *groups* of threes,—still it is certain that the extension of this power must have been one of the earliest steps in the progress of civilization. The system of numbering by decimal stages or "rests" has been very generally supposed to have been suggested, at any rate, by the use of the human hand as an instrument to assist the process of reckoning numbers. Indeed, it is quite possible that the structure of the human hand suggested not only the decimal, but the earlier supposed methods of counting by 'triads,' or 'threes,' the quinary, the quaternary, and the duodecimal modes of numeration. The first being suggested by the ten fingers and thumbs of the joined hands, the second by the 'three' joints, the third by the four fingers, the fourth by the fingers and thumb of one hand (the Akkadian name for 'five' is synonymous with that for 'hand'; Pinches, Proc. Soc. Bibl. Arch., June, 1882), and the duodecimal by the multiplication of the 'three' joints by the four fingers. It is singular, too, that the Babylonian sexagesimal unit of 'sixty,' or šuš, will result from the further multiplication of twelve by five (perhaps better of $2 \times 2 \times 3 \times 5$; see Pinches, Proc. Soc. Bibl. Arch., June, 1882, p. 116), and the still further multiplication of this result by 'ten,' gives the Babylonian 'nēr,' or 'six hundred.' At any rate, there are many curious facts which seem to indicate at least, this origin for the decimal system, and which also show the universal use of the human hand as a 'reckoning board.' It will suffice to mention a few of these only here. In Egypt, for example, in the hieroglyphic signs, the human hand and its portions were employed to signify measures of length. The cubit was divided into 'diti,' of which twenty-eight went to the royal, and twenty-one to the common cubit. One, two, or three 'diti,' were indicated by one, two, or three fingers respectively; four 'diti' by the human hand displaying four open fingers; five 'diti' by a similar figure with the thumb also displayed; six 'diti' by a closed fist; and eight by a reduplication of the sign for four 'diti.' Again, in general

necessarily, to the position in which they were written down. Each expressed the full number which it designated, whether accompanied by others or not.

numeration, the finger with the top joint bent designated ten thousand, and there are perhaps in the hieroglyphics other, though less palpable reminiscences, of the human hand. Another curious piece of evidence is suggested by a notice published by Mr. J. Fleet in the *Indian Antiquary* for 1875, vol. iv. p. 85. Mr. Fleet mentions Professor Hunfalvy's remarks at the Oriental Congress of the preceding year to the effect, that in a very considerable number of languages of the Turanian stock, the 'ring' finger is always termed 'the finger without a name.' Mr. Fleet illustrates this by quoting a curions anecdote recorded by a Sanskrit author with reference to the poet Kalidâsa and his eight contemporaries of literary fame at the Court of Kanouj, who were termed its "Nine Gems," and by it proves that a similar custom had existed for so long a period in India, that even at that date (the seventh century A.D.) its origin had been forgotten; for in reckoning these nine gems on the fingers, the writer says "Kalidâsa" was always reckoned first (on the little finger of the left hand), but no one was counted on the next (or ring finger), because none of his contemporaries could be reckoned as even second to him; and, adds this author, hence was assigned at last some reason for calling that finger 'anâmika,' or 'without a name.' Mr. Fleet, it is true, goes on to suggest that this may not be the true signification, and that the term might mean in Sanskrit 'unbent,' in allusion to the difficulty of bending that finger, but in the face of the Turanian parallel, this explanation can hardly stand. This ancient custom, however, may easily be accounted for by referring back to the origin of decimal notation on the hand. If the ten fingers and thumbs suggested the origin of the decimal notation, it is nevertheless evident that in using the hand as an instrument for reckoning, *one* finger would be superfluous, nine symbols only being required, as the tenth became the first of the new and next highest stage of the decimal series. One finger, therefore, would necessarily be 'skipped,' or laid aside. It is not perhaps easy to suggest any reason why the ring finger should have been specially chosen for omission, but it would be only natural that the omission should be by common custom of one selected finger, and if, as Mr. Fleet suggests, the process of counting commenced in ancient as it still does in Modern India, with the little finger of the left hand, then it would be natural that the calculator should wish to put his calculation right as soon as possible, and should therefore omit the finger next after the initial one, which would be of course the ring finger. These facts may suffice to illustrate the antiquity of counting on the fingers. Its wide general diffusion need hardly be pointed out. The Chinese, to this day, have a mode of counting up to 99,999 on the fingers of one hand alone, which will be seen illustrated on Fig. I. Plate I. The nine units are reckoned on the joints, commencing along the *outside* of the little finger; then counting four, five, and six on the joints at the back of the finger; seven, eight, and nine on the joints along the inside of the finger; the next finger is similarly used to represent the tens; the next the hundreds; the next the thousands; and the thumb for the tens of thousands. In England the venerable Bede describes another system, which he states to be great antiquity, while the practice of concealed bargaining by pressure of the fingers has been used from time immemorial and is still used among the nations of the East. In India (where the hands are concealed under a cloth), Tavernier (Voyages, Part II. pp. 326-7, ed. 1712) describes this mode of settling prices. Halhed says that (in Bengal) the practice is limited to counting up to fifteen; this may be an error, but even this would enable bargains to be made in pies, annas, rupees, and mohurs, and a limit of fifteen mohurs or 240 rupees would suffice for the requirements of most Bengal markets. In Barbary and Arabia the hands are manipulated under cover of the long sleeves of the burnous. Enough has, however, been said to indicate the probability of the derivation of the decimal system from the structure of the human hand, and to show that, at any rate, it is apparently the most primitive and simple and most

On the other hand, the main principle of the new method was the discovery, and application of, 'the value of position'; in other words, the discovery that the signs for the nine units only, when arranged in a certain strict decimal order, would suffice to express any number or series of numbers whatsoever.

This discovery rendered it possible to dispense with the signs of the older system for expressing the higher numbers, tens, hundreds, and thousands, and was, undoubtedly, the first and main step of the reform.

The next step, that which made the reform complete, and which resulted in our present beautiful and flexible system, was the invention of the 'zero,' that is to say, a sign for 'nullity,' to be employed when the number to be expressed contained no special indicator of any one or more of the steps in the decimal series represented. Not only, however, are these really independent discoveries, but it will be attempted to show presently that, as a matter of fact, the invention of the zero was

widely spread of all extant methods of numeration. On the other hand, however, it is not to be forgotten that other suggestions have been made as to the origin of this and especially of the quaternary, quinary, and duodecimal methods of notation, which are in themselves not improbable, particularly those derived from astronomy and the natural divisions of time. Indeed, as regards the sigu for 'five' employed in Egyptian hieroglyphics, such an origin is expressly assigned by the Egyptian priest Horapollo, and may be taken as correct. "Τὶ 'Αστέρα γράφοντες δηλοῦσι" . . . τὸν πέντε ἄριθμον, ἐπειδὴ πλήθους ὄντος εν οὐρανῷ πέντε μόνοι ἐξ αὐτῶν κινούμενοι τὴν τοῦ κόσμου οἰκονομίαν ἐκτελοῦσιν (Horapollo Hierog. liber i. c. 13, *apud* Cantor, M.B. p. 18 and note p. 17); that is to say, the idea of the five pointed stars was taken from the 'five' planets, then alone known to Egyptian observers. No doubt, too, the Egyptians used both quinary and quaternary methods of notation, for eight stars were used to represent 'forty,' and a single star with two to make seven. So the Egyptians early used quaternary multiples of the 'hen' or unit of capacity in their scale of measures of capacity (see Rossi, Grammatica Copto-Geroglyfica, p. 89 note and p. 97).

While thus referring back to the oldest pyramids for evidence as to the origin of decimal notation, it may not be out of place to remark that if the theory adopted by this paper be correct, all the signs of the Indian numerals may also be referred back directly or indirectly to the same source. This is even the case with the uuit signs, which it has been proposed to derive from the Bactrian alphabet, for since Prinsep assigned these characters to some form of the Phœnician alphabet, this point has never been questioned seriously by subsequent writers, and has been, indeed, supported by several of high authority (see Thomas, Num. Chron. N.S. vol. iii. p. 229, and Prinsep's Essays, vol. ii. pp. 144-162; also Cunningham, Successors of Alexander in the East, pp. 30-44), though they were modified to meet the requirements of an Aryan language, and perhaps also (as Dr. Bühler has suggested) of a Brahmanical liturgy. Again, the Vicomte de Rougé and M. Lenormant (Introduction à une memoire sur la propagation de l'alphabet Phœnicien, Paris, 1866, pp. 108-9) have made it almost certain that the Phœnician characters came, through the hieratic, from the Egyptian hieroglyphics.

considerably later in point of time than that of the 'value of position,' and for the present the inquiry will deal only with the latter.

It may be said with truth that from its earliest appearance the Indian system was founded on a decimal principle. It has been shown that the units were represented by a certain set of signs. With ten a fresh series of signs was introduced; then came a new symbol for the hundred and another fresh one for the thousand.

But this practice was not, as has been pointed out above, peculiar to the Indian system, in fact it may be said that it was common to all ancient systems of numeration, and is found in the Egyptian, Phœnician, Babylonian, Assyrian, and in all their derivative systems, and upon this decimal principle have been mainly founded all the ancient and modern systems of arithmetic.

It is found in its simplest, and probably its earliest recorded form on the monuments of the Fourth Egyptian dynasty, to which reference has already been made in Part I. There a single stroke represents unity, two strokes represent 'two,' three strokes 'three,' and so on as far as 'nine.' With 'ten' a new symbol appears; two of these signify 'twenty,' three 'thirty,' and so on up to ninety; at a hundred another new sign comes into use; another at a 'thousand,' 'ten thousand,' a 'hundred thousand,' and a 'million' respectively, that is to say, a fresh symbol is employed at every new decimal stage.

There is, however, one point of some importance to be incidentally noticed. The Phœnician, and all other systems derived from the Phœnician of anterior date to the discovery of alphabetical notation, seem, at any rate up to a very late time, to have possessed no separate and special sign for any number above the hundred. The thousand seems always to have been expressed by groups of the lower signs, and so on with higher numbers.

But while the ancient Indian numeration was thus decimal in its fundamental idea, it was also decimal in another sense, that is to say, the method in which its signs were arranged.

For though each numeral sign was, as has been said, self-contained, and expressed absolutely the number it represented, without any question of its position in reference to other signs, still, nevertheless, these signs were in practice[1] actually arranged in a decimal order, the highest numbers being written first (*i.e.* to the left), and the others following in regular decimal procession. Thus thousands were written first, then hundreds, then tens, and last units. Of course, if there were no hundreds in the series of numbers to be represented, then the tens followed the thousands, or if no tens, also, then the units would follow next upon the thousands. As a matter of fact, however, this arrangement had little *direct* connexion with the decimal principle, or at least was mainly determined by other causes. It is self-evident that a decimal notation by self-contained signs does not necessitate their being written in any fixed order at all, and is quite as consistent with an order proceeding from right to left, as with one proceeding from left to right. Indeed, the latter practice actually prevails in some methods of writing, notably in the Egyptian hieratic. The *cause* of the arrangement will, however, become obvious if it be borne in mind that (as has been already said) *all* numeral signs were in their inception merely shorthand modes of expressing *numeral words*, whether written or expressed by hieroglyphic signs. Numeral signs, therefore, when written, followed quite naturally in their disposition—(1) The arrangement which the language to which they belonged adopted for expressing numbers either orally or in writing; (2) The direction of the writing, whether from right to left, or left to right, which that language employed. For example, if a people (as the Indians did), in speaking and writing mentioned first the higher denominations of the decimal series, and then those next lowest, and if also they wrote from left to right, then in putting down the numeral signs, they would do this in the same order in

[1] As will be presently explained, in some rare instances the Indians arranged numbers perpendicularly one above the other—as, in fact, they did letters also; in either case, however, the first letters and the highest numbers occupied the uppermost positions; the fact does not, however, affect the general argument as respects the ordinary arrangement of the Indian numeral signs.

which they were spoken, and would write naturally the highest number first on the left, then the next highest, etc., as was, in fact, the case with the Indian numerals.

It is evident, also, that a similar result must follow if all the conditions are exactly reversed, that is to say, if when numbers are spoken or written, the units are first mentioned, and then the other higher decimal places in their successive order, and if at the same time the language is *written* in characters reading from right to left. To give an example of either case, it may be instanced that the Indians in speaking or writing, would say one thousand two hundred and twenty-two, and in writing also would begin to write from left to right; the numeral characters following this order would stand as 1222. The Arabs, per contra, would write (or say) two-and-twenty and two hundred and one thousand; but, as in writing, they begin on the right hand and go on to the left, the numerals following the order of the writing; the result is also 1222. Of course this is only an indirect effect of the decimal arrangement, the linguistic idiom and the mode of writing having at least an equal share in producing it.

Another factor also contributed in a most important degree to the simplification of the ancient Indian system. This, as will be presently shown, was doubtless the use of the abacus. Indeed, so important a part did this instrument play in the invention of the new method of numeration, that it will be necessary to go at some length into the consideration of its character and of its history, so that its action may be fully understood. It was a contrivance unquestionably of great antiquity, as will be gathered from what has been already said. The popular belief among the Greeks certainly was that it was introduced into Greece by Pythagoras, and Jamblichus,[1] though writing at a comparatively late date, no doubt represented what was current both among the Greeks and Egyptians, when he says that it was upon the abacus that Pythagoras taught both arithmetic and geometry, and

[1] *εἰs τὴν δέ ἀριθμῶν μάθησιν καὶ γεωμετρίας ενάγειν αὐτὸν ἀπειράτο επ' ἄβακος τὰς ἑκάστου ἀποδείξεις ποιούμενος.*—Jamblichus De Vitâ Pyth. cap. v. § 22.

as the same author[1] also says that Pythagoras also first taught the Greeks a particular form of proportional arithmetic, which was a Babylonian invention, it seems very probable the abacus—at any rate as an arithmetical instrument—was of Eastern invention. Indeed, as has been already said, Radulphus of Leon expressly declares it to be so, and there is no reason to doubt the fact, especially as its use seems to have spread to Eastern Asia at a very early period—for it has been known both in India and in China for a period probably long anterior to the Christian era. It has been already pointed out that the etymology most generally received connects the name of the instrument with an ancient Semitic word which signifies fine dust, and the form, therefore, which the instrument originally assumed was probably that of a board covered with fine dust.[2] The instrument on which Pythagoras taught *both* geometry and arithmetic must have been something of this kind, the board having probably a raised edge to retain the dust or sand with which it was covered, and being used lying flat. This latter view is supported by the fact that the word 'abacus' is used in several other instances in which the leading idea seems to be that of a flat slab, board, or table. Thus it is used in Latin to signify a sort of side table (cf.

[1] *εὕρημα δ' αὐτής φασίν ἔιναι Βαβυλωνίων καὶ διὰ Πυθαγόρου πρώτου εἰς Ἑλλήνας ἐλθειν.* (Jamblichus Comment. ad Nicomach: Arith., the second word ought apparently to be *δ'αυτῆς*.) See also Isidore Hispaliensis (Bishop of Seville) Origines, liber iii. c. 2. Numeri disciplinam primum apud Græcos Pythagoracis autumnant conscripsisse et deinde a Nicomacho diffusius esse dispositam quam apud Latiuos Appuleius deinde Boethius transtulisse (for the quotations in this note see Cantor, pp. 369 and 391). Porphyry, in his Life of Pythagoras, credits the Phœnicians with the invention, or at least perfection of arithmetic, while assigning that of geometry to the Egyptians, and of astronomy to the Chaldeans. *Γεομετρίας μὲν γὰρ ἐκ παλαιῶν χρόνων ἐπεμεληθῆναι Αιγυπτιόυς τὰ δὲ περὶ ἀριθμούς τε καὶ λογισμοὺς Φοίνικας· Χαλδαίους δὲ τὰ περὶ τὸν οὐρανὸν θεορήματα* (De Vit. Pythag. 56, ed. Krissler, p. 12). But the point is not of importance for the present argument; if the Babylonians or Chaldeans were far advanced in astronomy, they could hardly have made much progress without some considerable use of arithmetic, and Pythagoras, who is reported to have been carried as a prisoner into Babylon by Cambyses, and who spent a long captivity there, may well have learnt his arithmetic and the use of the abacus in that country.

[2] The idea may have arisen from some such practice as still obtains in many a village school in India, where the smallest boys are made to lie upon the ground and scrawl letters and figures in the dust or sand of the floor (sometimes on the ground outside) with a bit of stick till they acquire some familiarity with the shape of these; they are then promoted to the use of a writing-board. Of this more will be said when treating in Part III. of the Gobar numerals.

Cicero against Verres, Actio ii. Lib. iv. c. 15, "Ab hoc abaci vasa omnia ut exposita fuerint abstulit;" see also Juv. Sat. iii. p. 264). In architecture, also, the word has a special significance, meaning an ornamental moulding such as would be produced by the projecting edge of a slab placed over the top of a pillar or in any other similar position.[1] At any rate this cheap and primitive form of the instrument was early in vogue, and seems to have held its place down to a very late date. See as regards India the Preface to Taylor's Lilawati, quoted by Reinaud, "Memoire sur L'Inde," in which an Indian instrument is described as composed of *red* sand on a whitened board, the figures thus appearing as white on a red ground. In classic ages the original form seems to have survived, at least for popular employment, even side by side with others of an improved form invented later on. Thus Persius, Sat. i. 181:

> "Necque abaco numeros et secto in pulvere metas,
> Scit risisse vafer.

So also in the fifth century Martianus Capella:

> Sic abacum perstare jubet, sic tegmine glauco
> Pandere pulvereum formosum ductibus æquor.

On an instrument thus constituted work must have been done with some kind of "stilus," but in all forms the principle was the same. When used for arithmetical purposes parallel lines were drawn, usually (as will be argued presently) horizontally, and each of these signified one place respectively in the decimal series.

Thus the first (i.e. the lowest) line represented units, the second tens, and the third hundreds, the fourth thousands, and so forth. Probably not more than seven or eight such places were usually represented, though eventually lines were used (before, or below that which represented the units) to express fractions, or when (as it will be shown was the case) the instrument was used for monetary calculations, to

[1] Ἄβακες (or *abaci*) was also the term employed in the language of 'decorative art,' to signify the rectangular parallelograms or 'pannels' used in painting the walls of rooms.

show the sub-divisions of the standard unit, in terms of which these calculations were made. On the lines thus made the numbers to be represented were doubtless at first simply marked by scratches, in groups up to the number of nine; for the tenth of each series was always, of course, the first of the line next highest in the decimal series. But this simple form of the instrument was eventually replaced by others of a more permanent, and in some cases of a more portable character; boards of wood, and slabs of stone on which the lines which indicated the various stages of the decimal series were painted, or cut, were amongst the first used. On these, perhaps, the signs for numbers were originally marked by chalk or some similar material, but eventually these signs were replaced by pebbles or 'calculi' (whence, of course, the origin of the terms 'calculate' and 'calculation'), to which various references will be found in classic writers. Later on, especially in the days of Roman magnificence and luxury, the pebbles were replaced by counters, often constructed of the most valuable materials; possibly, however, these last may have been rather used for a game, which it is known was played with the abacus. The use of counters was probably already known in the second century B.C., for Polybius [1] has a curious passage describing courtiers as exalted or depressed in condition, at the will of the king, just as counters on the abacus are made to signify 'talents' or 'oboli' at the will of the person using the instrument. These counters were, apparently, at one time placed half-way between the two lines, to indicate an intermediate stage, and so to reduce the necessary number of counters, but this purpose was more completely effected by an invention, according to which the lines themselves were divided into two parts—one of which served to indicate half of each decimal series. This will be best explained by reference to the figure of an actual Roman abacus described in the Theatrum Arithmeticum of Leopold,

[1] *Ὄντως γάρ εἴσιν οὗτοι παραπλήσισι ταῖς ἐπι τῶν ἀβακίιον ψήφοις Ἐκεῖναί τε γὰρ κατὰ τὴν τοῦ ψηφίσοντος βούλησιν ἄρτι χαλχοῦν καὶ παραυτίκα τάλαντα ἴσχουσιν; οἵ τε περὶ τὰς αὐλὰς κατὰ τὸ τυοῦ βασιλέως νεῦμα μακάριοι, καὶ παρὰ πόδας ἐλεειναὶ γίγνονται.*—Polybius, v. 26, 13 (Cantor, p. 390).

as preserved in the Library of St.-Généviève at Paris, and of which a figure will be found on Pl. III. Fig. 4. On this each line was divided into a long and a short part, and (except on the lines set apart in this case for the fractional parts of the 'as') the long line employed four indicators, which each represented 'units,' and the short line only one indicator, which represented 'five'; thus four units on the long line represented four, without them the solitary indicator on the short line represented five, *with* the four 'nine,' with two the seven, etc. In this example the instrument itself is made of a plate of metal, and the lines are under cut grooves, in which the indicators (which are 'buttons' of metal) slide backwards and forwards at will. Other peculiarities in this particular instrument, however, require notice; the first set of lines is divided into *three* short, instead of one long and one short; and the first of those divided into two has five buttons in the longer part instead of four. This set of double lines is marked with a Greek 'θ' or 'theta,' while the other long lines bear respectively the Roman signs for one, ten, a hundred, a thousand, etc. The explanation of these latter facts clearly is that the instrument was specially intended for monetary calculations. The line marked by 'theta' represented the 12 'uncias,' or duodenary subdivisions of the 'as,' and the three short lines marked ·S· (semi uncia), > (sicilica), and 2 (duodecima), the further subdivision, into $\frac{1}{2}$, the $\frac{1}{4}$, and $\frac{1}{12}$ of the latter respectively. Another example of the Roman abacus is a still nearer approach to the common form of the abacus usually employed at this day in India, China, and Russia, being a frame (of wood) on which the lines themselves are represented (as in the Indian instrument, Pl. I. Fig. 2) by wires, which in the Roman example are bent at each end so as to rise on one side above the frame; and on these, as the Indian abacus, the indicators employed are moveable beads. This last form of instrument seems clearly intended for use in the hand, or hung up against a wall, and on this the lines *must*, therefore, have been used in a horizontal position, for the beads could hardly otherwise be kept apart, to show the number to be marked.

A Greek example was also found at Salamis in the year 1846, of which an engraving is given in Pl. III. Fig. V. (taken from Dr. Cantor's work). It is a slab of marble, on which is cut a parallelogram, nearly double as long as it is broad; within, and parallel with the shorter sides and with each other, are cut near one end eleven lines, which are divided in the centre by a single line at right angles, and at the point of its intersection with the central line of the eleven is cut a star, which is repeated in the middle of each half of the dividing line. Separate, and at a little distance from this set of lines, are cut five other parallel lines, which are not divided by any central line: At one end, that apparently intended to be the top, and immediately above the first set of parallel lines, are cut eleven Greek signs, which have no doubt been correctly interpreted by MM. Létronne and Vincent to signify 1000, 500, 100, 50, 10, 5, 1, *drachmas*, the highest sign being on the left; to the right of the one drachma sign are others intended to denote the 'obolos' or 12th part of the drachma, and the half and one-third of the obolos, and one which indicates the 'Chalchos' or one-sixth of the obolos. These signs are repeated on the side which is on the right hand if the inscribed end be placed opposite and furthest away from the spectator; on the opposite side also they are repeated, but with the addition of two higher signs which undoubtedly stand for 5000 drachmas, and for one talent=6000 drachmas. The signs at the sides are written with their lower ends towards the outside of the board, those at the top with the lower ends towards the inside of the board. The signs at the sides are not written against the several columns, but, on the contrary, nearly all opposite the blank space between the two sets of columns. The one fact which is clear is that the board must have been used with counters of some sort, and therefore lying flat. Indeed, looking to the heavy material (marble) of which it was composed, it was probably, if not permanently fixed, at any rate not intended to be much moved. It has been suggested that this particular instrument was intended either for use by a money-changer or public accountant, or for playing a kind of game to which allusion has been already made, and said by various classical writers to have been played on the abacus. The suggestions have their rise in the fact that the instrument, from its size and the way in which it is constructed and marked, seems intended to be used by more than one person at a time. It was certainly intended primarily, as was the Roman abacus above described, for monetary calculations. The division of the main set of lines into two parts may have been, as in the case of the Roman abacus already described, to reduce the number of counters required,[1] in fact, if such were the case, three counters would have sufficed (as the scale was *quinary*), two on one side and one on the other. The 'stars' were probably intended merely to assist the eye and to facilitate rapid calculations. On the other hand, the fact that the number of the signs at the upper end and on one side coincide with the number of the lines in the principal set, seems to show that these last were used for ordinary calculations; the other group of five lines without division were perhaps used for the rare cases in which sums above 1000 drachmas were the objects of calculation, and in which the numbers on the other side, expressing terms of 5000 drachmas and the talent, would come into play. These considerations, perhaps, make it more probable that the instrument belonged, as suggested, to some public accountant or money-changer, who, standing at the bottom, would read the numbers opposite to him at the upper end, while those with whom he was dealing stood on either side according to the magnitude of their accounts or dealings. It is not necessary at this stage to inquire more particularly whether, either by Greeks or Romans, the abacus was generally used with the lines in a perpendicular or in a horizontal position or indifferently in either. The Roman abacus, at least, seems probably to have been sometimes used horizontally, as has been already shown.

In any case it is clear that if the lines on an abacus mark-

[1] The Chinese abacus, the lines of which are used horizontally, has also a similar perpendicular dividing line. The Chinese methods of using this instrument, however, are peculiar, and it is not possible to discuss them here at length.

ing the decimal scale be placed or held in a *perpendicular* position, and if the counters used to represent the numbers be replaced by their equivalent unit signs written at the foot of each column, then these last, valued according to the decimal scale of the columns in which they stand, and read from left to right, will give the actual sum of the entire number represented, in other words, it would become palpable that unit signs alone, arranged in a decimal order, were capable of representing any series of numbers. That is to say, the ' value of position ' would at once be revealed, *cf.* Pl. I. Fig. 3.

That the value of position was thus actually discovered is not a mere conjecture. For in the earliest known examples of its use in Europe it will be shown that it was employed by the aid of a series of lines, which in fact represented those of the abacus in a perpendicular position. Indeed this figure was then often actually designated by the name of 'abacus,' though also called the 'arcus Pythagoreus,' and in French the ' tableau à colonnes.' It was in fact merely an abacus transferred to paper. The first fact which requires notice in reference to it is, that while it enabled those who used it to dispense with any higher numeral signs beyond those of the units, it did not require even the assistance of the modern sign for zero. The next point to be remarked is that it palpably thus became possible to express not merely one but several series of numbers on the same instrument, by writing them one above the other, and this fact would give immensely increased facility for arithmetical operations. As to the first point it will be best to quote the exact words of M. Woepcke in the Journal Asiatique, series vi. vol. i. p. 38 note, "Comme il será encore question à différentes reprises . . . du tableau à colonnes, comme d'un moyen de remplacer l'emploi du zéro, j'ajouterai une courte explication pour ceux d'entre les lecteurs que ne seraient pas tout à fait familiarisés avec cette matiére. Nous écrivons actuellement des nombres tels que les suivants,

305
84009076
1020084000

en faisant usage du zéro; mais on comprend que, si des lignes verticales étaient tracées d'avance sur la page ou l'on voudrait écrire ces nombres, par exemple, pour en faire l'objet d'un calcul, on pourrait se passer du zéro en écrivant

							3		5
		8	4			9		7	6
1		2			8	4			

cette notation est moins commode, mais aussi claire et aussi precise que la nôtre, pourvu que l'on convienne, une fois pour toutes, que les chiffres signifient les unités lorsqu'ils sont placées dans la première colonne a droit; les dizaines, dans la colonne suivante; les milles dans la troisième, et aussi de suite. Le tableau à colonnes fournit donc un moyen d'écrire tous les nombres, quelque grands qu'ils soient, au moyen des neuf chiffres, en donnant a ceux ci des valeurs différentes selon leur position, et sans faire usage du zero."

While thus showing how the tableau à colonnes enables the nine ciphers for the units to be employed without the use of a zero, the example given by M. Woepcke practically also shows how, on this 'abacus transferred to paper,' it became possible to deal with more than one series of numbers at a time, whereas on the abacus itself one set only could be shown. The 'tableau à colonnes' accordingly, as has been said, offered enormously enhanced facilities for all arithmetical operations, so much so indeed as practically to create an entirely new use for the instrument itself.[1] Having arrived at this point, we may now pass to the actual history of the marvellous reform with which we have to deal.

[1] It was possibly by drawing lines between each series of numbers that the 'tableau à colonnes' was eventually transformed into the 'exchequer table' or 'chequers.' This last is described by an English mediæval writer, Richard Fitznigel, as consisting of a space covered by a black cloth with white lines on it, drawn both transversely and perpendicularly about a palm apart, on which calculations were made by means of counters. The calculations in the extreme column to the right advancing by 'twelves' (for 'pennies,' as in the case of the columns for the subdivisions of the 'as' on the Roman abacus), the others by 'tens.' On this cloth the calculations of payments into the Royal Treasury were made, and the term 'chequers' is supposed to be derived from the mediæval term for a chess-board, or 'scaccum,' to which the tableau à colonnes in this shape bore a strong resemblance. See Edin. Review for 1811, vol. xviii. art. vii. p. 207.

It may be said, in the first place, that while the credit has been at different times, and by different writers, claimed exclusively for India, for the ancient, or for the later Greeks, some writers have been disposed to believe in a double invention both in the East and West.

Again, a further question arises whether, as has already been suggested, the two portions of which the invention has already been shown to consist, *i.e.* that of the value of position and of the 'zero,' were simultaneously discovered, or whether the one was older in point of time than the other.

It will suffice to say here that the conclusion which it is the purpose of this paper to maintain is briefly that the invention was, as a *practical* invention, at any rate, wholly Indian; that the discovery of the value of position, and of its use, was made a century or more before the discovery of the zero; and that these two inventions reached Europe (also separately and in their turn) the first certainly, the second possibly, if not probably, through Egypt.

To establish this position it is proposed to show that the Indians knew and used either portion of this invention at a date considerably anterior [1] to their use in Europe, and that the earliest and best authorities distinctly describe them as Indian.

On the other hand, it will be attempted to show that the Greeks, ancient and modern (though very nearly approaching them), were certainly ignorant of either invention, or at any rate never put them to any practical use, till very long after the time when they were in full operation in India; and that even when they appear (in their earliest shape) in Europe, they bear distinct and manifest traces of an Oriental, indeed of an Indian origin.

It will be perhaps convenient to deal with these two branches of the inquiry separately, and to take the claims of the Indian arithmeticians into consideration first. In doing so the direct evidence which favours their claims will be first cited, and of this first of all the proofs, afforded by the

[1] See M. Woepcke in Journal Asiatique, tom. i. series 6, pp. 247-248.

works of Indian and of Arab writers. Perhaps the most important of these is the evidence of the well-known traveller and historian, Masaudi, who visited India at the close of the tenth century A.D., and who, in his "Meadows of Gold" [French translation, Paris edition, 1861, vol. i. chap. vii. p. 150], says, "Un congrès des sages reuni par ordre du roi (of India) composa le livre du Sind Hind [Siddhanta] ce que signifie 'L'age des ages' . . . . Ils inventerent aussi les neuf chiffres qui forment le systême numerique Indien." The well-known painstaking accuracy of this writer, his early date and his opportunities, give great weight to his testimony that the nine ciphers are an Indian invention, though their attribution to the deliberations of a congress of sages requires, perhaps, confirmation, and is in itself hardly likely. Moreover, the fact thus stated is quite in harmony with the evidence direct and indirect of other Arab writers; at pp. 237, 238 of M. Woepcke's article, already cited, from the Journal Asiatique, will be found authorities to show that the Khalif Walid, who reigned from 705 to 715 A.D., forbad by a special edict the use of the Greek language in the public accounts, and directed the substitution of the vernacular language in the East and of Arabic in the West. He made, however, a special exception in favour of Greek letters as numeral signs, on the ground that the Arabic language possessed no numerals of its own, and in Egypt, also, the Coptic equivalents of the Greek alphabetic numerals, and the Greek methods of bookkeeping, were adopted by the Arabs in the public accounts. [See the authority from Theophanes, quoted by Cantor, pp. 416, 417.] It was not, apparently, till some sixty years later,[1] viz. in the year 773 A.D., that the Arabs became acquainted with the Indian ciphers and with the Indian methods of notation and arithmetic. They obtained this knowledge from a book presented by the envoy of an Indian monarch to the Khalif Al Mansur; I have endeavoured recently to show in the Numismatic Chronicle[2] that

[1] See Woepcke on the authority of the Târikh ul Hukamá, Journal Asiatique as above, and also pp. 472-480.

[2] Part II. of vol. ii. 3rd series, pp. 138-146.

this monarch was probably one of the Hindu kings of Kábul, at least that the modern Arabic numerals seem to be derived from the peculiar form of those then employed in that part of India. At that date the complete Indian system with the zero was, as will be shown presently, certainly in full use in India, and it must have been that system (employing the zero) with which the Arabs first came in contact; this seems clear from the excessive eulogiums lavished by them upon the new system of numeration and calculation, as being infinitely superior to the Greek systems, which we have seen were already known and used by the Arabs, a fact which could hardly be predicated even of the tableau à colonnes without the zero (at least for all purposes), much less of the ancient Indian system.

The Indian book thus obtained by the Arabs was translated by order of the Khalif, and served as the basis of an Arabic mathematical work by one of the learned men of his day, Mahomed bin Ibrahím al Fazárí. His work again, later on, was abridged by Mahomed bin Muṣa al Khwárizmi at some date slightly before 205 A.H.=820–21 A.D., whom a later writer [1] expressly describes as teaching in his work *Indian* arithmetic, while Avicenna in the tenth century and other authors invariably describe the modern decimal system of arithmetic, employing the nine ciphers and the zero, as 'Indian.' Indeed the etymological sense of the word which is now the common term all over the East for a numeral cipher is هندسة 'hindisah' or 'hindsah,' which means simply 'Indian.' Again, one later Arab author (Alkaṣadi, in his commentary on the Talkhis of Ibn Albanná), expressly discussing the Neo-Pythagoreans, describes the ciphers used by them as identical with the Gobar signs, which he says were of Indian origin (Woepcke, J. A. tom. i. ser. 6, pp. 58–60). The Arabic writers therefore, from the earliest times, without hesitation and in unbroken succession, attributed the invention of decimal arithmetic and of the signs with which it was accompanied, to the Indians. Nor was

[1] *Tárikh ul Hukamá.* See Woepcke's Traité sur l'Introduction de l'Arithmétique Indienne en Occident, p. 19.

this because they were unacquainted with any rival claims which could be put forward on behalf of the Greeks; for, as has been seen, so early as the very beginning of the eighth century, the Arabs knew and eagerly employed the Greek methods of arithmetic; and even as early as 901 A.D. the Almagest of Ptolemy was translated into Arabic by Ṭhabit bin Korrah; and it has been shown by the quotation from Albiruni, given from Mr. Burnell's note in Part I., that the Almagest was still used and regarded as a leading authority in the commencement of the eleventh century A.D. So far, therefore, as the evidence of Arab writers is concerned (and this is of great value, both from its date, its coherence and the independent character of those who give it), it may be said, not only that it supports the Indian origin of the modern numeration both with and without the zero, but that it practically refutes the claim of the Greeks even to a simultaneous invention. This is the more remarkable—for the Arabs, who were pretty certainly not ignorant of the Indian algebra, do not claim its invention for the Indians, but speak freely also of the Greek algebra, and seem to have adopted largely from either source. Indeed, Abul Faraj, who himself lived in the thirteenth century, calls Diophantus the contemporary of Justinian, and speaks of him in terms which imply that he was still in the thirteenth century the best of all known authorities on the subject of algebra.[1]

Indeed, the Indian origin of the new method of numeration, and of the signs which belonged to it, is not without direct support, even from the testimony of later Greek writers themselves. Thus Planudes, who wrote in the first half of the fourteenth century, says, speaking of the zero (which he calls '*τζιφρα*'): *τιθέασι δὲ καὶ ἑτερόν τι σχῆμα ὃ καλοῦσι τζίφραν κατ' Ἰνδόυς σημαῖνον ὀυδὲν· καὶ τὰ ἐννέα σχήματα καὶ αὐτὰ*

[1] That the Indians not only had a knowledge of algebra at a remote period, but made great progress in the employment of it, is doubtless true; but the Greeks also knew it at a very early date. (Diophantus can hardly have been its first originator among the Greeks, and have advanced *per saltum* to a stage beyond even the Indian algebra.) And though it is quite possible that, through the intercourse between the two nations, one may have borrowed from the other algebra and similar inventions, yet there is nothing to prove that it was indigenous with either, or may not even have been borrowed by both from some common source. (Cf. Reinaud, Memoire sur l'Inde, p. 303.)

'Ινδικά εἶσιν (see Cantor, p. 373 and Rechenbuch das M. Planudes, Gerhardt, C. J., Halle, 1865, p. 1). Neophytos too, writing about the same period, expressly speaks (Cantor, p. 418 and note 497) of the zero and its companion figures as of Indian origin.[1] There can therefore be no doubt, as far as the signs themselves are concerned, that their Indian origin was known and acknowledged in Europe at that date. The term of 'Indian arithmetic' was known, too, but it was also applied to certain special methods of working, which were later improvements on Al Khwárizmi's methods, of which more will be said presently, and it is possible that it may have been confined to these later methods only.

It may, however, be said that, so far as direct evidence alone is concerned, there is a fair body of testimony, and of testimony above all suspicion, and from various and wholly independent quarters, all distinctly affirming the purely Indian origin alike of the 'zero,' of the modern ciphers, and of the modern methods of decimal arithmetic.

The case, nevertheless, does not rest on direct testimony alone, however valuable or important. There is a still further and, if possible, more valuable and indirect evidence on this behalf, which it is now necessary to examine. Traces of the use of the new decimal arithmetic, at least of an arithmetic employing and based upon the value of position, are to be found in very early Sanskrit writers on arithmetic. The first of these is Áryabháta, who is known, from his own statement, to have been born at Kousámbhipúra (a town on the Jumna, situated not very far above the confluence of that river with the Ganges), in the year 475 A.D., and who may therefore be fairly assumed to have been writing and teaching in the very commencement of the sixth century A.D.

M. Leon Rodet has shown that the method which this writer employs and prescribes, for the extraction of square and cube roots, is practically identical with that of our

[1] Representations of these figures will be found on Plate I. Fig. 6. They will be seen to be for the most part derived from an Arabic model, though one set given by Cantor, from a MS. of Planudes, clearly comes direct from an Indian source. The chain of descent of these figures, and of the Böethian apices will, however, be more fully treated in Part III.

modern arithmeticians, or at least proceeds on the same principles and seems to presuppose a knowledge of the value of position; that is to say, he prescribes the breaking up of the series whose root is be extracted into groups of two numbers (or three for cubes), to be dealt with successively, a proceeding which seems to imply a knowledge of the value of position, and of the force which each cipher derived from its place in the general series. See Journal Asiatique, series vii. tome xiii. pp. 397, 405–8. Those who care to contrast the method set out by M. Rodet with the older Greek methods, will find the latter stated at length by M. Delambre, in his treatise on Greek Arithmetic attached to Peyrard's translation of the works of Archimedes, Paris, 1807.

But there is another passage in Árynabháta's work which also gives a further proof of his knowledge of position, though in order to show this a somewhat lengthy explanation is needed. He prescribes (if he did not invent) a method of numeration by a new set of 'aksharas,' made by assigning numerical values to the letters of the alphabet arranged in the method of Sanskrit grammarians according to their 'vargas' or phonetic classes,[1] and thus by means of the 'classified' consonants, twenty-five in number, the four semivowels, and the three sibilants, with the aspirate, he obtained signs for the decimal succession of numbers up to one hundred, that is, by the consonants up to twenty-five; then, for 30 and the succeeding powers of ten up to one hundred, by the semivowels and sibilants as shown below:

Classified Consonants.

Gutturals	K = 1,	Kh = 2,	G = 3,	Gh = 4,	Ng = 5
Palatals	Ch = 6,	Chh = 7,	J = 8,	Jh = 9,	Ṅ = 10
Cerebrals	Ṭ = 11,	Ṭh = 12,	Ḍ = 13,	Ḍh = 14,	Ṇ = 15
Dentals	T = 16,	Th = 17,	D = 18,	Dh = 19,	N = 20
Labials	P = 21,	Ph = 22,	B = 23,	Bh = 24,	M = 25[2]

Semi-vowels	Y = 30,	R = 40,	L = 50,	V = 60
Sibilants	Ṣ = 70,	Sh = 80,	S = 90,	H = 100

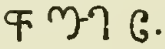
[1] This *principle* was probably known to the Indians long before. See remark by Dr. Bühler in Part I., but this particular application of it is new.

[2] It is evident that possessing signs both for the units and for thirty, for ten and for twenty, *i.e.* the intermediate places between twenty-five and thirty would be expressed by the use of these.

The passage in question, however, occurs with reference to the use of the vowels and diphthongs. Of these A´ryabháta prescribes the use of the *short* vowels only, that is to say (the 'a' being inherent in the other letters), of the *ĭ*, *ŭ*, *r*(*i*) and *lr*(*i*), and of the double vowels *e*, *ai*, *o* and *ou*. These are to be employed only in *connexion* with the others, to which they add a step of *two* decimal places each, and the passage is to the effect that these in succession, added to the other consonants, give birth each to a couple of 'khas.' Now 'kha' is a well-known term for the 'zero,' and is in its intrinsic meaning equivalent to 'ṣúnya,' the term usually employed; both in their primary sense signify 'emptiness,' 'a void.'[1]

A´ryabháta also uses the word 'sthána'=place, to signify the position of the numeral signs, a term which also may seem to imply a knowledge of fixed places in a decimal series. It was probably taken from the 'columns' of the abacus. This point however is not perhaps, in itself, of much force.

Another writer, Varáha Mihíra, living also in the sixth century A.D., but somewhat later than A´ryabháta, was the author of a work called the 'Brihat Sanhita,' and employs the word 'ṣúnya' in a method which pretty certainly shows that he must have had some knowledge of the value of position. I take the liberty of using a paragraph of a private letter from Dr. Bühler to myself, which puts the facts in a singularly neat and clear manner.

"I conclude from the occurrence of the word 'ṣúnya' in the writings of Varáha Mihíra that he knew the modern system. For if a man expresses (see Brihat Sanhita, viii. 20) the numbers three thousand seven hundred and fifty by the words,[2] the nought (emptiness), the arrows, the mountains, and the Ramas, it seems to me that he must have thought of 3750, and cannot have had in his mind .[3] If he had

[1] The force of the argument, as will be seen later on, rests mainly on the use of these terms. The actual employment of this mode of notation *might* have been suggested by a knowledge of the Greek 'octads,' as hinted by Reinaud, Memoire sur l'Inde, p. 303.

[2] These words are of course 'aksharas' or 'phonetic numerals.'

[3] =3000 =700 =50.

the latter before his eyes, he would have said, or used words equivalent to, the three thousand, the seven hundred, and the fifty. There are of course hundreds of similar instances in the Brihat Sanhita."

In other words, by employing four distinct and separate phonetic symbols to express a number which under the old system would only have required three such symbols, Varáha Mihíra shows that he was dealing with the modern, and not with the old system of numeration, and was at least acquainted with the value of position, which demanded the use of as many symbols as there were decimal places in the series of numbers to be expressed.

Perhaps these facts will be accepted as sufficient to show that the Indian mathematicians of the beginning of the sixth century A.D.[1] were at least acquainted with the value of position, and with the use to which it could be put for arithmetical purposes; and that the simplification of the Indian numeral system had at that date advanced by the initial and most important step. But was it then *complete?* did the writers then employing the terms 'súnya' and 'kha' use them in their more recent sense of 'zero'? and were they acquainted with that part of the invention also?

This is a point of very considerable importance. If it be conceded that they had *no* such knowledge, it will no doubt clear up a good many of the difficulties which have hitherto obscured the history of the simplification of the numeral system; an attempt will therefore here be made to show that such a supposition is at least rendered probable by the facts which are now known.

M. Woepcke, in the passage already cited, has shown that it is quite possible to use the value of position by means of the 'tableau à colonnes' without any zero; and, as will be explained later, there can be no doubt that it was first known in Europe under this form. But he seems to have taken for granted, that in India the zero was in-

[1] It is, no doubt, possible that similar evidence may be discovered as to the knowledge of still earlier writers; but it is enough for the purpose of this inquiry that the case goes back even as far as the first half of the sixth century A.D.

vented simultaneously with the value of position. Humboldt,[1] while he claimed for the ancient classic nations a knowledge of the value of position, admits that it was 'sterile,' and attributed the latter fact to the want of the knowledge of the zero. So far as I am aware, M. Leon Rodet[2] in the 'avant propos' of his paper in the Journal Asiatique of 1880, first suggested that A'ryabháta might have known the value of position without being acquainted with the zero, or at least might have known the value of position only as exhibited on the abacus; though even he seems to have *inclined* to a contrary opinion.

As Dr. Bühler and M. Woepcke both point out, the words 'ṣúnyá' and 'kha' mean 'emptiness,' and M. Rodet, J. A., series vii. tome xvi. p. 463, goes on to suggest that the word had originally reference to the 'place vide,' on the abacus, by which the function of the modern zero was certainly once fulfilled. "Les deux noms indiens de zéro शून्य *çúnya* 'vide' et surtout ख 'kha,' et ses synonymes व्योम 'vyôma' वियत 'viyat,' अम्बर 'ambara' (que j'ai relevés dans le Súrya Siddhanta), l'atmosphere, l'air, l'espace, conviennent admirablement à l'expression d'une 'case vide' beaucoup mieux qu'au nom d'un signe quelconque. Aben Ezra, dans son 'Traité d'arithmétique,' appelle le zéro (qu'il fait tout rond)

[1] "The method of the Pythagorean abacus as we find it described in Boethius' Geometry, is almost identical with the positive value of the Indian system, but that method, long unfruitful with the Greeks and Romans, first obtained general extension in the middle ages, especially after the zero sign had superseded the vacant space" (Kosmos, Murray's ed. vol. ii. p. 164). "Even the existence of the cipher or character for '0' is not a necessity for the simple positive value, as the scholium of Neophytus shows" (Kosmos, Murray's ed. vol. ii. p. lxxxi). "What a revolution would have been effected in the more rapid development of mathematical knowledge . . . . if the Brahman Sphines, called by the Greeks Calanos, or . . . . the Brahman Bargosa had been able to communicate the knowledge of the Indian system of numbers to the Greeks" (Kosmos, Murray's ed. vol. ii. p. 164).

[2] "Au moment que j'allais conclure et attribuer à A'ryabháta l'usage de notre système decimal ecrit, un scrupule m'est-venu: les calculs qu'il enseigne à faire peuvent s'effectuer conformément à son règle sur un abaque; le nom que les Indiens ses successeurs comme lui donnait au zero, a du être inventé à une epoque ou l'on faisait usage d'un abaque sur lequel le zero n'est marqué que par une place vide. A'ryabháta effectuait il ses calculs sur l'abaque, et, ... se contentait et de transcrire les resultes a l'aide d'un systéme de chiffres decimaux mixtes . . . ? Voila un point capital que je suis contraint de laisser sans solution, attendant que des documents nouveaux viennent nous fournir des éclaircissements qui nous manquent."—Journal Asiatique, series vii. vol. xvi. p. 443.

גלגל 'galgal,' une roue, un rond. Jamais on n'a rencontré en Sanscrit le zero[1] designé par चक्र 'cakra' (chakra) un cercle, ni par बिन्दु 'bindu' 'un point.' Ainsi ce nom de 'vide' et d'espace fait fortement pencher la balance du côté de l'abacus, du tableau à colonnes."

Other similar equivalents are given by Albiruni (J. A. series vi. tome i. p. 284), 'akáṣa,' 'gagana,' 'abra,' all meaning 'the heavens,' and in the Nouveau J. Asiatique, vol. xvii. p. 16, 'ananta' or 'space' given as another term. It is hardly too much to say, therefore, all the various 'aksharas,' by which the zero is designated in Sanskrit, convey one idea, and one only, under various different forms, viz. 'empty space,' and do not certainly indicate the use of any particular sign or figure. While therefore the use of these terms as arithmetical expressions wherever they are found, though it certainly involves at least a knowledge of the 'place vide,' and therefore of the value of position, does not by its own force seem to imply any knowledge of the sign for 'zero.' Dr. Bühler, indeed, informs me that he has found the word 'ṣúnya' used in inscriptions in the sense of a 'lacuna' in a MS., and has found sometimes actual lacunæ designated in documents of very ancient date by the points or dots which are now sometimes used for the 'zero,' but neither fact seems to derogate from the force of the argument above stated; indeed, the former rather strengthens it; as to the latter, it will be dealt with further on when treating of the original sign for zero.

If, however, the fact be admitted that at least as early as the time of Varáha Mihíra, that is to say, some time before the close of the sixth century A.D., the value of position was fully known and taught and used in India, it is a somewhat remarkable fact that, for all official purposes, such as grants, inscriptions, etc., the *old system of notation* was employed till well into the second quarter of the seventh century. A number of inscriptions of the Valabhi kings, executed in the fifth, sixth, and seventh centuries, exist, and

[1] This remark refers to the later forms of the Sanskrit zero the 'o' and the '•'—As to this, more will be said immediately.

even some of the Chalukya dynasties, all dated in figures, many of which are certainly of later date than Varáha Mihíra; two of the Valabhi grants, indeed, of Siladitya V. and VI., are probably of 631 and 637 A.D.,[1] and all belong to the old system.

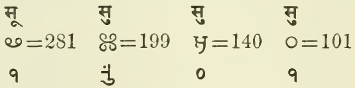
Now if the use of the value of position *with the zero* was known, and publicly and generally taught as early even as, say, 575 A.D., it is hardly likely that so convenient a system would have been ignored in official use for more than half a century, if not for more than a century. Indeed, it does not make its appearance in actual use for nearly half a century later still. The earliest example at present known is dated in 738 A.D.[2] On the other hand, when the new system with zero was once introduced, it seems to have almost immediately and completely to have superseded and swept away the older system, except, indeed, in one or two remote places not open to much external intercourse, such as Nepal, where neither the value of position, nor the newer and more convenient Western form of the numerals seem to have been introduced for several centuries later on. There is, however, one very remarkable exception to be made to this assertion; for among the Tamil and Malayalam speaking populations of Southern India the old system of notation was retained, is indeed retained to the present day; subject, however, to one fortunate modification, that is to say, that while the Tamil and Malayalam systems of numeration know nothing even now, (in their proper indigenous forms) of either zero or value of position, they have yet rejected the old signs for the powers of ten, replacing them by compounds of the several units differentiated by the sign for ten, the ten not being used however as a zero, but in one integral group with the unit which it differentiates. This change is important, and will supply a material link to the argument further on. Putting aside these exceptions, I have only been able to trace two

[1] These are dated in 441 and 447, which I have given in the Numismatic Chronicle reasons for believing to be in an era dating from 189 or 190 A.D.

[2] This grant, which is yet unpublished, is in the possession of Dr. Bühler, who kindly furnished me with a facsimile. It is one by Jaika Rashtrakúta of Bharuj and is dated in 794 'Vikramaya.' It was found at Okamaudel.

instances later than 738 A.D. of the use of the old method, one in a grant[1] of Govinda III., Rashtrakúta of Malkhéd, d. 730 Saka=808 A.D., in the body of which the old symbol for twenty occurs (in a slightly modified form). The other instance is a curious one, which was brought to light by Dr. Kielhorn, in his report on Sanskrit MSS. at Bombay for 1880-1. The oldest MS. which he found was written at the end of the eleventh century, and other MSS., all on palm-leaves, bore dates of the twelfth and thirteenth centuries. He says, "In nearly all of them the leaves, in addition to being numbered on the right-hand side with the ordinary numeral figures now in use, are also numbered on the left-hand side with the more ancient numerals mentioned by Pandit Bhagwanlál Indraji, in the Indian Antiquary, vol. vi. p. 42."[2] As a matter of fact, however, this system is not the *old* system, but a singular medley of the old and new, employing the 'aksharas' for 100 and 200, *written in modern Devanagari*, and in some cases the aksharas for the units. With these appear the old *numeral signs* for the powers of ten, while in some cases these are all mixed with the 'zero' and with modern units!

In all these cases the separate numerals are placed perpendicularly one over the other, the hundreds uppermost, the 'tens' in the middle, and the units lowest. Thus:



It will be seen that the hundred place is in every case represented by 'sú,' the akshara for 200, or 'su,' the akshara for 100, but rendered into the modern Devánágari. The tens are represented by the *old* signs in every case but one, in which they are replaced by the modern zero, while the units are sometimes shown in the Devánágari 'aksharas,' but usually in modern figures! Dr. Kielhorn says that there are indications that this system had ceased to be understood even when

[1] Found at Rádhanpúr in 1873-4. See Indian Antiquary, vol. vi. for 1877, p. 59.

[2] Professor Jacóbi has kindly favoured me with other similar examples from Jain books.

these MSS. were being written.[1] This remarkable survival, therefore, may to some extent be looked upon as a kind of mechanical imitation—retained perhaps out of some superstitious feeling—but no longer serving any useful purpose, and replaced for practical objects by the modern numerals which accompany it. The most remarkable point in it is the fact that the old letters are written one over the other, as if the idea of the value of position, which to some extent they possess, had been borrowed from the horizontally-held abacus.

Except in these isolated cases, however, the adoption of the new system, when *once* it is found in its perfect state, seems to have been singularly prompt and complete, and it is hardly comprehensible that if Áryabháta and Varáha Mihíra, and their immediate successors, had known and publicly taught the complete system in the early part or middle of the sixth century A.D., and had employed it in their written works, that its general adoption should have been so long delayed. On the other hand, if the value of position was known and used in India without the zero, it can only have been used with some such contrivance as the tableau à colonnes, and if the tableau à colonnes with its value of position was at first known alone, it is of course palpable that, however useful it might have been as an instrument for effecting arithmetical calculations, it was too clumsy a method for ordinary employment in indicating numbers and dates; and this fact would easily explain why, for a century or more, the two systems remained in full parallel use, though for different purposes.[2]

It may of course be objected that in no existing Sanskrit MS. is there any instance of the use of the tableau à colonnes; but in reply it must be said that no MSS. are extant of a date prior to, or indeed in any way approaching

[1] Dr. Kielhorn gives facts which seem to bear out this statement, in the succeeding pages of his report, to which it is only necessary to refer in this place.

[2] It may be remarked that Dr. Bühler has more than once drawn attention to a similar fact—disclosed by recently discovered inscriptions—viz. that the early Indians certainly employed *two* modes of writing contemporaneously—one stiff and formal for official purposes, the other cursive for general use.

that, when we *know* that the zero was actually in use, viz. 738 A.D. Of course, when the perfect system was known, all the older arithmetical works would ere long have been, when reproduced, rewritten in the form of a fresh recension, adapted to the new discovery. There are, however, some positive indications still traceable which seem to show that the 'tableau à colonnes' *was* once, and at a very early date, in use in India, but that it was also dropped at a comparatively early date. The first of these has been pointed out by M. Rodet in his paper, already quoted (J. A. vol. xvi. series vii. p. 463), in the following words: "Un autre fait sur lequel mon attention à été appelée tout recemment, vient encore, à mon avis, appuyer cette manière de voir (*i.e.* the view that the word 'ṣúnya' originally indicated only the 'place vide' on the abacus). On sait que dans la grande majorité des manuscrits arabes et persans ou l'on rencontre des calculs arithmétiques, ces calculs sont effectués dans des tableaux à colonnes,[1] auxquel il ne manque pour les rendre identiques aux '*abaci*' des calculations occidenteaux, que les 'arceaux' '*arcus*' qui surmontaient chaque colonne et les groupaient trois par trois. M. Cantor a qui je dois de connaître la presque universalité de cet usage, que je n'avais eu lieu de remarquer encore que sur quelques manuscrits, l'attribue à un emprunt fait par les Arabes aux Occidentaux. Cet emprunt serait d'autant plus étrange que ce mode de calcul assez peu commode à été de bonne heure abandonné en Occident, et que, des le xv^e^ siècle, les auteurs de traités de calcul ont supprimé les barres de séparation des colonnes, et

[1] L'emploi est formellement prescrit dans un traité d'Arithmétique, probablement assez ancien, qui fait partie du manuscrit 169 fonds persans de la Bibliothéque nationale. L'auteur (Mahmûd ben Mohammed 'Qíwám ul Qázy, de Valisthán, surnomme Mahmûd de Herat), ne manque pas de dire a chaque operation: "*Tariq é ʿamal ienan ast, ke* Jadúli *ʾasm kunand, ke ʿadad é sutûr e tûlî é ú matasavi e ʿadad é mafaradát é án ʿadad shavad ké,*" "la manière de faire cette operation est celle-ci: on trace un tableau dont le nombre des lignes (colonnes, bandes) en longitude (cette à dire comprises entre deux meridians d'une carte) soit egal au nombre des places du nombre que." Cet auteur n'efface pas les chiffres a modifier il ecrit le nouveau chiffre "*dar zîr é digar baʿ ad az Khaṭ i ké án ra Khat-i-é máhy khwánand*" au-dessous de l'autre après une ligne que l'on appelle 'linea occultans.' Cette dernière expression, empruntée a la grammaire syriaque, doit elle faire croire à une origine syriaque de 'jadûl' de notre auteur, (may not 'jadúli' rather mean a form for a 'magical table,' such as used for incantations, and amulcts, from the old Persian 'jádú' 'magic,' or 'witchcraft').

superposent leur chiffres, en barrant (non plus en effaçant) ceux que ne sont que d'un emploi transitoire, procédé déjà employé par Aben Ezra à Rodez en 1156. En voyant l'usage du 'tableau à colonnes'[1] répandu surtout en Perse et particulièrement dans le Khorâsân, tout à côté de l'Inde, je serais porté à croire bien plutôt que l'usage de ce tableau a été emprunté par les Persans orientaux aux Indiens en même temps que l'usage des chiffres. Et comme, ainsi qu'on va le voir tout à l'heure, j'ai de fortes présomptions pour admettre que les elemens de la notation numerique indienne ont eu une origine égyptienne, tout comme, suivant l'opinion qui tend à prévaloir, les apices de Boéce et de ses successeurs de l'occident, il n'y'aurait rien d'impossible à ce que les mathématiciens de l'Inde aient, comme ceux des pays latins, reçu l'usage du tableau à colonnes en même temps que celui des chiffres, de la même source à laquelle les Latins l'avaient emprunté, et que de l'Inde, l'emploi de ce tableau ne soit passé en Perse, puis dans toute l'école Arabe fondée en définitive par des Persans. Peut être si nous arrivons jamais à posséder le texte arabe du traité d'arithmétique d'Alkhwârizmi, dont l'opuscule publié par le prince Boncompagni (Algorismi de numero Indorum) ne saurait être une traduction fidéle, peut être, dis je, verrons nous se confirmer l'hypothèse que j'emets en ce moment sur l'emploi, dans les pays voisins de l'Inde, et partout dans l'Inde elle même, du 'tableau à colonnes,' de l'abacus, sur lequel les compartiments (sthânâni) repondant à tel ou tel ordre d'unités, qui manquait dans le nombre à ecrire, restaient 'vides,' çûnyâni=spacia vacua."

It will be seen from the above that M. Rodet has already divined from the facts before him that the use by the Arabs and Persians of the 'tableau à colonnes' was, in all probability, derived from India, though he was inclined to consider that Greeks and Indians both originally derived it from the Egyptians with the numeral figures. How far this last conjecture is probable will depend on the value to be attached to the remarks already made in Part I. It will be now

[1] I omit here a note by M. Rodet, which I hope to reproduce when the subject of the "Gobar" ciphers comes under consideration.

endeavoured to adduce further evidence in support of the remainder of M. Rodet's suggestion that the 'tableau à colonnes' was in very early use in India, and that the terms employed by later Indian writers to designate the 'zero' derive their origin from the 'place vide' upon it.

Allusion has been repeatedly made to the work of Mahomed bin Musa 'Al Khwárizmi,' written about the close of the first quarter of the ninth century A.D., and Reinaud ("Memoire sur l'Inde," p. 304) has the credit of first pointing out[1] that the mediæval term for arithmetical science 'Algorism' or 'Algorismus' was really a corruption of the title 'Alkhwárizmi,' 'the man of "Khwárizm," by which this writer was distinguished. The discoveries of Prince Buoncampagni and others have now placed this beyond question, and prove that Alkhwárizmi's work was known by Latin translations, at least in the twelfth century A.D.[2] The work of Leonard of Pisa further shows that the term 'Algorismus' was specifically used to designate a particular method of arithmetical working—itself an improvement on the 'abacus' or arcus Pythagoreus, but which also, in the time of this latter writer, had itself begun to be superseded by another yet more improved method; and this last, coming apparently directly from India, was specifically known as 'Indian.' It is proposed to extract from M. Woepcke's "Traité sur l'Introduction de l'Arithmétique Indienne en Occident" the description of Leonard of Pisa, and to abridge M. Woepcke's remarks on that passage, and then, taking the account given in the same work of Alkhwárizmi's mode of multiplication, it will be attempted to show that, while that author employed the 'zero,' yet that his method of working, which, as has already been said, was avowedly *Indian*, shows traces of having been at least invented on a tableau à colonnes, and it

[1] Though, as Prince Buoncampagni shows, he had been anticipated by a writer in the thirteenth century.

[2] See M. Woepcke, Journal Asiatique, series vi. vol. i. p. 518. M. Woepcke considers that it came probably through the school of Toledo, where Adelard of Bath studied in 1130, Robert of Reading in 1140, William Shelly in 1145, Daniel Morley in 1180 (all Englishmen), and Gerard of Cremona about the same time. M. Woepcke quotes Wallis, De Algebra. tract. hist. et pract. Operum Math. vol. ii. p. 1216.

will be further shown that Alkhwárizmi's methods were capable of use upon, and were indeed probably sometimes still employed up to a comparatively late date, with the 'tableau à colonnes' or chequer tables; if indeed that was not still their ordinary mode of employment, even when the use of the zero had rendered any tabular form no longer indispensable.

To begin, however, with the description of Leonard of Pisa. Speaking of it himself, he says, "Génitor meus . . . . me studio abbaci per aliquot dies . . . voluit . . doceri. Ubi ex mirabili magisterio in arte per novem figuras indorum introductus, scientia artis in tantum mihi pre ceteris placuit, et intellexi ad illam quod quicquid studebatur ex eâ apud egyptum, syriam, greciam, siciliam, et provinciam cum suis variis modis, ad que loca negotiationis tam postea peragravi per multum studium et disputationis didici conflictum. Sed hoc totum etiam et algorismum atque arcus pictagore quasi errorem computavi respectu modi indorum." Without going further, it may be seen that Leonard of Pisa thus distinguishes three distinct methods—the abacus, the algorismus, and the Indian method, which latter he proceeds to praise extravagantly and to announce his intention of describing, as, in fact, he proceeds to do. M. Woepcke remarks, "Quant aux arcs de Pythagore ce nom designe la méthode de l'Abacus telle qu'elle est décrite par Boèce et developpée dans les traités d'auteurs chrétienne du X^e et XI^e siècle. En effet nous avons vu que l'invention de cette méthode est attribuée par Boèce aux Pythagoreens, et que ceux-ci appelaient, d'après le même auteur, le tableau à colonnes la table de Pythagore. Dans les manuscrits des traités de l'Abacus on trouve que chacune de ces colonnes est surmontée d'un arc de cercle, et que de plus grands arcs embrassent les colonnes trois à trois. De là le nom d'arcus Pythagorae donné par Leonard de Pise à la méthode de l'abacus." [Traité sur l'introduction, pp. 15, 16.] Further on, at p. 46, M. Woepcke says, "La valeur de position est commune à tout ces systèmes; aussi bien à celui de l'abacus et de Boèce, qu'a celui des Indiens, soit dans la reproduction d'Alkhwárizmi, soit dans cette de Leonard de Pise et de Planude. Mais le système de l'Abacus et de Boèce n'emploi que neuf chiffres, tandis que les autres en emploient dix."

As regards the abacus and the so-called method of Boethius, further remarks may be deferred till the claims of the Neo-Pythagoreans to the invention of the simplified decimal unit system come under consideration. It is with the method only of Alkhwárizmi that the argument is at present concerned, and it remains now to show that, as has been just suggested, while this used the 'zero,' and therefore had no need of the tableau à colonnes, yet that its forms seem to bear traces of having been invented to suit the latter arrangement, and were capable of being used with it; indeed it is certain that they were occasionally, perhaps ordinarily so used.

Without going in detail into the method of Alkhwárizmi (which will be found discussed in full in M. Woepcke's two papers, to which reference has been made), it may suffice to refer to the rules prescribed for multiplication, and these are set out by M. Woepcke in the forms now reproduced. They are not of course given as the actual tables of Alkhwárizmi's work, which unfortunately are not available for reference. +

The first of these, however, is that which most closely touches the present point, and, as will be observed, according to it, the *first* products of multiplication are written down at the *top* of the form, at the *bottom* of which the multiplier and multiplicand are set forth. Now, as the rules require the products of multiplication to be harmoniously arranged *with reference to the decimal places of the multiplicand and multiplier*, it would manifestly be very difficult, if not impossible, to set down the two sets of figures at so great a distance apart, correctly and in their proper decimal places, without some such guide as the "tableau à colonnes" would afford.

+ The dividing lines between the columns are not given by M. Woepcke, but are added here for the sake of clearness

I.

4	9	7	7	6	4
				2	4
				6	
		1	2		
				8	
			2		
		4			
		1	2		
		3			
	6				
		8			
	2				
4					
		2	3	2	6
2	1	4			
	2	1	4		
		2	1	4	
			2	1	4

II.[1]

		7				G
		6	**7**			F
	9	5[2]	6	**6**		E
	8	1	4	4		C
4	2	8	2	8	**4**	A
		2	3	2	6	
2	1	4				
	2	1	4			
		2	1	4		
			2	1	4	

This is not so apparent with the second form, and it is also to be observed even of the first, that the directions in the original do not seem to necessitate the method adopted of writing the first methods of multiplication at the *top*, but the text of the rules is rather obscure and very possibly corrupt.

[1] This may be explained as below, *the thick* letters in the figure above expressing the ultimate product. The results are written without carrying the tens, etc., but these are set down (mentally) as follows: thus—

A. 2326× 4 = 8284, which write, carrying 1020.

B. 2326×10 = 23260 (of which write only 200000 before 8284) and carry nothing.

C. is A+B = 2144(4, which write, omitting the last four, which is already entered, and carry 10100.

D. 2326×200 = 464200, carrying 1000.

Add C. 2144(4 (N.B.—This is a mental operation not shown at all.)

Result = E : 234)8564(4, carrying 4200.

Write down above C., with the 8, however, in the line with C., and the 4 in a line with A., in which also the first '4' will be included. Now commence to add the sums carried; 1st, the 1020, from A., which makes the 5000 and the '40' in line E. 6000 and 60 respectively. Write the 6 of the 60 in line E., and the 6 of the 6000 in line F., and add the 10,100 from C. This will make the 80,000 into 90,000, and the 600 of E. 7000. Write the 9 in line E. and 7 in line F; then add the 1000 from D., which makes the 6000 in line F. into 7000; write the 7 in line G. which completes the operation.

[2] The figure '5' is substituted for the '2,' given at p. 23 of M. Woepcke's Traité, whence the example is taken, and which is clearly a typographical error.

The Persian MS. already described, as quoted by M. Rodet, seems, however, to favour the idea that the first result is to be set down at the top of the sum. By the first method it will be observed that the whole of the results of multiplication are fully set out from the first, and that nothing is set aside to be carried over to the next product, but this is not the case with the second method, by which a certain amount of carrying is necessitated. The use of a tableau à colonnes, of course, though practically necessary, renders the former method easy. The second is manifestly shorter and more compact; still it will be seen by those who attempt to work the sum that it is not *easy* even here to do so without the aid of the table. It seems, therefore, at least far from improbable that the peculiar forms which the Indian arithmetic assumed at first were due to the fact that the processes to which they apply were invented on, if not suggested by, the use of the tableau à colonnes.

The practice of "carrying," by which the modern systems have been so much simplified, probably was suggested by the continued use for arithmetical purposes of the ancient form of the instrument—a board covered with earth or sand. In fact, some of the early Arabic writers expressly describe the operations of arithmetic by directing the effacement of some of the results temporarily written down, and the substitution of those which come out of the final operations. Finally, the two systems of multiplication given above and the so-called Indian system of Leonard of Pisa may be taken as showing the successive steps by which the "carrying" process grew up.

But apart from this suggestion, the probability that Al-khwárizmi's methods were actually intended for use in the tableau à colonnes (and they were Indian methods) is greatly strengthened by the fact that they actually were so employed. For example, in Cantor's book, at pp. 144–45, will be found a description of a work, entitled the "Margarita Philosophica," published by one Gregorius Reesch at Freiberg in 1503, where the "Algorithmus" methods of calculation (they are described under that name) are applied to a table (of which a copy will be found at pl. iii. fig. 34 of Cantor's work[1]), on which the calculations are worked out with counters, and which is palpably a modified form of the tableau à colonnes. Again, it has been seen that the English chequer board, which was quite clearly only a form of the tableau à colonnes, was early used in England with counters at a time when Algorism was practically synonymous with arithmetic. Indeed, Chaucer, speaking of the Clerke of Oxenforde in the Miller's Story in the Canterbury Tales, connects the "stones" or "counters" with the "augrim," or algorismus:—

> "His almageste and his bokes grete and small,
> His astrolabe longing for his art,
> His augrim stones layen faire apart,
> On shelves couched at his beddes hed."

And, indeed, the practice of reckoning by counters certainly survived till the time of Shakespeare, who makes his clown in the Winter's Tale say, "Let me see! every 'leven wether—tods, every tod yields—pound and odd shilling; fifteen hundred shorn, what comes the wool to? . . . . I cannot do't without *counters*."—Act iv. Sc. 2.

In the absence of a perfect example of Alkhwárizmi's work with tables of examples, it may perhaps be allowable to put forward this inferential evidence that the Indian methods of arithmetic which he put forward were originally suggested by, founded upon, and employed upon the tableau à colonnes; and if so, this fact affords additional evidence that the earlier Indian arithmetic, which first employed the value of position, can hardly have possessed also the 'zero,' for that would have quickly rendered the tableau à colonnes unnecessary; and, in fact, this had disappeared in India apparently not long after the period when it first appeared in Europe.

Another argument which favours perhaps the notion that the Indians knew and used the tableau à colonnes, in the first instance *without the zero*, for purposes of calculation, may perhaps be drawn from the fact established by M. Woepcke in

[1] See Journal Asiatique, series vi. vol. i. p. 497, where a quotation is given.

his later essay (in Journal Asiatique, series 6, vol. i. p. 500), viz. that the Indians were acquainted with the "proof by nine," as shown by the treatises, both of Alkhwárizmi and of Avicenna, and which indeed is expressly declared by the latter to be an *Indian* method. It may, perhaps, not be deemed a very far-fetched hypothesis to suggest, that the invention of such a method would be most naturally prompted by a mode of working wherein 9 was the highest figure known, and played such an important part in the tableau à colonnes, as the highest though incomplete expression of the decimal series.

Before taking leave of this portion of the case, it is necessary also to say that the tableau à colonnes, as first found in Europe, and in the hands of the Neo-Pythagoreans, bears distinct traces in more than one respect of an Oriental origin. This point will be more fully set out when the case for the Neo-Pythagorean origin of the new decimal arithmetic presently comes to be examined.

If, therefore, the invention of the value of position was known to the Indians in the beginning of the sixth century, there is at least no proof that they discovered the zero simultaneously,—no evidence, indeed, of its use at all, prior to the commencement of the eighth century A.D. On the contrary, there are facts which seem strongly to indicate that the value of position was, during that interim, put to practical use in India by means of a written abacus or tableau à colonnes, such as was afterwards employed by the Neo-Pythagoreans, and which would hardly have been needed if the new system started in life already furnished with a sign for zero. Again, the approval with which the Arabs received the new system may be accepted as proof that they knew nothing like it before. They obtained it from India only in 776 A.D.; but they had already, some 70 years before, overrun and occupied the Indian province of Sind, and the resulting fact, that the use of the zero, at least, had not become generally known in Sind at the *commencement* of the eighth century A.D., is one which seems to limit pretty closely the earliest date of its invention.

It remains therefore to seek the origin of the zero in India itself, and it will now be attempted not only to show that this sign was of indigenous growth, but to indicate the manner in which it may have been originally suggested, the approximate probable period of this suggestion, and finally to trace it through the stages by which it reached ultimate perfection.

What has been said as to the intrinsic meaning of all the Indian names or 'aksharas' for zero, and the probable connexion of the idea which underlies them all with the 'place vide' of the tableau à colonnes, need not be repeated, though, of course, this evidence affords in itself a strong argument in favour of the Indian origin of the sign—an argument which is still further supported by the manifest derivation of all the European terms for this sign from the Arabic word صفر (ṣifr), which it need hardly be said is itself a direct and literal translation of the Sanskrit 'súṇya.' It has the exact intrinsic meaning, in fact, of súṇya, and since, as has been shown, the new Arabic arithmetic was avowedly derived from the Indian, the derivation of 'sifr' from súṇya is beyond doubt. The Neo-Pythagorean 'sipos' seems to be really only a partial transliteration of 'sifr,' or of its first syllable with a Greek substantive termination added, and it will be attempted in the sequel to show that the Neo-Pythagoreans in all probability derived their knowledge, of the zero at least, from India *through* the Arabs; though it is possible that the actual shape of the word they used, may have been adopted in order to bring it into some resemblance with the Greek *σιφλός* or *σιφνός*, which had the same meaning صفر 'sifr.' Be this as it may, the term in Planudes and Neophytos is *τζίφρα*, a term which is certainly not of *Greek* origin, and can hardly be anything but an attempted transliteration of 'sifr,' which the Greeks had converted into 'zifr,' either from a confusion between ('swad') ص and ('zwad') ض, or from inability to render the peculiar sound of the former. In Leonard of Pisa the word becomes 'zephyra,' whence the transition to zephiro, zefiro, zefro, and finally to 'zero,' is easy. On the other hand, the Greek *τζιφρα* would naturally in French become 'chifre' or 'chiffre,' whence undoubtedly our 'cipher,' or 'cypher.' So far as the European names of the sign go, therefore, they are clearly traceable through the successive stages of Latin, Greek, and Arabic to the Sanskrit, and no further back.

As regards the sign itself, it is to be remarked that there still exist in current use in Southern India (as has been already pointed out) two systems of numeration, the Tamil and the Malayalam, which to this day make no use of the value of position or of the zero, and which preserve entire the principles of the old Indian notation, indeed its details also, with one exception only; but that exception is a very singular one, and for our present purpose important and instructive.

The Tamil and the Malayalam both reject the arbitrary signs for the powers of ten (except the sign for 'ten' itself) of the older system, of which it has been suggested in Part I. that they were later additions to the Indian numeral system, borrowed or compounded from various sources.

These signs are in both Tamil and Malayalam replaced by a series of symbols which perform exactly the same functions, but which are in effect nothing but the unit signs, from 'two' upwards, *differentiated by* the sign for 'ten,' which is placed *after* them, whereas in writing eleven, twelve, thirteen, etc., the sign for ten is placed *before* the sign for the unit.[1]

But, as has been said, the ten, even when thus compounded, does not fully discharge the functions of 'zero.' The new combinations each form one new integral sign, and when used with the unit to represent such numbers as 21, 22, etc., each is written out at full length before the second unit signs, which are separately added. Thus, while the Tamil ௨ represents 'two,' and ௰ or ௰ stands for 'ten,' then ௨௰ in composition expresses '20'; but it is necessary in order to give '22' to write ௨௰௨, as if it was '20.2.' So in Malayalam ൨ is 'two,' and ൰ is 'ten,' ൨൰ is '20,' but ൨൰൨ is 'twenty-two.' The hundreds are similarly treated,

[1] I do not here speak of the Cingalese ancient numerals, still used for some purposes, and which present even a still closer resemblance to the ancient Indian modes of numeration, and are therefore shown in Pl. II. Table I.

൱ is the Malayalam 'hundred' (almost the identical sign of the Valabhi or Kshatrapah periods), ൱൨ is '102,' but ൨൱ is '200.' (See Pl. II. Table I.)

Some approximate deduction may be made as to the date of this first step towards the completion of the new Indian notation from the character of these signs. In the first place, this imperfect substitute for zero would seem necessarily to have been invented before the use of the true 'zero' was known; if it had been known, so partial a reform would scarcely have been adopted at all; and, as has been seen, the use of the zero seems to have been fully established in Upper India, at least, during the second quarter of the eighth century A.D.

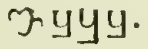
On the other hand, this use would seem from the Tamil form of the compounded numerals (if these have not been subsequently modified) to be later in date than the Valabhi inscriptions of the seventh century; for in these the 'aksharas' had hardly quite so wholly effaced the original shapes of the older signs, or so completely effected the conversion of the old numerals into the equivalents of the alphabetical forms, as is the case with the Tamil numerals. The Malayalam forms, too, point in the same direction; for some of the unit signs are palpably allied to the cursive forms, which are first found in official use with the new system and the zero, and can hardly be of much earlier date. It is true that these cursive forms when first employed for dates are so freely used as to lead to a belief that they were even then not entirely new; and this is exactly what might be expected if the new method of notation had been for some time employed, by means of the 'tableau à colonnes,' for purposes of general calculation, before the time when the addition of the 'zero' fitted it for *all* purposes, and led to its adoption even for official documents.

When, indeed, rapid calculations were thus facilitated by the new inventions, and therefore more widely applied to the general purposes of social life, the need of more simple and easily written signs than those which had grown up under the influence of the 'aksharas' would be soon felt, and it may be useful for purposes of illustration to anticipate a little the history of these changes, which more properly belongs to Part III., and to show how the three lower of the new signs were formed, and their close identity with the modern Hindi forms, १=1, २=2, and ३=3. The latter is a rather ornamental instance, but it is clear that they are all cursive forms of the ancient —, =, and ≡, the change being effected merely by writing them by a continuous stroke and without removing the pen from the surface on which it is writing. It is perhaps most probable that the invention took place in this way, and that it was indigenous. It is, however, to be remarked, that the same process had already long before established similar forms in Egypt, whence they *may* have been imported into India. I allude to the numerals specially employed in connexion with the Demotic writing (Sir Gardner Wilkinson says also in connexion with the Hieratic) to express *days of the month.*[1] These will be found in Pl. I. Fig. 6.

It may be, however, allowable, perhaps, to hazard another conjecture, which, if accepted, would indicate the way in which, the time at which, and the locality in which, this intermediate step towards the invention of the zero was first suggested.

If the interpretation which places the initial date of the Gupta era at 190 A.D., or some closely approximate date, be accepted as correct, then it will be observable that this era only reached its fifth century in 590 A.D. Except the Sáka, at that time hardly any other era seems to have been in use. The Seleucidan and Maurya eras, to whatever extent they had ever been employed, had by that time been apparently forgotten. The Vikramáditya era, even if (as is most probable) it were that in vogue among the Kshatrapah kings, had, when it had reached its fourth century, become so completely superseded by the Gupta and Sáka eras, that Mr. Fergusson and other writers have doubted whether it ever had any real existence as an ancient era. The Sáka and the

[1] See Wilkinson's Ancient Egyptians, vol. ii. p. 493, edition 1878; also Pihan, Signes de Numeration.

Gupta eras were those almost alone employed, and of these the monarchs who used the Sáka, seem rarely to have expressed it except *in words.* The Valabhi kings, who used pretty certainly the Gupta era, *invariably* expressed it in numerals. Before 590 A.D., however, according to this view, the date of the century, according to the Gupta era, would have been expressed in the old notation by the sign for a hundred differentiated by the old spur-shaped side strokes; it was only when *four* hundred had to be written that the differentiation began to employ the units in combination with the hundred figure. Thus when 444 came to be written, it would be .

Now to a person already acquainted with the method of notation according to the value of position, such a group would palpably suggest its simple expression by three consecutive unit signs for four. The difficulty would, however, still remain as to the expression by successive decimal places of the dates which had no unit place, such as 450.

Now, as has been seen, by the example of the Hindu Kábul forms, the Indian arithmeticians had boldly used already their method of differentiation, for the purpose of creating new and more convenient numeral signs; it would be a very natural step therefore for them to conceive the idea of units differentiated by a sign for ten placed after them, in order to supply a convenient arrangement by which the number of decimal places could be preserved, and the use of units according to the value of position could be made applicable to the expression of dates; indeed, for all purposes of written numeral notation whatsoever.

The existence of the Tamil and Malayalam forms, crystallized, as it were, in this first stage of transition, seems to indicate the actual reality of some process of the kind.

The new mode of notation, however, transferred to the tableau à colonnes, would at once supply a mode of filling the 'ṣúnya' or 'place vide' in the case of the 'ten'; and it can hardly be supposed that the Indian arithmeticians would have been so dull as not at once to perceive that the substitute which sufficed to fill the 'place vide' in the case of the column of tens, would fulfil the same office equally well in the column of hundreds, thousands, etc.; and thus the invention of the 'zero' would be completed!

That it was actually so brought about, and that it was actually in its original form nothing but the Indian sign for ten, there is some further evidence.

In M. Woepcke's memoir in the J. A., so often quoted, at p. 465, and in the following pages to p. 473, will be found an account of the method employed in certain Arabic MSS. for writing the *sexagesimal* zero. As to this zero, for the present it will suffice to say that Ptolemy certainly introduced a method of expressing the zero in the sexagesimal place (*but in no other*) by an 'omicron,' which, M. Woepcke contends (J. A., p. 466, note) with great probability, was a contraction of the Greek word '*ουδεν*' or 'nothing.' Now Ptolemy's Almagest, as has been already said, was known to the Arabs, and translated as early as the year 901 A.D., and several Arabic treatises were written at later periods on sexagesimal arithmetic, and on Ptolemy's astronomical methods employing sexagesimal notation.

The figures given by M. Woepcke as employed in these treatises for the purpose of rendering the sexagesimal zero, though in a somewhat conventionalized form (as might be expected, inasmuch as the actual MSS. quoted are of a comparatively modern transcription), bore so strong a resemblance to the several forms of the ancient Indian 'ten,' that I ventured to address M. Zotenberg, of the Bibliothèque Nationale, under whose charge these MSS. are placed, and he has at my request verified M. Woepcke's figures by comparison with the originals. In Table II. Pl. II. will be found both M. Woepcke's figures and those of M. Zotenberg's tracings, together with the signs for the Indian forms for ten, of which they appear to be reproductions.

The demand for cursive signs would, as was the case with the symbols for the units, tend to a reversion towards the simpler forms of the older signs, and the 'spurred' circle of the Naná Ghát "ten," may have thus become the original of our modern 'zero,' or, what is perhaps even more probable, the still simpler form of the Ptolemaic 'zero,' when it became known to the Indians (whether through the Arabs, or by earlier direct intercourse, which is quite possible), was finally adopted as the usual representative of the 'ṣúnya,' together with the 'bindu' or point which, as has been said, the Indians appear to have used to fill up *lacunæ* in MSS.[1]

The oldest figures directly derived from the Indian signs for ten, however, might well have been retained by the writers of Arabic versions of Ptolemy, and of similar works, to designate and, indeed, to distinguish the sexagesimal 'zero,' in regard to which 'cursive' writing was comparatively little needed.

It is to be observed that the new signs are first found in the upper part of Western India. Indeed, they have never fully established themselves or the new numeration of which they were the exponents, in Southern India; and neither were known for many centuries after their first invention in Nepál and the extreme East. There seems some reason, therefore, to believe that it was somewhere on the west coast of India that this great reform was completed.

The Indian claims to the invention, first of the value of position and of the zero, and the evidence which may be adduced in support of them, whether direct or circumstantial, have now been stated. Of course it is just possible that, as regards the value of position, the Indian knowledge of this, though certainly of early date, may relate back to a still earlier age than that here assigned to it, viz. the commencement of the sixth century A.D. It is possible also that this part of the invention *may* not be wholly indigenous to India, but may have come from some other Eastern source. The evidence against such an hypothesis is, indeed, so far simply negative; on the other hand, as will be presently shown, the claims made on behalf of the Greeks,

[1] The oldest actual example of the Indian 'zero,' with which I am at present acquainted, occurs on a coin in my own cabinet, of the Hindu Kábul series, which seems to read 707 (Gupta according to my view, and equal to 897 A.D.). Unfortunately the coin is in poor preservation, and the precise shape of the sign is hardly certain. It seems to be a kind of irregularly formed dot. See Numismatic Chronicle, vol. ii. N.S. for 1882, p. 111, pl. i. fig. 7.

for its first, or even for its independent invention cannot be sustained; it is pretty certainly *not* of Western origin. As regards the 'zero,' however, if the evidence adduced is deemed satisfactory, it must be deemed of *purely* Indian origin. In regard to the new signs of the units, moreover, direct evidence has been already given, which in itself would seem conclusive as to their wholly Indian origin. It will, however, be part of the subject which properly belongs to Part III. further to establish this more completely by tracing their genealogy directly through the various later forms, Gobár or Indian, in each case back to the ancient unit forms of the Náná Ghát rocks.

The next subject which it is necessary to examine is the degree to which the arithmetical knowledge of the two great classic nations of the West—the Greeks and Romans—had advanced during the time when this reform was being effected in India.

It has been necessary to show that the Arab authors, who ascribed the invention of the present system to the Indians, in effect denied the claims of the earlier Greeks; but what has been said shows that the latter had nevertheless a good independent system of numeration and calculation, capable of very extensive practical use. Fully to understand what this really was, would require an examination of the exhaustive treatise by M. Delambre, to which reference has been already made. It will suffice for present purposes to make use of an excellent review of that work (one also already quoted), which appeared in the *Edinburgh Review* of 1811, vol. xviii. Art. vii. (on the History of Numeration). It was written by the late Professor Sir John Leslie, and condenses into a brief space as much as will be necessary to show here.

Professor Leslie thus describes the ancient Greek method of multiplication, which may be selected as typical:—"In this process the Greeks appear to have followed the same method as that which was formerly practised with the cross multiplication of duodecimals and nearly corresponding to the ordinary treatment of compound numbers in algebra. They proceeded, as in their writings, from left to right. The product of each numeral of the multiplier with every numeral of the multiplicand, was set down separately, and these distinct elements were afterwards collected together into one total amount. For the sake of compactness these partial groups were often grouped or interspersed, though sometimes apparently set down at random. But still they were always noted, nor was any contrivance employed similar to that mental process of carrying successively tens to the higher places which abridges and simplifies so much the operation of modern arithmetic." These remarks will be confirmed by the following example:[1]

$$\frac{\sigma\xi\epsilon}{\sigma\xi\epsilon} = \frac{265}{265}$$

(I.) $\frac{\delta}{M}, \frac{\alpha}{M}\ \frac{\beta}{\iota}, \frac{\alpha}{\iota} = 40000,\ 10000 + 2000,\ 1000.$

(II.) $\frac{\alpha}{M}\ \frac{\beta}{\iota}, \frac{\gamma}{\iota}\ \chi, \tau = 10000 + 2000,\ 3000 + 600,\ 300.$

(III.) $\frac{\alpha}{\iota}, \tau, \kappa\epsilon \ldots = 1000,\ 300,\ 20 + 5.$

Which may be thus explained more fully:—

(I.) $\sigma \times \sigma = \frac{\delta}{M} \ldots$ or $200 \times 200 = 40000$

$\sigma \times \xi = \frac{\alpha}{M}\ \frac{\beta}{\iota}$ or $200 \times 60 = 10000 + 2000$ (12000)

$\sigma \times \epsilon = \frac{\alpha}{\iota} \ldots$ or $200 \times 5 = 1000.$

[1] For facility of reference, it may be well to set out the Greek system of alphabetical numerals as employed by their later arithmeticians, bearing in mind that it was not quite identical with the Hebrew or Arabic alphabetic methods. In the Greek system, after the first five letters, which were used to express the first five units, a special sign the 'epistemon' or 'ϛ' was inserted to represent six. The alphabetical order was then resumed till 'iota' represented 10; from this point the power of the letters rose by tens, κ representing 20, λ 30, and so on until ninety was reached, which was expressed also by a special sign, the 'koppa' or ϟ; then the ρ represented 100, from which the power of the letters rose by hundreds, thus $\sigma = 200$, $\phi = 500$, $\chi = 600$, $\psi = 700$, but the nine hundred had also its own special sign ϡ, or ϡ, termed 'Sampi.' But the thousand introduced a new mode of marking, the power of a thousand being given to the nine first units by inserting an iota beneath them, thus $\frac{\alpha}{\iota} = 1000$, $\frac{\gamma}{\iota} = 3000$. The tens of thousands were expressed by the letter M (or Μυ) for Μυρί, similarly subjoined to the unit letters, thus $\frac{\alpha}{M} = 10{,}000$, and $\frac{\delta}{M} = 40{,}000$. Of the modes of expressing yet higher numbers, whether by octads, or tetrads, or otherwise, mention will be made in the text, and special signs were also used to mark certain fractions. The mode of writing fractions, however, does not bear on the subject immediately under discussion.

+ cf. ... on the signs for the thousands & ten thousands sh^d be omitted

(II.) $\xi \times \sigma = \frac{\alpha}{\mathrm{M}} \; \overset{\beta}{\iota}$ or $60 \times 200 = 10000 + 2000$ (12000)

$\xi \times \xi = \frac{\gamma}{\iota} \; \chi$ or $60 \times 60 = 3000 + 600$ (3600)

$\xi \times \epsilon = \tau$ ... or $60 \times 5 = 300$.

(III.) $\epsilon \times \sigma = \frac{\alpha}{\iota}$ ... or $5 \times 200 = 1000$

$\epsilon \times \xi = \tau$ ... or $5 \times 60 = 300$

$\epsilon \times \epsilon = \kappa\epsilon$ ... or $6 \times 5 = 25$.

The separate addition of the figures in the several groups gave of course the final result of the operation. Professor Leslie sums up the case at p. 203 thus: "The Greek arithmetic, therefore . . . . had attained, on the whole, to a singular degree of perfection, and was capable, notwithstanding its cumbrous structure, of performing operations of considerable difficulty and importance. The great and cardinal defect of the system consisted in the want of a general mark analogous to our cipher, and which, without being of any value itself, should serve to ascertain the rank and power of the other characters by filling up the vacant places in the scale of numeration."

"Yet were not the Greeks altogether without such a sign, for Ptolemy in his Almagest employs the small '*o*' to mark the accidental blanks which occurred in the notation of sexagesimals." [1]

This extract will alone suffice to show that the ancient Greeks were practically ignorant of the employment of the value of position in ordinary arithmetical processes, and knew only a sexagesimal zero, which, though a true zero, was capable only of employment in a few exceptional cases, and was never used with the ordinary decimal arithmetic. They had, however, a system of 'octads' and 'tetrads' for expressing numbers of very high value, which in its methods came very close upon a discovery of the value of position. In fact, it amounted to an assignment of value by position to *groups* of figures, which it failed to give to separate figures, even to those *within* the groups themselves. A further step

[1] The passage in which Sir J. Leslie gives his views as to the origin of this sign is omitted, as the explanation already adopted from M. Woepcke seems, for the reasons he gives, preferable.

towards simplifying the arithmetical treatment of these groups, by dealing with their so-called 'radicals' (or πυθμένες), approached even more closely still to the discovery both of the value of position and of the decimal zero.

The octads constituted a method of expressing any high numbers by collecting the alphabetical signs in groups of eight or 'octads,' decimally arranged; that is to say, not only were the figures within the groups decimally arranged according to the ancient Greek method, which was the same as the older Indian, but the groups themselves were placed in regular decimal order, the group of lowest value (which, in fact, bore the normal values of the signs) being placed to the extreme right, that group which was next highest in value standing on the left. Nevertheless the old self-sufficing signs were used *in* the group without value of position. Of the octads, Professor Leslie's article says that Archimedes used the idea to explain how it was possible to denote infinitely great series of numbers, "being aware of the theorem that the product of two numbers will have the sum of its numbers determined by the sum of their separate ranks—a conclusion which he deduced from the nature of a geometrical progression." It is, at the same time, clear that although the mathematical result may have been so presented to the mind of the philosopher, yet he could hardly have invented the actual method of setting it down, had he not been aware of the mode of *writing* numbers according to the decimal arrangement, the origin of which, from the combination of the various methods of speaking and writing, has been already discussed. Professor Leslie goes on to say (p. 196), "The fine speculation of the Sicilian astronomer does not appear, however, to have been carried into effect. Apollonius, who certainly holds among the ancients the next rank as a geometer, revised that scheme of numeration, simplified the construction of the scale and reduced it to commodious practice."

In others words, instead of the cumbrous 'octads,' Apollonius employed 'tetrads,' or groups of 'four' figures. In actually writing the groups, moreover, these were separated either by brackets or by a point; thus, to take an example of 'tetrads'—

(ρν) ζͅϡπδ) or ρν.ζϡπδ stood for 150.7984.

(ᾳϡϟα) (εͅσιδ) or αϡϟα.εͅσιδ stood for 1991.5214.

It will be observed that *within* each group the normal method of notation is still retained, and the lowest figure in the second group was exactly one decimal place above the highest decimal place in the group on the right.

The *πυθμένες* were also introduced by Apollonius. Professor Leslie (p. 197) gives the following example, which sufficiently explains their 'character and object.' "Suppose it were required to multiply '*κ*' and '*τ*' or 20×300. Instead of these, take the lower characters $\beta = 2$ and $\gamma = 3$ (the *πυθμένες*), which were called radicals, and multiply them, the product is the epistemon or ϛ, or six, which multiplied successively by ten and a hundred gives *χ* or 6000 for the result." After thus explaining the functions of the *πυθμένες*, Professor Leslie adds, "As that very important office which the cipher performs by marking the rank of the digits was unknown to the Greeks, they were obliged when the lower periods failed to repeat the letters '*Mυ*' or the contraction of *Mυρία*; thus to signify 37,0000,0000,0000, they wrote λζ *Mυ, Mυ, Mυ.* Where units (or monads) had to be expressed, Diophantus and Eutocius prefixed the contraction of *Mυ.*"

It must be admitted that with the use of the radicals (at least for the purpose of arithmetical calculations), and the use of the myriad signs to represent accumulated places of decimals, the Greeks came within 'almost a measurable distance' of the great discovery of the value of position. Professor Leslie goes on to say, indeed (p. 204), "Had Apollonius classed the numerals by 'triads' instead of 'tetrads,' he would greatly have simplified the arrangement and have avoided the confusion arising from the admixture of punctuated letters expressive of the thousands. It is by this method of proceeding by periods of three figures, or advancing by thousands, instead of tens, that we are enabled most expeditiously to read off the largest numbers . . . . . . It would have been a most important step to have exchanged these triads into monads by discarding the letters expressive of tens and of hundreds, and retaining only the first class, which with its inserted epistemon, should denote the nine digits: the iota, which signified ten, now losing its force, might have been employed as a convenient substitute for the cipher."

But, though the evidence thus adduced shows that the Greeks came very near the discovery of the value of position, it shows also that they approached it by methods wholly different from that with which the Indians did actually reach it, as was also the case with the knowledge of the decimal zero; but, as has just been said, these very facts militate against the belief that they ever had any real practical knowledge of either one or the other. For they actually approached the complete discovery so very closely in principle, that had they been acquainted with the abstract fact that numbers could be expressed always to any extent by a decimal arrangement of the unit signs only, their progress would hardly have been arrested at the stage to which alone it can be shown to have arrived, and it must ultimately have reached the full perfection of the Indian reformed method.

And while there is no evidence that they ever arrived at this stage of knowledge, there is very strong presumptive proof that they did not, until at least a comparatively late period, for they certainly continued not only to employ the mode of numeration, which, as shown above, takes no real heed of the value of position or of the decimal zero, but it can be shown that they also retained in use methods of arithmetic which were inconsistent with such knowledge.[1]

[1] It is hardly necessary here to refer to the supposed discovery announced by Niebuhr (as having been established to the satisfaction both of Playfair and of himself) of the Arabic numeral signs and of the zero (the decimal zero) used according to the true value of position, in a Greek MS. (a palimpsest in the Vatican Library), which is supposed to be of the seventh century Supposing even the fact as stated to have been correctly ascertained, still so far as the figures themselves and the value of position are concerned, these still might well, looking at the date of the MS., have had an Indian origin, although the discovery would have militated against the comparatively late date which has been assigned

Here it may be observed that Sir John Leslie's paper, as well as that of M. Delambre, on which it is founded, and which is probably to this day the most complete sketch of ancient Greek arithmetic, embraced a complete survey, not only of the arithmetic of Ptolemy, but of that also of his successors and commentators, *e.g.* of Theon of Alexandria (father of the celebrated Hypatia), who flourished in the latter part of the fourth century A.D., and of the still later Eutocius of Ascalon, whose commentaries were certainly not written earlier than the fifth century A.D. No trace, therefore, of the value of position or of the decimal zero, can be found in Greek arithmetic up to that date. The Greek mathematicians were, moreover, early studied by the Arabs. Euclid, Diophantus, and Ptolemy were soon known to them; the Almagest of the latter was translated by Thábit bin Korrah, who died in 288 A.H. = 901 A.D., and it can hardly be believed that they were ignorant of the best and latest commentaries on these authors, yet they certainly were unable to discover among them any knowledge of either of these inventions, for, as has been shown, they universally ascribe their acquaintance with them to communications from the Indians, the earliest of which, as we have seen, took place about 776 A.D. Moreover, until just before that date, they continued to use both the Greek alphabetic numerals and the Greek mode of accounting in their books of the public revenue.

If these arguments be conclusive against the possession by the ancient Greeks of a knowledge of either branch of the reformed system of numeration till after a date when its use was already well established in India, the same thing may be practically said of the Romans, whose arithmetic was avowedly derived from the Greek, as shown by the quotation already given from Isidore of Seville. The only ground, indeed for a different opinion, is a passage said to occur in a fragment of the Geometry of Boethius, a fuller examina-

to the zero. Professor Spezi has, however, demonstrated by a careful re-examination of the MS. itself, that Niebuhr's decipherment was clearly erroneous, and that in fact the supposed numerals, so far as they are numerals at all, are the ordinary Greek alphabetical numerals. Cantor, pp. 386-388 and note p. 248.

tion of which will be made in connection with the claims of the Neo-Pythagoreans. Neglecting for the present, however, this passage, it may fairly be said that if either Greeks or Romans had a theoretical knowledge even of the value of position, or at least of the possibility of expressing any series of numbers by a distinct arrangement of units only, it was a knowledge justly described by Humboldt as unfruitful. It is, however, quite *possible* that the facts which seem to have suggested its discovery in India were less prominently brought to general notice among the nations of Europe. Of course *if* the abacus was generally used perpendicularly, the idea itself can hardly have altogether escaped notice. But it has been shown that the two Roman instruments which have been preserved seem certainly best adapted, on the whole, for use horizontally, and even the Salaminian abacus appears to have been used in a horizontal position by the person actually working it; and, as the Indian and Chinese instruments have always been so used, it seems likely, at least, that this was the general mode of using the instrument also in Greece, Rome, and Egypt. Even then it seems difficult to believe that the ordinary form of instrument can have been much used without suggesting the discovery. It may be remarked, however, that the scale of the Salaminian abacus is quinary and not decimal, and if that was the usual form of the Greek abacus, it would of course conceal the idea from casual observers. The clumsy Roman notation, too, may in their case have helped to conceal the underlying principle; thus 847,986 written as VIII.IV.VII.IX.VIII.VI. would hardly suggest a decimal succession so clearly as $\eta\ \delta\ \zeta\ \theta\ \eta$ ϛ. Still after all it is difficult to believe that the abstract idea of the value of position was wholly unknown to the ancients, especially to the Greeks, even though not utilized by them; and perhaps the most probable explanation is that suggested by M. Martin (quoted by Woepcke, Journal Asiatique, vol. i. series 6, p. 236): "Ce qui a empéché les Grecs d'arriver à ce changement si simple qui avait été pourtant un perfectionnement notable, c'est qu'ils en etaient precisement trop prés pour en sentir vivement le besoin." Indeed, in the new system, and especially in the new system without the zero, there would perhaps have appeared to the Greeks, at any rate at first sight, no very appreciable advantage over the method which they were already employing, or at least none sufficient to warrant the setting aside that which was familiar to and sufficient for them. Under these circumstances perhaps it would be not very strange if these nations should both alike have failed to take any practical advantage of the value of position, even if known to them as an abstract truth. At least it would be less strange than the fact, which is to this day still a living fact, that a large proportion (even though a minority) of the population of Southern India till this day adhere to the principle at least of the ancient Indian numeration, and neither employ the value of position nor the zero, though these both have been known and used by Hindu races living in close proximity with them, for at least a thousand years, and have been employed among themselves officially for many centuries by the Mahommedan rulers who governed them.

Leaving the classic western nations, however, the next set of claimants for the honour of discovering the modern systems of numeration and arithmetic are the Neo-Pythagoreans, of whom, for the immediate purposes of the argument, it will for the present be sufficient merely to assume that they were a school or sect of philosophers who professed, among other things, specially to represent the disciples of Pythagoras and to preserve or revive his doctrines and teachings. They especially, also, affected scientific knowledge, came into existence really *about* the first century of the Christian era, and, like many other similar philosophic sects, had their head-quarters at Alexandria, in Egypt. So much will suffice for the object of discussing their claim to the original invention, or at any rate to the independent invention of the value of position with all its resulting advantages. Hereafter (in Part III.), when dealing with the share which these philosophers took (and it was a very important one) in the propagation of the new system in the West, it will be necessary to enter somewhat more fully into the history and character of the Neo-Pythagorean sect. Their claims to the present discovery rest, it may be said, mainly on the allegations of certain mediæval writers, and on the fact that they certainly did possess and teach at a very early period, both the value of position and a set of special ciphers or signs for the units employed, of which, as will be shown, they probably introduced the knowledge into Europe, and which they apparently claimed as an integral part of the Pythagorean teaching, or which were at least supposed to be such by those whom they taught.

These claims received additional force from the discovery during the seventeenth century of certain MSS. of the Geometry of Boethius, containing a description of methods which indisputably involved a knowledge of the value of position, and which were accompanied by certain peculiar signs termed 'apices,' used to represent the units. Both these inventions were unhesitatingly attributed in the MSS. to the Neo-Pythagoreans. Now, as Boethius was certainly put to death in 525 A.D., it is clear that if the MSS. really represent his actual words, they practically decide the question, and show that at this early date the Neo-Pythagoreans possessed both the value of position and the 'apices.' It becomes therefore of great importance to examine this position, and to test it both by external and by internal evidence. That is, external evidence as to the genuineness of the MSS. of Boethius, and the internal evidence as derived from the methods themselves as represented in them. As to the first question, it was soon pointed out by various writers that while the earlier part of the MSS. seem undoubtedly to represent, with more or less accuracy, the real teachings of Boethius, that the latter part of the first book of the 'Geometry' (to which this description belongs) was omitted in several of the MSS. of best authority, and that it was, moreover, couched in language the style of which did not well accord with that of the rest of the work. As to the second question, M. Woepcke also showed in his paper of 1863 that the 'apices,' or peculiar numeral signs used, were really of Indian origin, as was proved not only by their manifest correspondence with the Indian signs, but by their express identification as Indian by an early Arab writer.

The Neo-Pythagoreans used also certain peculiar names for the units, which Radulphus of Laon considered to be of Chaldean origin, and to have been introduced by Pythagoras with the abacus. It will be attempted to show that these too are partly Indian, and all or nearly all of Oriental character, while some of them are certainly far later than the time of Pythagoras, and some of the signs even more modern than those of Boethius himself. Lastly, it will be attempted to show that the peculiar form of the "arcus Pythagoreus" used by the Neo-Pythagoreans is clearly not of Greek but of Oriental origin. In short, it will be attempted to show, not only that there is insufficient proof of the genuineness of the MSS. of Boethius, but that the methods of the Neo-Pythagoreans, as set forth in them and by other mediæval writers, are essentially such as could not have been handed down from the date of Pythagoras, or have (all of them) existed even in the time of Boethius, but that they bear internal and conclusive marks of being derived from the Indian method as it existed just before the invention of the zero.

M. Woepcke, in his paper in the Journal Asiatique of 1863, accepting the Indian system as at once completed by the invention of the zero, was led to adopt a curious theory in order to account for the appearance of distinct traces of the new Indian notation, but *without* the zero, in the earliest Neo-Pythagorean methods (Journal Asiatique, vol. i. ser. 6, pp. 78, 79, and pp. 243-48). He thus reviews the case:—"En somme, si l'on examine, signe pour signe, les chiffres du Manuscrit d'Altdorf[1] d'une part, et les anciennes initiales des numératifs Sanscrits d'autre part,[2] la coincidence des deux suites de signes me parait telle qu'il est impossible de la considerer comme purement accidentelle. Mais si elle est la conséquence et la marque d'une affinité réelle, elle ne peut signifier qu'une chose, a savoir que les Néo-Pythagoriciens

[1] The MS. of Altdorf is that of Boethius, in which this passage was first discovered in full.

[2] This refers to Prinsep's theory that the Indian numeral signs were in reality the initial letters of their written equivalents, a theory which has long since been abandoned, and which has been dealt with virtually in the discussion as to Aksharas in Part I.

d'Alexandrie ont reçu de l'Inde les signes que certains d'entre eux employaient dans leurs operations d'arithmétique pratique. . . . . .

Je viens de dire que les nouvelles méthodes remplacérent chez les Arabes occidentaux la tableau à colonnes par l'emploi d'un dixième signe, c'est à dire, du zero . . . . Cette circonstance nous permets en même temps de nous faire une idée plus exacte de la manière dont les Neo-Pythagoriciens reçurent de l'Inde la forme de leur chiffres, fait que nous revêlent les figures de ces chiffres d'après les documents, placés çi-dessus sous les yeux du lecteur. . . . . . Il faut en conclure qu'il n'arriva à Alexandrie que des rapports plus ou moins vagues touchant le fait d'une existence de dix signes employés dans l'Inde, et propres à exprimer tous les nombres imaginables, en prenant une valeur de position ; et que ces rapports étaient accompagnés de listes représentant les figures des signes au moyens desquels on pouvait réaliser un effet si extraordinaire. Les Neo-Pythagoriciens cependant, familiarisés avec l'étude des nombres, devaient reconnaitre aisément que la même idée se pratiquait au fond sur les machines à compter, en usage depuis longtemps chez les Grecs et les Romains. Il ne pouvaient pas manquer de comprendre que les signes merveilleux de l'Inde etaient le moyen de transformer l'abacus manuel en un abacus écrit, et le syncretisme Alexandrin amoureux du prestige mysterieux qui entourait les idées et les symboles venus de loin, et surtout de l'Orient, amalgama les figures Indiennes avec les pratiques Grecs et Romaines dans le système de numération et de calcul dont nous trouvons l'exposé dans le passage de Boéce. Mais il faut prouver encore que rien nous empêche d'admettre que l'emploi des dix signes, avec valeur de position, ait existé dans l'Inde et ait pu être transporté de la en Alexandrie, centre de civilization Néo-Hellénique dans les premiers siècles de notre ère."

If, as it has been attempted to show above, the value of position with the 'abacus écrit' was known in India as early as the very beginning of the sixth century A.D., and if the decimal zero was probably *not* known or invented till after Alexandria was already in the power of the Arabs, and no longer the centre of Neo-Hellenic civilization, or of Neo-Pythagorean philosophy, M. Woepcke's complicated suggestion becomes unnecessary; and the natural deduction, that the Neo-Pythagoreans received the discovery in the precise shape, in which it was actually used in India at the time, affords a simple and sufficient explanation of all existing facts.

This explanation, therefore, it will be endeavoured to support by an examination in detail of the arguments already enumerated above.

It is clear, at any rate, that the older Greeks and Romans cannot be said to have *practically used* the value of position, whatever theoretical knowledge they may have possessed of that method of expressing numbers, except so far as the question is affected by the celebrated passage of Boethius, to the existence of which attention was virtually drawn by Vossius and Weidler during the year 1727 from the MS. at Altdorf.[1]

The authenticity, as already said, of this passage has been the subject of much learned discussion, which is very impartially summed up by M. Woepcke at p. 39-44 of his Memoir in the Journal Asiatique of 1863. M. Martin, in the Revue Archéologique (1856-57), has maintained the genuineness of the passage, while it has been assailed by Mr. Halliwell in his Rara Mathematica, as being an interpolation, which is not found in two at least of the best MSS. of Boethius; and this view is confirmed by the criticisms of Lachman and Boeckh, founded mainly on the fact that the language of the whole passage differs entirely from that of the rest of the work. To this argument I would venture to add another, which, if accepted, seems conclusive. It is hardly likely that a mere transcriber should alter the actual numerical signs given by Boethius, as the very signs employed by the Neo-Pythagoreans. They are given, that is, not as showing the signs in current use when the MS. was written, but as those

[1] The passage had been printed as early as 1499, and again in two or three later editions, but in a corrupt and unintelligible condition.

specially belonging to the Neo-Pythagoreans of the time of Boethius himself. Now if we turn to the facsimile of them, which M. Woepcke has given at p. 75 of his Memoir in the Journal Asiatique, and at p. 10 of his Memoire sur l'Introduction d'Arithmétique Indienne, it will be seen that the sign for the cipher 'four' is unquestionably a copy of the *Arabic* form of that cipher; which, again, it has been shown, is a Mahommedan corruption of the peculiar *Northern* Indian or Hindu Kabul form, which certainly was not known to the Arabs, even in its original shape, till 776 A.D., 250 years after Boethius died! The other ciphers, as will be more fully shown hereafter, are all also of Indian origin.

The evidence therefore of this passage, on which so much has been built, can hardly in itself be admitted as showing that Boethius stated that the Neo-Pythagoreans knew and used even the value of position at the date when he lived. But while the statement which has thus been considered cannot be accepted as that of Boethius, or as showing that either the Romans or the Neo-Pythagoreans knew the value of position and the peculiar signs for the units in the fifth century, the general question of the internal evidence to be derived from the Neo-Pythagorean methods themselves remains to be considered, and as to these the MS. of Altdorf affords instruction which is of considerable value. It was probably actually written in the eleventh century, and the knowledge which it claims for the Neo-Pythagoreans may therefore be accepted as that which was really possessed by them at that period, and probably also at a considerably earlier date. This comprised a knowledge of the 'tableau à colonnes,' arcus Pythagoreus, or 'written abacus' (involving a knowledge of the value of position), the use of the unit numerals only, the employment of peculiar signs for these (really of Indian origin), and the employment also of these instruments and methods for arithmetical calculation, according to modes based on the principles of modern decimal arithmetic.

[1] See Pl. IV. Table III., where sets of these "Apices" are given from various sources.

The whole system of the Neo-Pythagoreans seems indeed to have been at that time put forward as derived by regular devolution from the teaching of Pythagoras. To what extent this claim was based on the truth, and what part the Neo-Pythagoreans themselves had in its invention, will be discussed in the sequel. It is here traversed only so far as regards the first invention of the arcus Pythagoreus, or written abacus, and the value of position, the use of the forms of arithmetic which that rendered possible, and of the special unit signs.

These signs, it may be remembered, were not always employed, they were replaced sometimes by counters, sometimes by the Greek alphabetical numerals. But the special signs were also known, and owing, no doubt, to their distinctness and incapability of confusion, and to the facility with which they were written, these always were largely employed, and eventually superseded all the other modes of marking numbers, and from them it will be shown that our modern numerals unquestionably descend. It has already been asserted that these Boethian 'apices' are of Indian origin, on the authority of an Arabic work (the Commentary of Al Kalaṣadi, Woepcke, J. A., vol. i. series vi. p. 38), who particularly says that the Pythagorean signs are identical with those of the Gobar, and that these came from India. This fact will further be established when the forms themselves and their descent from the Indian originals comes under full discussion in Part III. For the present it will suffice to refer to what has just been said as to the late form of the Neo-Pythagorean 'four.' When the derivation of these forms comes to be more fully considered, it will be found to throw much light on the origin and history of the forms themselves and of the system to which they are attached, and even on the part played by the Neo-Pythagoreans in the introduction of these last into Europe, and even on the approximate date of this event. Meanwhile, some attention may be bestowed on another point of some interest, viz. the names bestowed by the Neo-Pythagoreans on their 'apices' or unit signs. These names being entirely different from those belonging to any European language, have already been the subject of much speculation. It has been seen that Radulphus of Laon so early as the twelfth century assigned both to these names, and to the abacus with which they were associated, a Chaldean origin. As regards the abacus itself, it has been already shown that the assertion is in full accordance with what seems the consistent course of Greek tradition, and there is, *per se*, no improbability in it. Pythagoras is said to have used the abacus to teach his arithmetic, which, in part, at any rate, is described as of Babylonian invention, and the abacus in its very name seems to bear traces of Eastern origin; it was pretty certainly widely known all over the East at a very early date, and it may very possibly have been invented in—at all events it seems to have reached Greece from—Babylon. The system of Pythagoras was, no doubt, the foundation of all the early Greek arithmetic, and was fundamentally the same as that which (though with improved methods) the earlier Neo-Pythagoreans used and taught. There is nothing therefore impossible in the suggestion that some reminiscence, at least, of the names of the Chaldean units may have survived also, though in a more or less corrupted form, to Neo-Pythagorean times. In fact, some of these names have already been pretty clearly identified with those belonging to certain of the Semitic languages, viz. Hebrew and Arabic, and through these may be traced back to ancient Assyrian originals. The whole of the Neo-Pythagorean names of the nine units will therefore now be given (from M. Woepcke's paper in the J. A., mainly taken from the fragments of Boethius; see also notes, Cantor, M. B. p. 414), together with the Hebrew, Arabic, and ancient Assyrian equivalents,[1] and an attempt will be made to trace the connexion between these last, and the Neo-Pythagoreans, and to account for the differences when these have no resemblance to their ancient Assyrian or to their Arabic and Hebrew equivalents.

[1] I am again indebted for these (in the form now generally accepted) to the kindness of Mr. Pinches of the British Museum.

Neo-Pythagorean.	Assyrian.	Hebrew.	Arabic.
1 *igin*	*êštin*,[1] *êdu* or *aḫad*	*ăkhad* אחד	*ahad* احد
2 *andras*	*šana*	*shanăim* שנים	*asnín* اثنين or *aṣnáni* اثناني
3 *ormis*	*šalšu*	*shĕlŏshah* שלשה	(*t*)*ṣalṣ* ثلث
4 *arbas*	*arba* or *irba*	*arbayah* ארבעה	*arba* اربع
5 *quimas*	*ḫamšu*, *ḫaššu* or *ḫauša*	*khamissah* חמשה	*khams* خمس
6 *calcis, caltis* or *χαλχος*	*šiššu*	*shissa* ששה	*sat* ست
7 *zénis*	*sibu*	*shibah* שבעה	*sab'a* سبع
8 *temenias*	*samnu*	*shemónah* שמונה	(*t*)*ṣamán* ثمان
9 *celentis*	*tišu*	*tishah* תשעה	*tisa* تسع

[1] 'š' is the Hebrew ש 'sin,' and 's' the Hebrew Sameth or Semeath ס, ḫ is the Arabic خ (rarely ح) and the Hebrew ח (kheth) and on the principle adopted in the Hebrew and Arabic columns, may be read as "kh."

It is to be remembered that the language of the ancient Egyptians was, like the Assyrian, of the Semitic stock, and some of the Assyrian terms for the numerals show strong resemblance to the Egyptian; thus the Assyrian 'sana' two, 'sissu' six, and 'samnu' eight, are palpably the same as their equivalents in Egyptian 'sen,' 'sas,' and 'sesennu.' Even the Assyrian 'arba' four, 'sab'a' seven, and 'tisa' nine, may be perhaps severally identified with Egyptian 'aft,' or 'avt,' 'sefech' or 'sevech,' 'sechef,' or 'sechev,' and 'peset' or 'psit'; there exists indeed further evidence of this connection, but important as the subject is, it is not possible to pursue it further here. It might be thought that the Neo-Pythagoreans, so closely connected by their founder and by their long settlement at Alexandria with Egypt, may possibly have got these terms direct from the Egyptians, but the 'arba,' 'quimas,' 'temenias,' and still more notably the 'zenis,' so obviously come through the medium of Arabic or Hebrew (in all probability the latter), which themselves descend from the ancient Assyrian, that there seems no room for such an hypothesis.

Now the resemblance between the Neo-Pythagorean 'arbas' and the 'arba' of the Assyrian, Hebrew, and Arabic vocabulary has long since been pointed out, as also that of 'quimas,' with khams and khamissah, which come from the Assyrian 'ḫamšu' or 'khamṣu'; so also the 'temenias' of the Neo-Pythagoreans is palpably the Arabic 'tsamán,' and the Hebrew 'shemónah,' which are practically identical with the Assyrian 'samnu' [see Woepcke, J. A., vol. i. series vi. pp. 47–52), quoting MM. Vincent, Martin, Bienaymé, and others]. The 'igin,' 'one,' may perhaps be derived from a hardened form of the Assyrian 'estin,' as in certain Aryan languages the 's' and 'sh' pass readily into 'k' or 'kh,' which again, in the later Semitic languages, is readily interchangeable with 'g,' and the 't' of the root might easily have been dropped for euphony, and thus 'êštin' would become successively 'ektin' or 'iktin,' 'igtin' and 'igin.' Similarly, if the sibilants of 'šiššu' (the Assyrian six) be hardened, it would become 'khikhkhu' or 'khakhku,' in Greek *χιχχυ* or *χαχχυ*, and an 'l' introduced for euphony in lieu of the middle *χ* would make the word *χαλχυ*, and with a Greek termination *χαλχος*. It is possible that even 'celentis' may come from 'tisu' by some such process, at any rate there is no better derivation for it;

> "Mais il faut avouer aussi,
> qu'en venant de la jusqu'à ici,
> il a bien changé sur la route."

It is clear, however, that if the sibilants which prevailed in the old Assyrian terms were all to be hardened after the same fashion, the result would be too great a similarity between the names of several of the units. For example, the result in the case of Assyrian 'salšu' or three would be actually identical with that arrived at in the case of the 'šiššu' or six. The Neo-Pythagoreans would seem, therefore, to have sought elsewhere for appellatives to fit the 'two,' the 'three,' and the 'seven.' The name they adopted for the latter, *Ζηνις* or 'Zenis,' is nothing but the transliteration into Greek letters of the Hebrew or Arabic word 'zain,' which in those languages designates the letter 'z,' and z being the seventh letter of their alphabet, has, when used as a numeral, the power of 'seven.' This fact shows clearly that in this instance the Neo-Pythagoreans borrowed from one or other of the Semitic languages which employ the comparatively recent form of *alphabetic* numerals. The equivalents chosen for 'two' and 'three' by the Neo-Pythagoreans are even more instructive still, for they prove almost conclusively that the Neo-Pythagoreans had access to Indian sources and made use of them. At least the nearest, indeed the only parallels of the Neo-Pythagorean 'two,' 'Andras,' and of the Neo-Pythagorean 'Ormis' or 'three,' in any known language, are the Tamil[1] 'Iraṇḍu' and 'Muṇṛu,' 'two' and 'three' respectively, and the resemblance is here so close that it is hardly to be doubted that the Neo-Pythagoreans did adopt these terms from a Southern Indian source. Prompted by the 'syncrétisme Alexandrin,' as M. Woepcke describes it, "amoreux du prestige mysterieux qui entourait les idées et les symboles venus de loin, et surtout de l'orient," the Neo-Pythagoreans seem to have followed, as far as they could, the traditional Pythagorean names, and when these could not conveniently be clad in a Greek dress, they went to other Oriental sources to supply the deficiency, and amalgamated all into one cabalistic and mysterious series. As will be shown hereafter more fully, such an arrangement was exactly in accordance with what might have been expected of them.

Less stress need be laid on the similarity of the methods of the Neo-Pythagorean arithmetic with those of the earlier Indian methods as shown in Alkhwárizmi's methods; for both are simply the natural methods of working the value of position by the Arcus Pythagoreus, but there is *one* peculiarity of the latter table, as used by the Neo-Pythagoreans, to which attention must be drawn, as being in all probability a mark

[1] It may be objected that these words might have come not directly from the Tamil, but from some older Dravidian form lingering more to the West. But the words for 'two' and 'three,' in what are deemed the older Dravidian tongues, such as the Biluch, differ almost wholly from "Andras" and Ormis. The Malayalam approaches rather more closely, but the Tamil affords the nearest analogues.

of its Oriental derivation. This is the fact that its columns were connected together in groups of '*threes*;' *i.e.* the columns for units, tens, and hundreds are collected under one semi-circle, and those for the thousands, tens of thousands, and millions, under another arc or semi-circle, and so on. It is this grouping by 'triads' which Dr. Cantor calls the "Roman method," and he points out the distinction between this method of grouping and the ancient Greek method of grouping by 'octads' or 'tetrads.' But it seems to be found at an early period in Oriental countries, whither it could hardly have come from *Rome*, and where an explanation of its use is found in the passage already quoted in Part I., from Sibth al Máridíní, to the effect that *the* primitive mode of reckoning comprised only units, tens, and hundreds, a remark which it has been said appears to refer to the ancient Phœnician non-alphabetical mode of numeration, and its derivatives (chiefly Asiatic), in which there was no *separate* symbol for any number above the hundreds, all others being expressed by *groups* of numbers.[1] It is clear that if the Arcus Pythagoreus had been of Greek parentage, the grouping would naturally have been rather by 'tetrads' or 'octads,' and that this peculiarity of its structure is therefore a palpable indication of its Oriental origin.

It is indeed this Neo-Pythagorean method of grouping in triads on the Arcus Pythagoreus, that has given rise to our modern method of similarly grouping numbers in triads, by commas or dots, for facility of calculation, as, for example, when we write 469,367,000.[2]

To sum up the case, therefore, the Indian claim to invention of the value of position and the zero rests first on the distinct and direct testimony of Arab historians, and other Arab writers, to that effect; on the certainty that it was practically used by the Indians at a date considerably anterior to that at which it can be really shown to have been used by

[1] The limit may be a survival of the primeval plan of counting by groups, but this question cannot be discussed now.

[2] That this trinal mode of grouping is a point of some importance may be seen from Professor Leslie's words already quoted on p. 92.

any other people; and the Indian claim to the 'zero' rests on exactly similar grounds. But the invention of the 'zero' was probably subsequent to the other, inasmuch as for all public and official purposes the old system of notation was certainly used in India long after the value of position was known, as would naturally be the case if the former was first known without the zero, and capable of employment only with the 'tableau à colonnes.' Moreover, the Indians appear, from the methods in which their early arithmetic was cast, thus to have used the value of position at first on a 'tableau à colonnes,' which would not have been necessary if they had simultaneously discovered the zero. Again, the use of the tableau à colonnes in Persia, and especially on the immediate confines of India, at a very early date, had already induced M. Rodet to suggest that this contrivance had its origin in India. In addition to these facts, all the many terms by which the 'zero' is known, seem to be derived from the 'place vide' or 'tableau à colonnes,' and can hardly have designated anything else. Moreover, systems of numeration still actually exist in India, which seem to show the intermediate step by which the invention of the zero was apparently suggested, viz. the substitution of unit signs, differentiated by the sign for 'ten,' in replacement of the arbitrary and cumbrous signs for the powers of 'ten' of the older Indian system. Lastly, the ancient forms of the sign for 'ten' seem to have been retained to a late period by Arabic writers, in order to represent at least the sexagesimal or Ptolemaic zero, the Greek sign for which had (perhaps on account of its more convenient form) been apparently adopted at a very early period to replace, as the sign of the decimal zero, the clumsier forms of the old Indian sign for ten.

The ancient Greeks, whether they knew or did not know, as a mere abstract and curious scientific fact, the power of the unit signs arranged in decimal order to express any series of numbers, at any rate made no practical use of the fact; and though they had a sexagesimal zero, it is clear they never had a decimal zero. Their leading arithmeticians neither used nor taught the use of either invention down to the sixth century, by which time the value of position at least was well known and employed in India. Indeed, it is clear that the Greeks knew or at least used neither commonly, down to the middle of the seventh or beginning of the eighth century; for, as has been shown, up to the latter period their Arab conquerors were avowedly content to copy in their public accounts the Greek notation, which they found in use in Syria and Egypt, and which they afterwards distinguished from the new notation by describing the latter as '*Indian*'; and by the beginning of the eighth century the Indians had already commenced (or were on the point of doing so) the general use of the nine units *with* the zero.

As regards the Romans, too, it has been shown that their arithmetic was at least in principle borrowed from the Greek, through Apuleius and Nicomachus; and there is nothing beyond the so-called passage of Boethius on which any separate claim on their behalf to a knowledge of the value of position can be sustained. If, for the reasons given above, that passage is admitted to be the spurious interpolation of a much later writer, then the claims of the earlier Romans must stand or fall with those of the early Greeks. In discussing hereafter the real claims of the Neo-Pythagoreans, in connection with these discoveries, it will be necessary to notice the old Pythagorean methods somewhat more fully; for the present it may suffice to sum up the case by saying that there is no trustworthy evidence to show that the Greeks or Romans knew the *use* even of the value of position, down to the time when, in India, both that and the zero were alike known and used.

No doubt, as will be shown, it is probable that about this time the Neo-Pythagoreans learned at any rate the value of position; but as it was palpably unknown to the older Greek and Roman writers, it is not from them that the Neo-Pythagoreans could have derived their knowledge. Nor is it probable that it was from any esoteric tradition handed down from Pythagoras or his immediate successors that this knowledge can have come. The whole school of Pythagoras, old and new, seem especially to have devoted themselves to the teaching of mathematical science, and it is hardly likely that doing so they should have concealed its best processes, and kept them for esoteric use. Nor, indeed, if they had wished to do so, could these inventions well have been handed down for many centuries without being ever divulged or employed for vulgar use.

If then the Neo-Pythagoreans cannot be shown to have derived them from any earlier Western source, can the invention have been made by them independently? Unquestionably the value of position as an invention might well have been independently discovered by two or more sets of persons in possession of all the antecedent conditions, those which have been set out at length in the beginning of this paper, and which the Greeks and Neo-Pythagoreans possessed in common with the Indians. But, as a matter of fact, they did not apparently know, or use the invention, until a date by which they might well have obtained it from India, and for this (as will be shown presently) ample facilities existed.

Again, when it appears first in Neo-Pythagorean hands, the new method is accompanied by various signs and tokens of Oriental, and indeed of Indian origin. The 'apices' or signs which the Neo-Pythagoreans claim as their special property, are certainly Indian in their form; this is shown not only by the direct evidence of Greek and Arab writers, but also by the internal evidence of their shapes, as will presently be more fully proved. Moreover, the Neo-Pythagoreans employ for these signs and for the units certain quasi-cabalistic names, all, or nearly all of which, are apparently of Oriental descent, and, in two cases, of distinctively Indian derivation.

M. Woepcke, as has been seen, pressed by the difficulty that the Neo-Pythagoreans at first seem to have been unacquainted with the zero, but to have known only the value of position, and as he held the belief that the Indian reform included from the first a knowledge of the zero, was induced to invent a theory supposing a partial acquisition of the discovery from India by the Neo-Pythagoreans, supplemented by a partial re-discovery on the part of the Neo-Pythagoreans themselves of the use of the nine units on the abacus. If, however, as it has been attempted to demonstrate, the first Indian invention did *not* include or employ the zero, while in all probability it did in its first stage and for some time later employ the 'arcus Pythagoreus,' the very form of which really attests its Oriental derivation, then M. Woepcke's theory is no longer needed to reconcile the undoubted facts. Indeed these will, more clearly than ever, favour the theory of a direct importation from India. The Neo-Pythagoreans in short will be found to have used in their early state of knowledge only what the Indian writers knew and taught in the first stage of their discovery, and to have used it, not with Greek forms, but with those derived altogether from Oriental, largely from Indian, sources. Under these circumstances it does not seem extravagant to claim the credit of both stages of the invention for the Indians. To the Neo-Pythagoreans, on the other hand, belongs the merit of the first introduction of the reformed method into Europe, and this it will be the main purport of the concluding part of this paper to trace out and discuss. But both the new notation, and the improved arithmetic which it rendered possible, were pretty certainly both Indian in their inception. Indeed it may be said that they were both fully developed in India, for the final shape in which the common arithmetical processes appear to have been transmitted from India, viz. those which Leonard of Pisa expressly distinguishes as 'Indian,' really left but little for European mathematicians to improve, so far as the processes themselves extend.

There is nothing to show to what individual, or individuals rather, we owe these reforms. In the passage of Masaudi already quoted, there is an allusion which seems to be intended to designate Aryabháta; but if this be its meaning, it seems rather to indicate him as the great teacher of the new system (which he seems to have been) rather than its inventor; further than this our present knowledge does not enable us to go. But whoever the separate discoverers of the use of the value of position and of the 'zero' may have been, it is hardly too much to say that their inventions have probably done more than any others—not perhaps excepting even those of printing and of the steam engine—to advance the progress of scientific knowledge and of material civilization. Had modern students been confined to the lumbering processes of the older Greek arithmetic, it would hardly have been in their power to work out the intricate calculations on which our astronomy, chemistry, mechanical knowledge, indeed all branches of scientific knowledge and research, so largely depend.

It has, however, been here attempted to identify, if not the individual inventors, the nation to which they belonged; and it may be perhaps said that it is possible to indicate within certain wide limits the locality of its birth. As has been seen, even to this day, the new system is not accepted by a large minority at least of the inhabitants of Southern India, and it can therefore hardly have been indigenous there. It was for a long time also unknown in the extreme East, in the hills for example of Nepál. The inscriptions which Dr. Bühler has published from that province show clearly that the older system of notation remained long in use there, after the new one had been established elsewhere. It may be said therefore that the reform of the old system must have arisen north of the Vindhya and west of the Himalayan ranges. On the other hand, the peculiar shapes of the numerals associated with the new system in the earliest inscriptions seem to make it at least probable that it did not arise amongst a race who used the Northern Indian forms. It may be said, therefore, that the reform was matured and perfected south of the Indus. If any portion of the credit is to be assigned to Aryabháta as regards the value of position—then, since he was a native of Kausámbhi—it is possible that this portion of the discovery arose somewhere in 'Madhya-désa,' *i.e.* in the valley of the Ganges and the Jumna. On the other hand, if the conjecture as to the connection of the first suggestion of the zero with the notation of the Gupta era is admissible, then this final step may, as has already been said, have belonged to Western India, and this is the rather probable as all the earliest instances of the use in inscriptions of the new system, have come from places in the neighbourhood of the western coast, as Okamandal, Bharuj, Morbi, etc.

This point, however, is not material to the general course of the history of the Indian numerals and of the peculiar arithmetic with which they are associated. They have now been traced from their first origin down to a stage at which they may be called practically complete. If the arguments used in this paper are correct, this stage has been reached, as in the case of almost all other important dicoveries, by a process of gradual evolution.

The semi-savage, who counted upon his fingers and recorded the results of his calculations in rows of mere scratches upon the sand, gave the first hint of the abacus. So the rude numeral signs composed of groups of simple lines themselves were gradually superseded by other more compact and convenient symbols. These, applied to the abacus with its primitive decimal system, led to the discovery of the value of position. Out of this again arose the Arcus Pythagoreus or 'written abacus,' with its accumulation of various series of numbers; and from this, in quick succession, came the new methods of decimal arithmetic; and lastly the invention of a sign to fill the 'place vide,' the 'súnya' or 'zero'; and the zero finally released the new notation and arithmetic from the trammels of the abacus, and rendered them perfectly applicable to all the purposes of social life.

In Part III. it will be attempted to trace by external evidence, how in all probability the Neo-Pythagoreans received in Egypt the Arcus Pythagoreus and the value of position without the zero, probably not very long before the Neo-Hellenic civilization of that country, and specially that of Alexandria, was overthrown by the Arabs. After this it will be attempted to demonstrate more particularly than has hitherto been done the connection of their numeral signs with those of the Indian system, and to indicate also the part played by the Arabs in the introduction of the zero, and in the formation of the 'Gobár' numerals.

In conclusion, it will be endeavoured, by a comparison of the Indian signs with those of the other derivative systems, to throw some light on the period at which the latter were received from India, and thus by the internal evidence of the signs themselves in some measure to corroborate the conclusions drawn from other testimony as to the channels and period of the propagation of the new discoveries among the nations of the West.